Bumpy Metrics and Branch Points of Minimal Spheres and Tori

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Two approaches to minimal surfaces in Riemannian manifolds:

1. Geometric measure theory (works in complete generality).

2. Global analysis on infinite-dimensional manifolds (works only for minimal surfaces of dimension two in an ambient space of arbitrary dimension).

We favor the second approach which emphasizes the relationship between minimal surfaces and nonlinear partial differential equations.

As mentioned yesterday, the simplest second-order linear partial differential equation of elliptic type is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Here we assume that f is vector-valued, that is, that it takes its values in $\mathbb{R}^N$.

The simplest way to make this equation nonlinear is to consider maps into a curved ambient manifold,

$$f : \Sigma \longrightarrow M \subset \mathbb{R}^N,$$

and replace the partial differential equation by

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)^\top = 0,$$

where $(\cdot)^\top$ represents orthogonal projection into the tangent space. Here Σ is a Riemann surface and (x, y) are locally defined conformal coordinates on Σ.

A solution to this nonlinear partial differential equation is called a *harmonic map*. The equation for harmonic maps can also be written in terms of the Levi-Civita connection D on the ambient Riemannian manifold M:

$$\frac{D}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{D}{\partial y}\left(\frac{\partial f}{\partial y}\right)=0.$$

If $z=x+iy$, we can also write this as

$$4\frac{D}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}\right)=0,$$

$$\text{where} \quad \frac{\partial f}{\partial z}=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right).$$

Let $E = f^*TM \otimes \mathbb{C}$. According to a theorem of Koszul and Malgrange, there is a unique holomorphic structure on the complex vector bundle E such that a section W of E is holomorphic if and only if $DW/\partial\bar{z} = 0$, where D is the pullback of the Levi-Civita connection. Thus the equation of harmonic maps simply says that

$$\frac{\partial f}{\partial z} \quad \text{is holomorphic}$$

as a section of E. This suggests that complex analysis (Riemann surfaces and holomorphic vector bundles over them) should play a major role in the theory of harmonic maps of surfaces into general ambient manifolds.

Of course, the holomorphic section

$$\frac{\partial f}{\partial z} \quad \text{of} \quad E = f^*TM \otimes \mathbb{C}$$

depends on the conformal parameter z on Σ. For an invariant description of harmonic maps, we can demand that the section

$$\frac{\partial f}{\partial z}dz \quad \text{of} \quad E \otimes K = f^*TM \otimes K$$

be holomorphic (where again K denotes the holomorphic cotangent bundle of Σ). The locally defined holomorphic sections

$$\frac{\partial f}{\partial z} \quad \text{of} \quad E$$

generate a holomorphic line subbundle L of E and

$$\frac{\partial f}{\partial z}dz \quad \text{is a holomorphic section of} \quad L \otimes K.$$

Recall the variational formulation given yesterday: Consider the energy

$$E : \mathsf{Map}(\Sigma, M) \times \mathcal{T} \to \mathbb{R},$$

which is defined by

$$E(f, \omega) = \frac{1}{2} \int_{\Sigma} |df|^2 dA.$$

Here Σ is a Riemann surface of a given genus g and $\mathcal{T}$ is the Teichmüller space of conformal structures on Σ. The norm of df and the area element on Σ are calculated with respect to any Riemannian metric within the conformal equivalence class selected by ω. (E does not depend on the choice of metric on Σ, only its conformal equivalence class.)

For fixed choice of ω, we also define

$$E_\omega : \mathsf{Map}(\Sigma, M) \to \mathbb{R}, \quad E_\omega(f) = E(f, \omega).$$

Critical points of E_ω are just harmonic maps, harmonic with respect to the conformal structure $\omega \in \mathcal{T}$. Critical points of the two-variable function E are not just harmonic, but it turns out that they are also weakly conformal, that is,

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle = \left\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} \right\rangle,$$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = 0.$$

We can also express this in complex form as

$$\left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial z} \right\rangle = 0,$$

when the Riemannian metric has been extended in a complex linear fashion to E.

In terms of conformal coordinates (x, y) on Σ, the formula for energy can be written

$$E_\omega(f) = \frac{1}{2}\int_\Sigma \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dxdy.$$

Thus it is really the celebrated Dirichlet integral. The ω-energy E_ω is closely related to the area function

$$A : \mathrm{Map}(\Sigma, M) \to \mathbb{R},$$

which is defined by

$$A(f) = \int_\Sigma \left|\frac{\partial f}{\partial x} \wedge \frac{\partial f}{\partial y}\right| dxdy.$$

To see that critical points for the two-variable energy E are also critical for A, one can reason as follows: It is a simple fact from vector algebra that

$$|v \wedge w| \leq \frac{1}{2}(v \cdot v + w \cdot w)$$

with equality holding only if

$$\langle v, v \rangle = \langle w, w \rangle, \quad \langle v, w \rangle = 0.$$

It follows from this that

$$A(f) \leq E(f, \omega)$$

with equality holding if and only if f is weakly conformal. It follows that a conformal harmonic map is a critical point for area, that is, a *minimal surface* in the classical sense.

Conversely, if $f : \Sigma \to M$ is a minimal immersion, and Σ is given the conformal structure induced by f, then f is harmonic.

However, critical points of our variational problem need not be immersions—branch points can occur, for example, when the ambient manifold M is Kähler.

Branch points correspond to zeros of the holomorphic section

$$\frac{\partial f}{\partial z} dz.$$

One way in which branch points can occur is in branched covers of minimal surfaces of smaller area.

We say that a nonconstant minimal surface $f : \Sigma \to M$ is a branched cover of a minimal surface $f_0 : \Sigma_0 \to M$ if there is a holomorphic map $g : \Sigma \to \Sigma_0$ of positive degree such that

$$f = f_0 \circ g.$$

(Recall that a branched cover $g : T^2 \to S^2$ can be identified with a meromorphic function on T^2. Whenever minimal two-spheres $f_0 : S^2 \to M$, they can be covered by minimal tori with branch points that persist under generic perturbations.)

Definition A nonconstant minimal surface $f : \Sigma \to M$ is *prime* if it is not a covereing (possibly branched) of a nonconstant minimal surface $f_0 : \Sigma \to M$ of smaller area.

It is natural to ask: What the properties of solutions to this nonlinear partial differential equation when the Riemannian metric on the ambient space is chosen to be generic? By generic metric, we mean that the metric belongs to a countable intersection of open dense subsets of the space of metrics when it is given an appropriate Sobolev norm. This notion of generic has been much used in the study of the Yang-Mills equation, the Seiberg-Witten equations, and other nonlinear partial differential equations of geometric interest. Generic properties of solutions to nonlinear partial differential equations are those which persist under perturbation in the data.

The technique used for studying generic properties is the Sard-Smale theorem from the theory of infinite-dimensional manifolds.

Bumpy Metric Theorem. For generic choice of Riemannian metric on the compact manifold M, all prime minimal two-spheres and all prime minimal tori in M are free of branch points and lie on nondegenerate critical submanifolds in the sense of Bott, each such submanifold being an orbit for the action of $PSL(2,\mathbb{C})$ in the case of the torus, $S^1 \times S^1$ in the case of the torus.

A preprint which gives a proof of this theorem is available at:

http://www.math.ucsb.edu/ moore/bumpy.pdf

Let us explain the definition of nondegenerate critical submanifold appearing in the statement of the theorem. We take the case of the torus for example. As mentioned yesterday, a difficulty in the projected Morse theory of minimal tori is that critical points for

$$E : \mathcal{M} = \mathrm{Map}(T^2, M) \times \mathcal{T} \to \mathbb{R}$$

can never be nondegenerate in the usual sense, because E is preserved by the group action

$$\phi : \mathcal{M} \times G \to \mathcal{M},$$

where $G = S^1 \times S^1$ is the torus group of rotations on the two factors.

Let $\mathcal{M} = \text{Map}(\Sigma, M) \times \mathcal{T}$.

A *nondegenerate critical submanifold* of $\mathcal{M}$ is a finite-dimensional submanifold N such that

1. every $p \in N$ is a critical point for E.

2. if $(f, \omega) \in N$, then $T_p N$ is the set of $X \in T_{(f,\omega)}\mathcal{M}$ such that $d^2 E(p)(X, Y) = 0$, for all $Y \in T_{(f,\omega)}\mathcal{M}$.

Our Bumpy Metric Theorem states that after a small perturbation of the metric, it can be arranged that all prime minimal tori lie on such submanifolds.

The case of the sphere is simplest in some ways, since the Teichmüller space of the sphere is a single point.

To understand Map(S^2, M) near a critical point f, we need to linearize the Euler-Lagrange equations at a critical point.

Just as the Euler-Lagrange equations for harmonic maps come from the first derivative of E, the linearization comes from the second:

$$d^2 E_\omega(f)(X,Y) = \int_{S^2} \langle L_f(X), Y \rangle dA,$$

for $X, Y \in T_f$Map(S^2, M). The second order elliptic partial differential operator L_f is called the *Jacobi operator*.

By definition, a *Jacobi field* is a solution

$$X \in T_f \mathrm{Map}(S^2, M) \quad \text{to} \quad L_f(X) = 0.$$

Holomorphic sections of the holomorphic line bundle L are always Jacobi fields, and the dimension of the space of such holomorphic sections is

$$6 + (\text{total branching order}),$$

in the case of the sphere. The Bumpy Metric Theorem implies that for generic choice of metric, the *only* Jacobi fields are holomorphic sections of L in the case where the domain is S^2, and the dimension of this space of Jacobi fields is exactly six.

To see why holomorphic sections of L are Jacobi fields, we can extend $d^2E_\omega(f)$ to a complex bilinear form on E. An integration by parts (carried out in the article by Micallef and Moore of 1988) shows that

$$d^2E_\omega(f)(W,\overline{Z}) = 4\int_\Sigma \Big[\Big\langle \frac{DW}{\partial \bar{z}}, \frac{DZ}{\partial \bar{z}} \Big\rangle \\ - \Big\langle \mathcal{R}\left(W \wedge \frac{\partial f}{\partial z}\right), \overline{W} \wedge \frac{\partial f}{\partial \bar{z}} \Big\rangle\Big]\, dxdy,$$

$\mathcal{R}$ being the curvature operator.

The first term on the right vanishes when V is holomorphic, the second when V is a section of L.

Definition. The *Morse index* of a nondegenerate critical submanifold $N \subset \mathsf{Map}(S^2, M)$ is the dimension of a maximal subspace on which $d^2E(f)$ is negative definite.

Here is a rough idea of the proof of the Bumpy Metric Theorem in the case where $\Sigma = S^2$:

In many ways, the hardest step consists of showing that for generic metrics, minimal spheres do not have branch points, but suppose that we have already done that.

Let $Met(M)$ denote the manifold of C^{k-1} Riemannian metrics on M. Let

$$\mathcal{S} = \{(f,g) \in \text{Map}(S^2, M) \times Met(M) \\ \quad |f \text{ is harmonic for the metric } g \\ \qquad \text{and } f \text{ has no branch points}\}.$$

There are two steps to the proof:

- $\mathcal{S}$ is a submanifold of Map$(S^2, M) \times Met(M)$.
- The projection $\pi : \mathcal{S} \rightarrow Met(M)$ onto the second factor is Fredholm of Fredholm index six.

We then apply the Sard-Smale theorem to show that almost all elements $g \in Met(M)$ are regular values for π. If g is a regular value for π, all the minimal two-spheres in $\pi^{-1}(g)$ lie on one-dimensional nondegenerate critical submanifolds.

This proof outline is identical to the proof of corresponding theorems in Yang-Mills and Seiberg-Witten theories.

The Bumpy Metric Theorem (together with other facts about harmonic and α-harmonic maps) can be used to prove:

Corollary 1. For generic choice of Riemannian metric on the compact manifold M, all prime minimal two-spheres and all prime minimal tori in M are immersions with transversal crossings. In particular, if the dimension of M is at least five, they are imbeddings.

Corollary 2. If the compact manifold M has finite fundamental group and dimension at least five, then for generic choice of Riemannian metric on M, there are only finitely many prime minimal two-spheres with energy less than a given bound. Moreover, there are only finitely many minimal tori and Klein bottles with energy below a given bound and conformal structure in a given compact subset of Teichmüller space.

The proofs of the corollaries also makes use of the α-energy introduced by Sacks and Uhlenbeck (Annals of Math., 1981). The *α-energy*, for $\alpha > 1$, is the function

$$E_\alpha : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \to \mathbb{R},$$

defined by the formula

$$E_\alpha(f, \omega) = \frac{1}{2} \int_\Sigma [(1 + |df|^2)^\alpha - 1] dA.$$

For fixed choice of ω

$$E_{\alpha,\omega} : \mathrm{Map}(\Sigma, M) \to \mathbb{R}, \qquad E_{\alpha,\omega}(f) = E_\alpha(f, \omega),$$

satisfies condition C in a suitable completion of $\mathrm{Map}(\Sigma, M)$. Note that $E_\alpha \to E$ as $\alpha \to 1$.

The right completion of $\mathrm{Map}(\Sigma, M)$ is the space of maps of Sobolev class $L_1^{2\alpha}$ and is denoted by $L_1^{2\alpha}(\Sigma, M)$. One can show that $L_1^{2\alpha}(\Sigma, M)$ is a Banach manifold with a complete Finsler metric. It is therefore possible to develop a Liusternik-Schnirelmann theory for $E_{\alpha,\omega}$ which established existence of α-energy critical points in many cases.

In the limit as $\alpha \to 1$, sequences of α-energy critical points tend to approach bubble trees, consisting of a base minimal surface, together with a collection of minimal two-spheres connected by a tree of curves.

For example, here are two theorems obtained via this approach by Sacks and Uhlenbeck:

Theorem 1. Any compact Riemannian manifold whose universal is not contractible contains at least one minimal two-sphere.

(This is an analog of a classical theorem of Liusternik and Fet for closed geodesics.)

Theorem 2. If M is compact and simply connected a basis for $H_2(M;\mathbb{R})$ can be represented by minimal two-spheres.

Our Bumpy Metric Theorem and its corollaries show that the two-spheres in Theorems 1 and 2 are imbedded for generic choice of metric.

Existence results of this type—in which we demand that the minimal surfaces have a specific topological type—seem to be easier to obtain via the parametrized viewpoint than via geometric measure theory.

The Bumpy Metric Theorem and its corollaries provide part of the foundation for our goal of using the α-energy to develop a partial Morse theory for minimal tori, and more generally, minimal surfaces of other genus.

The simplest extension concerns minimal spheres and tori of small energy. After renormalization, we can suppose that the Riemannian metric on M satisfies the condition that all of its sectional curvatures are ≤ 1.

In this case, the curvature of any minimal surface in M must satisfy $K \leq 1$, and it follows from Gauss-Bonnet that if f is minimal

$$E(f) = A(f) \geq \int_\Sigma K dA = 2\pi\chi(M),$$

$\chi(M)$ being the Euler characteristic of M.

In particular, the energy or area of any minimal two-sphere must be at least 4π. This prevents bubbling of minimal two-spheres when the energy is $< 8\pi$. Following an article of ours in Math. Ann. (vol. 288, 1990), it is possible to establish equivariant Morse inequalities for

$$\mathrm{Map}(\Sigma, M)^{8\pi-} = \{f \in \mathrm{Map}(\Sigma, M) : E(f) < 8\pi\}.$$

These enable us to show the existence of at least

$$\binom{n+1}{3}$$

geometrically distinct prime minimal two-spheres in suitably pinched metrics on n-spheres, for example.

To get better results of this type, we need a better understanding of bubbling.