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# Galvano-magnetic phenomena today and forty years ago

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**Abstract**

In this review we discuss the basic aspects of the theory of galvanomagnetic phenomena in metals developed by I.M. Lifshitz and his collaborators, assuming a general form for the electron energy spectrum. This theory was the foundation of the spectroscopic method for the examination of the topological structure of the energy spectrum of the charge carriers in metals and in conducting media with a metallic type conductivity (synthetic metals). We analyze the connection of these works made forty years ago with the present state of the studies of electron processes in conductors.



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## 1. Introduction

Galvano-magnetic phenomena such as the Hall effect and the change of the conductivity of a sample in a magnetic field, have been studied experimentally for more than one hundred years. In the 1950s the theory of galvano-magnetic phenomena (TGMP) in metals was developed on the assumption of a general form for the electron energy spectrum and an arbitrary nature of the scattering of the charge carriers. From our point of view TGMP is an important chapter in the electron theory of metals.

The 1950s were a time of significant development in the electron theory of metals. During that one decade

- the microscopic theory of superconductivity was constructed,
- the Fermi-liquid theory was founded,
- the relations between many experimentally observed phenomena and concrete characteristics of the electron energy spectrum, such as the shape of the Fermi surface (FS) and the distribution of the velocities on it, were determined.

The papers, which linked together experimental results with the structure of the FS and predicted previously unknown structurally sensitive phenomena, formed a new part of the electron theory of metals, fermiology. One of the most important achievements of fermiology is the formulation and solution of the inverse problem of the reconstruction of the electron energy spectrum from experimental data.

The methods based on experimental data concerning the behavior of metals in strong magnetic fields showed themselves as the most effective. Among them the TGMP occupies a notable place. Since galvano-magnetic phenomena are very sensitive to the form of the electron energy spectrum, they were successfully used as a simple and reliable spectroscopic method for the reconstruction of the topology of FS. A striking feature of fermiological papers is the application of geometric images related to FS. In recent years the topological aspect of the TGMP was subjected to deep mathematical examination [1]. The wish to bring the mathematical results closer to physicists was one of the motives for writing this review.

Naturally, during the more than forty years after the first publications on the TGMP,[1] the electron theory of metals and, in particular, the theory of galvano-magnetic phenomena was improved and developed. Many papers describing the behavior of electrons in a magnetic field seem to continue the forty-year-old investigations directly. The authors did not have in mind to write a more or less complete review of the papers on GMP that were published during the last forty years. One of the goals of this review is to draw the attention of the readers to the state of the physics of metals in the 1950s, and to try to locate TGMP among the papers on the theory of metals. We hope that our review will help to shed light on one of the chapters of solid state physics. It must be taken into account that our opinion is a view from within the problem. Such a foreshortening does not always allow seeing a true and, above all, complete picture, although as much as possible we tried to be unbiased.

[1] With regard to the present context, we consider Refs. [2–4] as the first papers on the TGMP. In what follows, the abbreviation TGMP relates to these papers only.

## 2. Prehistory

Apparently, the series of Kapitza's papers [5–9] has to be considered as the beginning of a detailed methodical study of galvano-magnetic phenomena. In these works with increasing magnetic field a linear growth of the resistivity was detected for bismuth type semimetals (Bi, As, Sb) and a large group of metals (Li, Na, Cu, Ag, Au, Be, Mg, Zn, Cd, Hg, Al, Ga, Tl, Ti, Zr, Sn, Pb, Th, V, Ta, Gr, Mo, Te, W, Mn, Ni, Pd, Pt).

The dependence of the resistivity of metals on the value of a magnetic field **H** was discovered by Patterson and his collaborators [10] as far back as in 1902, then by Grumnach [11], Laws [12], and Roberts [13]. For every metal examined, except for the ferromagnetic ones, the resistivity $\rho$ increased proportionally to $H^2$ as the magnetic field was increased:

$$\frac{\Delta\rho}{\rho} = \frac{\rho(H) - \rho(0)}{\rho(0)} = \alpha H^2, \quad \alpha > 0 \ . \tag{2.1}$$

For the rather high magnetic fields of the order of 2–3 T in his time, Grumnach observed in cadmium a transition from the square-law dependence of the resistivity on a magnetic field to a linear one.

In these experiments the magnetoresistivity (the relative change of the resistivity in a magnetic field) $\Delta\rho/\rho$ was only several percent. Undoubtedly, it was of interest to trace the change of the resistivity in higher magnetic fields. For this purpose Kapitza created several sources of strong magnetic fields. Their original design allowed obtaining a current pulse with a flat peak during some thousandths of a second, so that the produced magnetic field of the order of 30 T was practically constant during this interval of time. In such a field the magnetoresistivity reached 20–50% which allowed determining the law of variation of the resistivity in a magnetic field accurately.

In Kapitza's experiments the longitudinal magnetoresistivity (the magnetic field **H** was parallel to the current density **j**) reached a saturation in strong magnetic fields, and, beginning with a field $H_c$, the quadratic increase of the transversal magnetoresistivity ($\mathbf{H} \perp \mathbf{j}$) was replaced by a slower, approximately linear, growth

$$\frac{\Delta\rho}{\rho} = \beta H \ . \tag{2.2}$$

This linear dependence of the resistivity on the magnetic field was called the Kapitza law.

The parameters $\alpha$ and $\beta$ increase as the temperature is lowered. From Kapitza's experiments at room temperature, and at the temperatures of solid carbonic gas and liquid air, it follows that $\alpha$ and $\beta$ are approximately proportional to the fourth or fifth power of the inverse absolute temperature. The presence of impurities or imperfections of the crystalline lattice shifts the transition to the linear dependence into the region of higher magnetic fields, that is, leads to an increase of the characteristic field $H_c$.

The behavior of the transverse magnetoresistivity, found by Kapitza, could not be explained not only in the framework of the classical theory of metals, but also by the newly-born quantum theory of those years [14–17] (in the monograph by Sommerfeld and Bethe [18] there is a complete bibliography and a description of the state of the theory of galvano-magnetic phenomena in the 1930s, as well as comments on Kapitza's experiments).

Those Kapitza papers [5–9] stimulated investigations of electron phenomena in metals. The experiments of Meissner and Scheffers [19] have to be mentioned. They found that at low temperatures in highly purified gold the transition to the linear growth of the resistivity with a magnetic field was already observed for fields of the order of several thousand gauss.

Before the publication of Kapitza's first paper [5], in Leiden Shubnikow and de Haas examined the magnetoresistivity of rather pure samples of nearly single crystal bismuth obtained with the aid of the Tamman–Obreimov–Shubnikow method [20]. The dependence of the resistivity on the value of the magnetic field obtained by Shubnikow and de Haas [21] was more complex than in Kapitza's experiments, tending to be a nonmonotonic one. The substantial difference of the results obtained in the two outstanding physical schools (in Cambridge and in Leiden) was the reason for carrying out the subsequent experiments.

In Leiden there were no magnetic fields as high as the ones Kapitza had, so they tried to increase the influence of a magnetic field on the dynamics of the charge carriers by increasing the mean free path of the conduction electrons. After chemical clearing and repeated recrystallization the bismuth samples turned out to be perfect single crystals (according to to-day's (modern) estimations, they did not contain dislocations). On such Bi samples the oscillation dependence of the resistivity on the inverse value of a magnetic field was discovered, first at the temperature of liquid hydrogen [21], and then at the temperature of liquid helium [22]. In that way an effect, which was later called the Shubnikow–de Haas effect, was discovered. The amplitude of the oscillations decreased as the temperature was raised, and at the temperature of liquid nitrogen the oscillations were no longer detectable. Kapitza, carrying out his experiments at the temperature of liquid nitrogen and higher, could not detect the oscillatory dependence of the magnetoresistivity on $1/H$. However, his experiments stimulated Shubnikov and de Haas to examine the magnetoresistivity of Bi accurately and thus promoted the discovery of oscillation effects.

Later on the Shubnikow–de Haas effect along with the de Haas–van Alphen effect [23] (the oscillations of the magnetic susceptibility in a magnetic field) were one of the main sources of information about the electron energy spectra of metals [24].

It should be noted that for a comparatively long time the oscillation effects of Shubnikow–de Haas and de Haas–van Alphen were interpreted as one of the anomalies of bismuth among its other unusual properties. It was shown by investigations of Shoenberg and his pupils (Mond laboratory, Cambridge) and also of Lazarev and Verkin (Ukrainian Physico-Technical Institute, Khar'kov) that the oscillatory dependence of the magnetization on the inverse value of a magnetic field, as well as other quantities describing the properties of metals, is a general property of all metals.

It was found that for all metals, except those from the 1st and the 5th groups of the Mendeleev periodic table, the spectra of the de Haas–van Alphen and the Shubnikow–de Haas oscillations were very complicated. They consisted of a large number of harmonics with significantly different periods. The existence of large periods attracts one's attention.

The interest in the properties of metals in a magnetic field led to the discovery of a phenomenon that played an important role in developing the TGMP. In single crystals of pure gold for different orientations of a magnetic field with respect to the crystallographic axes Justi and Scheffers detected different laws of the change of the resistivity as the magnetic field was varied [25]. For zinc and cadmium Lazarev, Nakhimovich and Parfyenova [26,27], as well as Borovik [28,29], saw the same behavior of the magnetoresistivity for quite a number of metals. Borovik also observed a linear

growth of the resistivity in a magnetic field of a single crystal bismuth sample (the samples, examined by Kapitza, were polycrystals). He saw the boundary of the linear growth: with further increase of the magnetic field the linear growth of the resistivity was changed into a quadratic one. At the same time, the linear growth of the magnetoresistivity was detected in a fairly large interval of magnetic fields. Apparently, Borovik was the first who suggested that in some cases the linear growth of the magnetoresistivity was a change-over region between two quadratic functions, as was the case in Bi, or, for some other metals, between a quadratic function and saturation. In single crystals of highly pure gold Alekseevskii and Gaidukov examined thoroughly the behavior of the magnetoresistivity depending on the direction of the magnetic field **H**. They found that for some orientations of the magnetic field the resistivity tended to saturation, and for others it increased proportionally to $H^2$. It turned out that the averaged magnetoresistivity calculated with regard to measurements for four different orientations of the magnetic field, depended linearly on the value of a strong magnetic field [30].

Undoubtedly, the paper of Kohler [31] belongs to the prehistory of TGMP. Kohler formulated a semiempirical rule named after him, the Kohler rule. According to the Kohler rule the magnetoresistivity $\Delta\rho/\rho$ is a function of the effective magnetic field $H_{\text{eff}} = H(\rho_{300}/\rho_0)$. Here $\rho_0$ is the resistivity of the examined sample for $H = 0$, which at low temperatures, of course, depends on the presence of impurities and dislocations. In other words, $\rho_0$ characterizes the given sample and the conditions of the experiment; $\rho_{300}$ is the resistivity at room temperature ($T = 300$ K). Since at $T = 300$ K the main mechanism of dissipation of electron fluxes is the scattering of the charge carriers by phonons, $\rho_{300}$ does not change for different samples. It is a characteristic of the given metal, but not of a particular sample.

When the Kohler rule is used the results of measurements of the resistivity at different temperatures and on different samples of the same metal can be "collected" in a uniform curve.

Here and in what follows we have in mind macroscopic samples, whose sizes are much greater than both the mean free path $l$ of the charge carriers, and the characteristic radius $r_H$ of the curvature of their trajectories in a magnetic field.

The Kohler rule was evidence that GMP had to be described by examining the classical motion of conduction electrons under the action of the Lorentz force. Indeed, up to the factor that is the same for all the samples of the same metal, the effective magnetic field $H_{\text{eff}}$ is proportional to the ratio $l/r_H$. In other words, $\Delta\rho/\rho$ is a function of the ratio $l/r_H$! This conclusion is verified directly by an intensification of the influence of a magnetic field on the resistivity, when the quality of the samples is improved and/or the temperature decreases.

We described schematically the basic experimental results relating to the topic we are interested in, namely, the influence of a magnetic field on the resistivity of metals.

The foundation of the understanding of electron properties of metals is the band theory, based on the Bloch theorem and the Fermi–Dirac statistics. In the middle 1950s the nature of the metallic state was understood, and the theory of many electronic phenomena in metals was developed: the electronic heat capacity linearly depending on the temperature and the temperature dependence of the resistivity were calculated (in particular, the sharp drop of the resistivity of normal metals was explained); an adequate explanation of the Wiedemann–Franz law was found and the reasons for the existence of departures from the Wiedemann–Franz law were clarified; the interaction of electromagnetic radiation with metals was described for all experimentally accessible frequency regions. The ascertainment of the nature of the metallic glance has to be considered an achievement of metallooptics.

In one paragraph it is impossible to list all the achievements of the electron theory of metals. By the 1950s monographs and text books were published that allowed to study the electron theory of metals without referring to journal publications.

However, when emphasizing the achievements of the electron theory of metals, two points have to be in mind.

1. The structure of the electron energy spectrum was understood simultaneously with the formulation of the principal theses of the band theory. Already in the monograph by Sommerfeld and Bethe [18] published in 1933, it was shown that when the density of the charge carriers was of the order of one per atom (i.e. when the conduction band was approximately half filled) the FS passes through the entire reciprocal lattice, intersecting the boundaries of the first Brillouin zone. In Figs. 23–25 of the monograph [18] the possible types of FS of metals with the face-centered cubic lattice are presented. An experimental confirmation of these "pictures" was obtained after many years, in particular, on the basis of the results of the TGMP.

At that time for lack of reliable numerical methods and because of the complexity of the electron energy spectrum one was forced to resort to simplifications.

One of the ways was to use the Drude–Lorentz–Sommerfeld (DLS) model [18]. According to this model the charge carriers in metals are a degenerate gas of free (noninteracting) electrons. In order not to disregard the influence of the crystalline lattice completely, the effective mass of the charge carriers $m^*$ was introduced. It was not equal to the mass $m_e$ of a free electron outside the crystal. This way was called the effective mass method.

The partiality to the effective mass method was so excessive that the effective mass was introduced even when it was not necessary. For example, the coefficient of proportionality between the electron heat capacity and the temperature, equal to the density of states at the Fermi surface, was rewritten in terms of the effective mass, and in comparison with experiments the ratio $m^*/m_e$ was used. "The trace" of such an approach is in the monograph by Ashcroft and Mermin [32, see Table 2.3].

A second way was based on the possibility to calculate many characteristics of metals without resorting to a concrete definition of the electron energy spectrum. The potentiality of this approach was shown in the monograph by Peierls [33] and especially clearly in the brochure "Conductivity of Metals" by Landau and Kompaneets [34]. In the latter the Bloch and the Bloch–Gruneisen results concerning the temperature dependence of the resistivity were reproduced on the basis of some qualitative ideas about the electron and the phonon spectra of metals. Today, when numerical methods allow us to carry out first principle calculations for many characteristics of metals, it is interesting to read the statements of authors, who used the qualitative approach, concerning the impossibility to calculate the constants entering the answers, and the necessity to be satisfied with their order of magnitude estimates.

2. The DLS model and the band theory are based on the assumption that the interaction between conduction electrons is weak. However, "it must be mentioned once more, there is no reason to suppose that the interaction of electrons in metals is really weak" (a citation from [34]). The self-consistent Hartree potential takes into account the interaction between electrons only partially. Beginning in the 1930s there were many attempts at a more reasonable consideration of the interaction between electrons. All these attempts did not lead to the creation of a constructive theory. Only after Landau founded the Fermi-liquid theory [35], and Silin used its ideas in the electron theory of metals [36], little by little it was understood that in real normal (nonsuperconducting) metals electrons formed a quantum Fermi-liquid.

Under our more than schematic presentation of the principles of the electron theory of metals, up to now we have not mentioned the theoretical description of the influence of a magnetic field on the properties of metals. There were many attempts to describe GMP in metals. In all the publications, preceding the TGMP, their authors started from the DLS theory, sometimes modifying it in order to bring the theory nearer to the experimental results.

It is known that a magnetic field affects electrons, first, because an electron has a spontaneous magnetic moment (spin), and, second, because of the action of the Lorentz force. The paper by Pauli [37], where the temperature-independent paramagnetic susceptibility of conduction electrons was calculated, and the paper by Landau [38], where the diamagnetic susceptibility of the electron gas was discovered were, apparently, the first publications concerned with investigations of the influence of a magnetic field on electronic properties of metals. Often, when mentioning the paper [38], the main attention is paid to the quantization of an electron's motion in the plane perpendicular to the magnetic field (the discrete energy levels relating to this quantization, were called Landau levels). In fact, the principal part of this paper is the discovery and calculation of the diamagnetism of the electron gas. Let us note that in the same year, 1930, the quantization of an electron's motion in a magnetic field was obtained by Frenkel and Bronshtein [39]. Their paper turned out to be forgotten. In 1979, almost 50 years after publication of Landau's paper [38], Peierls in his book "Surprises in Theoretical Physics" [40] devoted a special section "Electron Diamagnetism" to this work, where he mentioned that "many people expressed doubts" (p. 103) and concluded the section with the words "The surprise lies... in the ease with which the problem can be discussed, in spite of its apparent complexity..." (p. 105). For our part we would like to pay attention to the possibility, noted in [38], "of a complex nonlinear dependence of the magnetic moment on the field" when the condition $\mu H \ll kT$ is violated; $\mu$ is the Bohr magneton. However, as was said in this connection in [38], "just due to the periodicity an observation of nonlinear phenomena is hardly possible, since because of the nonhomogeneity of real fields an averaging always takes place" (p. 52, Selected works of Landau, v.1). Of course, Landau did not know about the works of Shubnikov and de Haas, who published their results in the same year, 1930. But Landau knew the results of Kapitza (the work of Landau was done during his tenure in the Cavendish laboratory in Cambridge, where at that time Kapitza worked). Landau concluded his publication with the statement that the linear growth of the resistivity in a magnetic field (the Kapitza law) could take place if $r_H \ll l$, i.e. when electrons were relatively free. The last inequality Landau rewrote in the form

$$H \gg ec\frac{N}{V}R, \tag{2.3}$$

where $N$ was the number of electrons in a sample of volume $V$, and $R$ was the resistivity of the sample. Landau stated that this inequality "is in good agreement with the critical field in Kapitza's experiments, and this can be considered as a confirmation of the theory. I did not succeed in constructing a quantitative theory" (p. 55 of Selected works).

The complete presentation of Landau's work, made in 1930, where on the basis of the DLS model he developed the theory of the de Haas–van Alphen effect, was published as an appendix to an experimental work of Shoenberg [41] in 1939 only (at that time Landau could not submit the work for publication himself, since he was under arrest).

There is a special section in Peierls' book" [40] devoted to the de Haas–van Alphen effect. It entered the "Surprises..." due to "the surprising feature of this situation" with a small parameter

that demanded the use of Poisson's summation formula in place of the usual expansion into a Taylor series. After Landau's work [41] an application of Poisson's formula when analyzing quantum oscillation effects became generally accepted. Of interest is the "Historical note", with which Peierls concluded this section.

> "Historical note: The oscillatory behavior was noticed by Landau in his first paper on diamagnetism, but he regarded it as unobservable in practice. The discovery of the oscillations in Bi by de Haas and van Alphen therefore seemed a complete mystery. The present author, having missed or forgotten Landau's remark, then suggested the quantized orbits as the origin of the effect, and illustrated this by some rough numerical calculations, which were later extended by Blackman. The use of the Poisson summation formula was suggested by Landau. For a full account see D. Shoenberg [24]."

According to Landau's theory the oscillatory dependence of the magnetic moment on the inverse value of the magnetic field takes place when $\mu H \geqslant kT$, and the period of oscillations $\Delta(1/H)$ is proportional to $n^{-2/3}$, where $n$ is the density of the charge carriers responsible for the oscillations. For Bi and other metals from the 5th group the sizes of the periods did not contradict an estimate of the number of the charge carriers obtained by independent methods. To explain the oscillations with long periods that were the only ones which turned out to be well observed by the time of the formulation and even publication of the theory [41], it was necessary to assume that anomalous groups of electrons existed in metals with the number of the charge carriers per $\mathrm{cm}^3$ some orders of magnitude less than $n_e$, the conduction electron density accurately estimated from the measurements of the heat capacity, the resistivity, and the Hall "constant". At the beginning of the 1950s the existence of anomalously small groups of charge carriers was taken as one of the mysteries of the metallic state.

It is important to note that the part of the magnetic moment of a metal linear in the magnetic field is $(\varepsilon_F/\mu H)^{1/2}$ times smaller than the oscillation amplitudes of the magnetization, and for $\mu H \ll \varepsilon_F$ ($\mu H$ is of the order or smaller than $T$) quantum corrections to the magnetoresistivity are much smaller than "the classical" part of the resistivity, i.e. the resistivity calculated without taking account of quantization of the energy levels of the conduction electrons.

The assumption concerning the sufficiency of a classical description of electron motion under the action of the Lorenz force was verified by Titeica [42] when calculating GMP in the framework of the DLS model with regard to quantization of the energy of the charge carriers in a magnetic field. The measure of quantization of the energy spectrum is the distance between the quantized levels $\Delta\varepsilon \cong \mu H = e\hbar H/mc$. The quantization of the charge carriers' energy is not essential when $\mu H \ll kT$ and the temperature diffusion of the Fermi distribution function eliminates the influence of the quantization. Titeica treated the electric current in a quantizing magnetic field $\mathbf{H} = (0, 0, H)$ as a drift of the centers of oscillators $x_0 = c p_y/eH$, and showed that the quantum corrections to the conductivity were very small for $\mu H \ll kT$. The analogy between the wave functions of electrons in a magnetic field and oscillators of the frequency $\Omega = eH/mc$ followed from the Schrödinger equation [38,39].

Before Akhieser [45,46] in calculations of the oscillation effects as functions of the inverse value of the quantizing magnetic field the spin of the electrons was not taken into account. When $\mu H \geqslant kT$, taking account of the spin changes the structure of the oscillating part of the characteristic, although

it does not change the period of the oscillations. It can be said that Rumer completed the calculations with the aid of Titeica's method carrying out a correct and detailed analysis of quantum corrections to the classical part of the magnetoresistivity [43,44].

All the calculations showed that the oscillations in a magnetic field quantum corrections to the monotonically varying with $H$ part of the resistivity were negligibly small as long as the distance between the quantized levels $\mu H$ was much smaller than the Fermi energy $\varepsilon_F$.

The classical analysis with the aid of the DLS model allowed to obtain results describing some of the features of the observed GMP. However, this often required a specific complication of the model. Thus, the growth of the resistivity in a magnetic field turned out to be consistent with the DLS theory if a significant anisotropy of the mean free path of the charge carriers was assumed. When "the holes" moving in a magnetic field like positively charged particles were introduced along with electrons, it was possible to obtain a magnetoresistivity $\Delta\rho/\rho$ increasing quadratically in strong magnetic fields, if the number of electrons $n_e$ was equal to the number of holes $n_h$, and tending to saturation, if $n_e$ was not equal to $n_h$. In this case when the parameters of the problem were estimated in a reasonable way, the value $\Delta\rho/\rho$ of the saturation could be of the order of unity [47].

The inability of the DLS model to describe the giant anisotropy of the magnetoresistivity of many metals, and to explain the nature of the Kapitza law, stimulated attempts to found a GMP theory free of unjustified assumptions. The impossibility to understand the origin of the Kapitza law was accepted as a challenge. It seemed that the Kapitza law (the linear growth of the resistivity in a magnetic field) contradicted the Onsager principle of the symmetry of kinetic coefficients [48], according to which the resistivity had to be an even function of a magnetic field. In that case, when a series expansion of the function $\rho(H)$ in powers of $H$ or $1/H$ is permissible, there is no place for a linear dependence of $\rho$ on $H$.

Undoubtedly, in the development of TGMP, the work by Lifshitz and Kosevich [49] played an important role. They constructed a theory of the de Haas–van Alphen effect resorting to no specifications of the dispersion law of the conduction electrons. The starting point of their work was the condition of quantization of the areas

$$S(\varepsilon, p_H) = \frac{2\pi\hbar|e|H}{c}(n+\gamma)\ , \tag{2.4}$$

where $S(\varepsilon, p_H)$ was the area of a closed cross-section of the isoenergetic surface $\varepsilon(\mathbf{p}) = \varepsilon$ by the plane $p_H = \mathbf{pH}/H = \text{const}$, $n = 0, 1, 2\ldots$, and $\gamma$ was a positive number smaller than one (usually $\gamma = 1/2$).

Condition (2.4) generalizes the Landau quantization rule [38] to the case of an arbitrary dependence of the charge carriers' energy on the quasimomentum. When speaking about the role of the work [49], the importance of an example must be emphasized: first, it was found that to obtain a result describing a complex phenomenon, it was not necessary to define the dispersion law concretely, and, second, that the answer could be obtained in terms of descriptive geometrical images relating to the FS. In this particular case the relation between the period of oscillations $\Delta(1/H)$ and $S_{\text{extr}}(\varepsilon_F)$ was determined to be

$$\Delta\left(\frac{1}{H}\right) = \frac{2\pi|e|\hbar}{cS_{\text{extr}}(\varepsilon_F)}\ . \tag{2.5}$$

In Eq. (2.5) $S_{\mathrm{extr}}(\varepsilon_F)$ is the area of the planar cross-section of the FS $\varepsilon(\mathbf{p}) = \varepsilon_{\mathbf{F}}$ that is extreme with respect to $p_H$. Up to now for different metals Eq. (2.5) is successfully used for reconstruction of the shape of the FS.

For an arbitrary dispersion law the condition of quantization of the areas of the plane cross-sections of isoenergetic surfaces, Eq. (2.4), was obtained by Lifshitz (presented at the session of Ukrainian Academy of Sciences in 1951) and independently by Onsager [50]. In both cases the results of the experiments carried out in that time counted. Lifshitz was well informed about the results of Lazarev, Verkin and Rudenko [51] (Ukrainian Physical and Technical Institute, Khar'kov) and Onsager knew the results of Shoenberg [52] (Mond Laboratory, Cambridge). Lifshitz and Onsager understood that the quantization of the areas (2.4) made it possible to use quantum oscillations for the reconstruction of the electron energy spectrum. The possibility to set a rigorous problem of reconstruction of the electron energy spectrum from experimental data is one of the most important achievements of the theory of the de Haas–van Alphen effect. In the case of a convex FS, according to the Lifshitz–Pogorelov theorem [53], the knowledge of all its central cross-sections defines the shape of FS uniquely. The experimental discovery of oscillations of the magnetization of polyvalent metals with substantially different periods was the evidence that their FS were rather complex and consisted of some closed sheets of different sizes and, possibly, along with closed surfaces, there were surfaces passing through the entire reciprocal lattice. For such metals the solution of the inverse problem with the aid of the measured periods of the quantum oscillations of the magnetization, turns out to be not uniquely defined and is fraught with considerable difficulties. However, it allows producing the outline of the FS clearly.

After the work by Lifshitz and Kosevich [49], for us (for I.M. Lifshitz himself and his pupils) the word-combination "an electron with an arbitrary dispersion law" became a sort of "a password". It does not mean that the dependence of the electron energy $\varepsilon$ on the quasimomentum $\mathbf{p}$ can be an arbitrary one. It only means that we do not know it and, consequently, cannot use the explicit form of the function $\varepsilon(\mathbf{p})$.

A new trend in the electron theory of metals appeared. A need arose to calculate "everything that one could" using the idea of an electron with an arbitrary dispersion law. Naturally, preference was given to the problems where the answer was independent of, or weakly depended on, the scattering mechanism. First, because the problems, where the answer depended on the specific scattering mechanism, were very difficult for analytical solution. Often they could not be solved without a concrete definition of the dispersion law. Second, it was convenient to use the phenomena that were not sensitive to the scattering mechanism for the reconstruction of the energy spectrum on the basis of the experimental data. At that time the spectroscopic potentialities of different phenomena in metals were interesting for us.

The fundamental nature of the band theory forced stressing the difference between quasimomentum and momentum. In particular, this was necessary in order to explain the umklapp processes introduced into the theory by Peierls [54]. When these processes were not taken into account, it was impossible to examine the electrical conductivity of pure metals at low temperatures [56]. On the other hand, the difference between quasimomentum and momentum complicated the analysis of an electron's motion in external (with regard to the crystal) fields.

An important progress in the electron theory of metals was achieved due to the application of the semiclassical model, where the difference between quasimomentum and momentum was neglected, and for the analysis of the motion of conduction electrons in external electric $\mathbf{E}$ and magnetic $\mathbf{H}$

fields "an ordinary" classical equation with the Lorentz force on its right-hand side was used:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\left\{\mathbf{E} + \frac{1}{c}\left[\frac{\partial\varepsilon(\mathbf{p})}{\partial\mathbf{p}}, \mathbf{H}\right]\right\} \quad . \tag{2.6}$$

It is not a simple problem to derive this equation in the case of an arbitrary and rather complicated dependence of the charge carriers energy on their quasimomentum.

The following citation from the monograph "Solid State Physics" by Ashcroft and Mermin [32] (the first edition was published in 1976, i.e. more than 20 years after TGMP) gives a faithful characteristic of the psychological atmosphere not only in the 1950s, but also much later: "The reader who is dissatisfied with the very incomplete and merely suggestive bases we shall offer for the semiclassical model is urged to examine the broad array of mysteries and anomalies of free electron theory that the model resolves. Perhaps a suitable attitude to take is this: If there were no underlying microscopic quantum theory of electrons in solids, one could still imagine a semiclassical mechanics (guessed by some late 19th-century Newton of crystalline space) that was brilliantly confirmed by its account of observed electronic behavior, just as classical mechanics was confirmed by its accounting for planetary motion, and only very much later given a more fundamental derivation as a limiting form of quantum mechanics." (vol. 1, p. 215).

A justification of the semiclassical method is outside the framework of our review. The boundaries of the semiclassical approach are well-known. From an examination of the coordinate operator $\hat{\mathbf{r}}$ in the quasimomentum representation (see, for example, [57–59]) it follows that application of the semiclassical equations is well justified when the size of the classical trajectory of an electron in the external force field exceeds the lattice constant noticeably.

In the 1950s the gas approximation was used in the solution of concrete problems of the electron theory of metals feeling that many electron properties of normal metals (in particular, GMP) were described correctly by it. Later on it was shown that all the formulae for the galvanomagnetic characteristics derived in the framework of "the gas" approach were *strictly correct* accepting that the dispersion law of the charge carriers took the interaction between electrons into account. This means, that a conduction electron is a quasiparticle, and the collective of electrons is the Fermi liquid [59–61].

For the TGMP it is important that either the trajectory of an electron in the momentum space is open or closed. Open trajectories of electrons with the Fermi energy are possible in the case of an open FS. The presence of even a small layer of open cross-sections of the FS by the plane $p_H = \text{const}$ affects the asymptotic behavior of the magnetoresistivity in a strong magnetic field ($r_H \ll l$) substantially. After the TGMP was developed, this property was used in studying the topology of FS with the aid of galvanomagnetic measurements.

In the 1950s there was no reliable information about the shape of the FS of metals. Naturally, isoenergetic surfaces with energies that are close to the lower or the upper boundaries of an energy band, are closed. In the absence of the degeneracy they are ellipsoids with high accuracy, and the elements of the effective mass tensors of electrons or holes are constants. The FS of the majority of metals with a small charge carrier density are of this type. Between the ellipsoidal isoenergetic surfaces there must be isoenergetic surfaces with a complex topology. Most probably they are open surfaces. Therefore, as a rule, the FS of metals with a high conductivity are complex, and mostly they are open. The magnetoresistivity of such a metals is rather sensitive to a change of the orientation of the magnetic field with respect to the crystallographic axes. As we already mentioned, a possibility

of the existence of open FS was declared as far as the 1930s (see [18]). It looks like before the TGMP no one took notice of the importance of this statement.

## 3. Semiclassical description of galvanomagnetic phenomena

In metals with the number of conduction electrons of the order of one per atom for the main group of the charge carriers the distance between the energy levels quantized by a magnetic field is much smaller than the Fermi energy. Consequently, as we mentioned above, the semiclassical approach can be used when describing nonequilibrium processes. In accessible magnetic fields the ultraquantum limit ($\mu H \geqslant \varepsilon_F$) can be reached only for an anomalously small group of electrons which, in addition, have a small effective cyclotron mass. However, usually their contribution to the smooth part of the varying in a magnetic field-dependent conductivity is small. An exception is a group of semimetals (bismuth, stibium, arsenic) with anomalously small number of charge carriers.

In the framework of the semiclassical approach to obtain the relation between the electric current density

$$\mathbf{j} = \frac{2}{2\pi\hbar^3} \int e\mathbf{v} f(\mathbf{p})\, \mathrm{d}^3 p \tag{3.1}$$

and the electric field $\mathbf{E}$, the kinetic Boltzmann equation for the charge carriers distribution function $f(\mathbf{p})$ of the charge carriers can be used:

$$\left\{ e\mathbf{E} + \frac{e}{c}[\mathbf{v}, \mathbf{H}] \right\} \frac{\partial f}{\partial \mathbf{p}} = W_{\mathrm{col}}(f) \ . \tag{3.2}$$

Here $e$ and $\mathbf{v} = \partial\varepsilon(\mathbf{p})/\partial\mathbf{p}$ are the charge and the velocity of the charge carriers with the energy $\varepsilon(\mathbf{p})$, $\hbar$ is the Planck constant, $c$ is the speed of light, and $W_{\mathrm{col}}(f)$ is a collision integral describing the scattering of the charge carriers.

If nonlinear effects with respect to electric field are not taken into account, it is sufficient to find the solution of the kinetic equation (3.2) linear in the weak perturbation of the electron system. In this approximation the kinetic Boltzmann equation takes the following form:

$$\frac{e}{c}[\mathbf{v}, \mathbf{H}] \frac{\partial \Psi_i}{\partial \mathbf{p}} + \hat{W}\{\Psi_i\} = v_i \ , \tag{3.3}$$

where $\Psi_j(\mathbf{p})$ is a vector function describing the deviation of the electron distribution function $f(\mathbf{p})$ from the equilibrium Fermi function $f_0(\varepsilon)$,

$$f(\mathbf{p}) = f_0(\varepsilon) - eE_j \Psi_j(\mathbf{p}) \frac{\partial f_0(\varepsilon)}{\partial \varepsilon} \ , \tag{3.4}$$

and $\hat{W}\{\Psi_i\}$ is a linear integral operator acting on the desired functions $\Psi_i$. It is related to the collision integral $W\{f\}$ by the formula

$$\frac{\partial f_0(\varepsilon)}{\partial \varepsilon} \hat{W}\{\Psi_i\} = -W_{\mathrm{col}} \left\{ \Psi_i \frac{\partial f_0(\varepsilon)}{\partial \varepsilon} \right\} \ . \tag{3.5}$$

For elastic collisions $\hat{W} = -W_{\mathrm{col}}$. However, for inelastic collisions of electrons (for example, with phonons) the derivative $\partial f_0(\varepsilon)/\partial\varepsilon$ cannot be taken outside the integral sign, and the kinetic equation for the phonon distribution function has to be added to Eq. (3.3). In such a way the dragging

of phonons by electrons can be described. Although in metals the role of the dragging effect is not very important, when the interaction between electrons and phonons is taken into account, the correct values of the elements of the conductivity tensor cannot be obtained without the simultaneous solution of equations for both the electron and phonon distribution functions.

The solution of the kinetic equation (3.3) allows writing the electric current density in the Ohm law approximation,

$$j_i = \sigma_{ij}(\mathbf{H})E_j = -\frac{2e^2}{(2\pi\hbar)^3}\int v_i \Psi_j(\mathbf{p})\frac{\partial f_0(\varepsilon)}{\partial\varepsilon}\,\mathrm{d}^3 pE_j \ . \tag{3.6}$$

When electrons from several energy bands take part in the conductivity, in expression (3.6) for the conductivity tensor $\sigma_{ij}(\mathbf{H})$ the summation over all the bands must be carried out.

The elements of the conductivity tensor are kinetic coefficients, and are subject to the Onsager symmetry relations [40]:

$$\sigma_{ik}(\mathbf{H}) = \sigma_{ki}(-\mathbf{H}) \ . \tag{3.7}$$

Relations (3.7) can be obtained as a consequence of the collision operator being a Hermitian operator and $[\mathbf{v},\mathbf{H}]\partial/\partial\mathbf{p}$ being an anti-Hermitian operator. The resistivity tensor $\rho_{ik}=(\hat{\sigma}^{-1})_{ik}$ that is the inverse of the tensor $\sigma_{ik}(\mathbf{H})$ possesses the same symmetry property.

Every second rank tensor can be decomposed into symmetric and antisymmetric tensors (with respect to the subscripts). Then

$$\begin{aligned} &\sigma_{ik} = s_{ik} + a_{ik}; \quad \rho_{ik} = \rho^s_{ik} + \rho^a_{ik} \ ; \\ &s_{ik} = s_{ki}; \quad a_{ik} = -a_{ki}; \quad \rho^s_{ik} = \rho^s_{ki}; \ \rho^a_{ik} = -\rho^a_{ki} \ . \end{aligned} \tag{3.8}$$

In a magnetic field the evenness of the elements of the symmetric tensors and oddness of the elements of antisymmetric tensors with respect to $\mathbf{H}$ follow from the Onsager relations:

$$\begin{aligned} &s_{ik}(-\mathbf{H}) = s_{ik}(\mathbf{H}); \quad \rho^s_{ik}(-\mathbf{H}) = \rho^s_{ik}(\mathbf{H}) \ ; \\ &a_{ik}(-\mathbf{H}) = -a_{ik}(\mathbf{H}); \quad \rho^a_{ik}(-\mathbf{H}) = -\rho^a_{ik}(\mathbf{H}) \ . \end{aligned} \tag{3.9}$$

The symmetric part of $\sigma_{ik}$ ($\rho_{ik}$) describes the conductivity (the resistivity), and the antisymmetric part describes the Hall effect.

The symmetric and antisymmetric parts of the resistivity tensor are expressed in terms of the elements of $s_{ik}$ and $a_{ik}$ according to

$$\rho^s_{ik} = \frac{s(\hat{s}^{-1})_{ik} + a_i a_k}{s + (\mathbf{a}\hat{s}\mathbf{a})}; \quad b_i = \frac{(\hat{s}\mathbf{a})_i}{s + (\mathbf{a}\hat{s}\mathbf{a})} \ , \tag{3.10}$$

where $s$ is the determinant composed of the elements of the tensor $s_{ik}$, $(\mathbf{a}\hat{s}\mathbf{a})=a_i s_{ik} a_k$, and $a_i=\varepsilon_{ikl}a_{kl}$, where $\varepsilon_{ikl}$ is the unit antisymmetric tensor of the third rank.

In real space every symmetric tensor of the second rank can be transformed to principal axes, and a dual vector (in our case, the Hall vector) is in one-to-one correspondence with every antisymmetric tensor.

A full description of GMP must include

- three principal values of the tensor $\rho^s_{ik}$;

- the directions of its principle axes;
- the direction and the absolute value of the Hall vector **b** that is orthogonal to the current density vector[2] **j** by definition.

Usually only the resistance

$$\rho(\mathbf{H}) = \rho_{ik}\frac{j_i j_k}{j^2} = \rho^s_{ik}\frac{j_i j_k}{j^2}\ , \tag{3.11}$$

and the Hall field

$$E_{Hi} = \rho^a_{ik} j_k \tag{3.12}$$

are measured as functions of the magnetic field.

In the normal (not the superconducting) state a current cannot flow through a conductor without dissipation. Therefore the principal values of the tensors $s_{ik}$ and $\rho^s_{ik}$ do not go to zero, and they are always positive. This is a consequence of the law of entropy growth. The orientation of the principal axes of these tensors, of course, depends on the value of the magnetic field. When $H = 0$ it is determined by the symmetry of the crystal, and in a strong magnetic field one of the axes (the axis $x_3$) is almost parallel to the vector **H**, or it is directed straight along the magnetic field, if the field is oriented along a symmetry axis of the crystal.

For arbitrary directions of the vectors **j** and **H** the Hall field $\mathbf{E}_H$ has components both in the plane perpendicular to **H** and along **H**. If **H** and **j** are along some symmetry directions, the Hall vector is often perpendicular not only to the vector **j**, but also to the vector **H**, and the Hall effect is described by the Hall "constant"

$$R = \frac{\rho_{xy}(\mathbf{H}) - \rho_{xy}(-\mathbf{H})}{2H}, \quad H_z \equiv H; \quad H_x = H_y = 0\ . \tag{3.13}$$

The measurement of the Hall constant allows estimating the number of charge carriers. The mobility of the carriers is estimated from the value of the resistance.

Using the relation between the symmetry of the tensors $\rho^s_{ik}$ and $\rho^a_{ik}$ and their evenness, it is easy to verify that in a weak magnetic field (formally, when $H \to 0$) the series expansion of the elements of the tensor $\rho^s_{ik}$ includes even powers of **H**, and the Hall vector $\mathbf{b} = (\rho^a_{yz}, \rho^a_{zx}, \rho^a_{xy})$ odd powers of **H** only. Then in a weak magnetic field the magnetoresistivity is a quadratic function of the magnetic field ($\Delta\rho_{ik} = \lambda_{iklm} H_l H_m$), and the Hall vector is proportional to the magnetic field ($b_i = R_{ik} H_k$). In a weak magnetic field the elements of the tensors $\lambda_{iklm}$ and $R_{ik}$ do not depend on **H**. Note, the principle of the symmetry of kinetic coefficients does not demand the symmetry of the tensor $R_{ik}$. Therefore, if the symmetry of a crystal (when $\mathbf{H} = 0$) allows the absence of the symmetry of the material tensor $R_{ik}$, in such conductors the Hall effect must have some unusual ("strange") properties (see Ref. [55]).

The TGMP is based on the assumption that in a magnetic field every trajectory of an electron contributes separately to the conductivity tensor. Naturally, collisions can lead to a transition of

[2] As a rule, in experiments with metals, the current density **j** is fixed and, by varying the magnetic field **H**, the measurements of different components of the electric field strength **E** are carried out. Consequently, the elements of the resistivity tensor $\rho_{ik} = \rho^s_{ik} + \rho^a_{ik}$ are an intrinsic characteristic of GMP. The separation of even and odd parts of the tensor $\rho_{ik}$ can be done by the reversal of the magnetic field direction: $\rho^s_{ik} = [\rho_{ik}(\mathbf{H}) + \rho_{ik}(-\mathbf{H})]/2$; $\rho^a_{ik} = [\rho_{ik}(\mathbf{H}) - \rho_{ik}(-\mathbf{H})]/2$.

electrons from one trajectory to another one, and even from one sheet to another sheet. However, the main results of the TGMP were related to strong magnetic fields corresponding to $\Omega \gg 1/\tau$, where $\Omega$ was the characteristic frequency of rotation of a conduction electron in a magnetic field, and $\tau$ was the averaged relaxation time. The problem was analyzed by means of an expansion in terms of $(\Omega\tau)^{-1}$, and as the zero approximation the collisionless limit was used. In this case the transitions between different trajectories and sheets are not important (concerning some exceptions of this "rule" see below).

In the TGMP it was emphasized that the results of the analysis depended weakly on the nature of the dissipative processes described by "an ordinary" (a classical) collision integral. In this connection it was assumed that the probabilities of the transitions did not depend on the magnetic field at all. A justification for this assumption was the estimate $\hbar/r_H \ll p_F$ (here $r_H$ was the radius of the orbit of an electron). In order of magnitude this estimate gave the same result as the one that allowed neglecting the quantum oscillation effects ($\mu H \ll \varepsilon_F$).

Since in the TGMP the collision integral does not depend on the magnetic field at all,[3] the reason why a magnetic field affects the conductivity and resistivity tensors is the classical motion in the magnetic field of an electron with a complex dispersion law $\varepsilon = \varepsilon(\mathbf{p})$. It is described by the Newton equations with the Lorentz force:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{e}{c}[\mathbf{v},\mathbf{H}]; \quad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}; \quad \mathbf{v} = \frac{\partial\varepsilon(\mathbf{p})}{\partial\mathbf{p}} \ . \tag{3.14}$$

"The nontriviality" of these ordinary equations is only in "the unusual" dependence of the electron energy on the momentum.

A trajectory of an electron in momentum space is the line of intersection of the isoenergetic surface $\varepsilon(\mathbf{p}) = \varepsilon$ with the plane $\mathbf{pH} = \text{const}$. The energy $\varepsilon$ and the projection $p_z$ of the momentum are integrals of the motion. In the **r**-space the projection of the trajectory of an electron onto the plane perpendicular to the magnetic field is similar to its trajectory in the **p**-space with the similarity ratio $c/eH$, if the latter is rotated through the angle $\pi/2$.

In place of $p_x$, $p_y$ and $p_z$ it is convenient to define the position of an electron in the **p**-space by the integrals of motion $\varepsilon$ and $p_z$, and to choose the time $t$ of an electron motion along the trajectory in the magnetic field as the third coordinate. The time $t$ is counted from an arbitrary point on the trajectory. In terms of these variables the kinetic equation (3.3) and the elements of the conductivity tensor have the form:

$$\frac{\partial\Psi_i}{\partial t} + \hat{W}\{\Psi_i\} = v_i \ , \tag{3.15}$$

$$\sigma_{ij}(\mathbf{H}) = -\frac{2e^3H}{c(2\pi\hbar)^3}\int \mathrm{d}\varepsilon\,\frac{\partial f_0(\varepsilon)}{\partial\varepsilon}\int \mathrm{d}p_z \int \mathrm{d}t v_i\Psi_j(\mathbf{p}) = \langle v_i\Psi_j\rangle \ . \tag{3.16}$$

As a rule, the temperature dependence of the classical characteristics of GMP is due to the dependence of the mean free path of the charge carriers $l$ on the temperature $\Theta$, i.e. this dependence

[3] A proviso: in magnetically ordered conductors there are substantial singularities of GMP. The possibility of collisions with magnons and spin fluctuations is among them. The probabilities of these processes are functions of the magnetic field. In what follows there is no further mention of magnetic materials.

is taken into account through the collision integral. If the main inelastic scattering is the scattering by phonons, just as for $H=0$ the Debye temperature $\Theta_D$, which is much lower than the degeneracy temperature of the electron gas, serves as a characteristic temperature. Then, even at temperatures exceeding $\Theta_D$, when calculating the elements of the conductivity tensor in most cases $-\partial f_0/\partial\varepsilon$ can be replaced by the $\delta$-function $\delta(\varepsilon-\varepsilon_F)$.

In metals all the conduction electrons take part in the charge transport, but "deep" electrons (the electrons with $\varepsilon<\varepsilon_F$) do not affect the kinetic characteristics. The kinetic characteristics depend significantly on the shape of the FS, the velocities of the Fermi electrons, and the relaxation properties of the charge carriers with the Fermi energy $\varepsilon_F$ only. However, when the Fermi energy is close to one of the van Hove singularities $\varepsilon_k$, the replacement of $-\partial f_0/\partial\varepsilon$ by a $\delta$-function is correct only under the satisfaction of a rather severe constraint: $\Theta\ll|\varepsilon_F-\varepsilon_k|$. This circumstance has to be taken into account when analyzing galvanomagnetic characteristics of specific metals. The most obvious example of a complication of the temperature dependence of the galvanomagnetic characteristics is their "behavior" in the vicinity of an electronic topological transition of the $2\frac{1}{2}$ order [74,75].

At low temperatures $\Theta\ll\Theta_D$, when conduction electrons are scattered mainly by impurities and other defects of the crystal, and the collisions of electrons and phonons are very rare, in calculating the elements of the conductivity tensor $\sigma_{ij}$ one can manage without taking account of the kinetic equation for phonons. In this case one must seek a solution of the kinetic equation in the basis of the eigenfunctions of the integral operator describing the collisions of the charge carriers with the impurities. However, sometimes to find the dependences of the resistivity and the Hall field on the value and orientation of the magnetic field, it is sufficient to use the $\tau$-approximation for the collision integral, that is to replace $\hat{W}(\Psi_i)$ by $\Psi_i/\tau$. In the $\tau$-approximation the solution of the kinetic equation (3.15),

$$\Psi_i(t,p_H,\varepsilon)=\int_{-\infty}^{t}v_i(t',p_H,\varepsilon)\exp[(t'-t)/\tau]\,\mathrm{d}t'\ ,$$

allows obtaining the elements of the conductivity tensor without a lot of work,

$$\sigma_{ik}=\frac{2e^3H}{c(2\pi\hbar)^3}\int\mathrm{d}\varepsilon\delta(\varepsilon-\varepsilon_F)\int\mathrm{d}\,p_H\int_0^T\mathrm{d}tv_i(t)\int_{-\infty}^{t}v_k(t')\exp[(t'-t)/\tau]\,\mathrm{d}t'\ , \tag{3.18}$$

and it is not difficult to analyze the dependences of the magnetoresistivity and the Hall field on the magnetic field in all regions of the magnetic field.

For a collision integral of an arbitrary form the asymptotic behavior of galvanomagnetic characteristics can be analyzed rather completely in the regions of strong ($\Omega\tau\gg1$) and weak ($\Omega\tau\ll1$) magnetic fields.

According to the TGMP the asymptotic behavior of the magnetoresistivity differs fundamentally in the cases of a closed and an open FS. This was used in investigations of the topology of FS with the aid of galvanomagnetic measurements.

In momentum space the nature of an electron trajectory depends not only on the form of the FS, but also on the direction of the magnetic field $\mathbf{n}=\mathbf{H}/H$ with respect to the crystallographic axes. This allowed plotting angular diagrams, which described the dependence of the asymptotic values of the resistivity on $\mathbf{n}$ in a strong magnetic field. Namely, the angular diagrams gave the necessary information for the solution of the inverse problem and allowed determining the shape of the FS from the accumulated data concerning the asymptotics of the elements of the resistivity tensor.

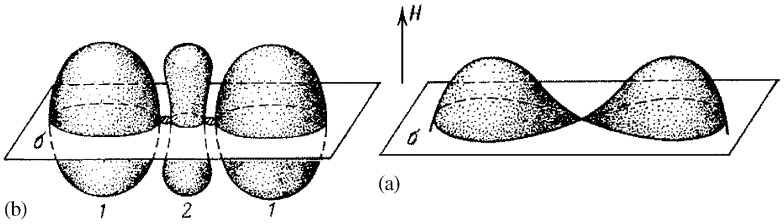



Fig. 1. (a) Self-intersecting intersections of a nonconvex FS $\epsilon(\mathbf{p}) = \epsilon$ by a plane $p_H = \mathbf{pH}/H = \text{const}$; (b) Isoenergetic surfaces belonging to the bands 1 and 2 are placed in such a way that classical trajectories in different bands approach: magnetic breakdown is possible.

A qualitative estimate of the contribution of closed and open trajectories to the elements of the conductivity tensor can be easily done without solving the kinetic equation, but with the use of the relation between the mobility and the diffusion coefficient of the degenerate electron gas (an analog of the Einstein relation). The contribution of closed trajectories to $\sigma_{xx}$ and $\sigma_{yy}$ is inversely proportional to $(\Omega\tau)^2$, and for open trajectories (along the direction of the openness) it weakly depends on the magnetic field [59]. The contributions of both open and closed trajectories to $\sigma_{zz}$ are approximately equally weakly dependent on the magnetic field. It should be noted that the declaration of "weak dependence" does not exclude the change of the resistivity by several times. As a matter of fact, this only means that when $\Omega\tau \gg 1$, the value of the resistivity does not depend on $H$ (saturation takes place).

In the cases when the layer of open trajectories is of the same order as the layer of closed trajectories, the contribution of the open trajectories (at least to one of the elements of the conductivity tensor) is much greater than the contribution of the closed trajectories.

The argument used to justify the possibility of neglecting quantum effects when describing the motion of conduction electrons in a magnetic field, was the inequality $\mu H \ll \varepsilon_F$, which showed that even in the case of a finite motion in the plane perpendicular to **H**, the spacing between the levels is much smaller than the Fermi energy. This argument is rather cogent for an isolated orbit. However, the energy spectra of the majority of metals are very complex. For classical trajectories this results in the possibility of their coming together in the **p**-space. The last circumstance changes the estimate of the possibility to neglect quantum effects. Before we will write down the relevant inequalities, let us note that there are two different *types* of the approach of trajectories (Fig. 1).

1. If FS is nonconvex (it has necks, dents, and so on), there is always a whole domain of orientations of a magnetic field for which the semiclassical trajectories turns out to be close to each other: it is obvious that near a self-intersecting cross-section of the FS by the plane $p_H = p_c$ there is a layer of coming together trajectories (Fig. 1a). The thickness of this layer $\delta p$ is much smaller than the characteristic sizes of the FS. Azbel' [62] analyzed the influence of tunneling between close trajectories belonging to the same sheet of the FS on the electron energy spectrum for $H \neq 0$. When the areas $S_1$ and $S_2$ of nearby cross-sections of the FS are incommensurable, in addition to quantum oscillations of thermodynamic and kinetic quantities in $1/H$ calculated by Lifshitz and Kosevich [49], specific quantum oscillations appear. Their period is inversely proportional to some algebraic combinations of the areas $S_1$ and $S_2$. These oscillations are formed by the charge carriers from

a small region on the FS where $p_H$ is close to $p_c$. Their amplitude is smaller than the amplitude of the "ordinary" oscillations relating to extremal (in $p_H$) areas of cross-sections of the FS.

2. An approach of sheets of the FS belonging to different bands is possible. In this case often the approach is realized in a sufficiently wide layer of the values of $p_H$ (Fig. 1b). In a magnetic field, the tunneling between close trajectories belonging to different bands was called *magnetic breakdown* following Cohen and Falicov [63], who discovered this phenomenon. The probability of magnetic breakdown $W_H$ increases as the magnetic field is increased [64,65]. Often its value reaches unity at comparatively low magnetic fields, if $\mu H \gg \Delta^2/\varepsilon_F$, where $\Delta$ is the minimal height of the energy barrier separating one classical trajectory from the other. The most interesting phenomena relating to magnetic breakdown take place when $0 < W_H < 1$ (see the review papers [66–71] and lists of references therein.)

Magnetic breakdown significantly changes the spectrum of quantum oscillations. First, periods appear that are inversely proportional to the sum of the areas bounded by complex trajectories composed of the parts of classical trajectories along which electrons move in a weak field (when $W_H = 0$). Second, for a developed magnetic breakdown the spectrum of oscillations of kinetic characteristics (an analog of the Shubnikov–de Haas effect) differs from the spectrum of oscillations of thermodynamic characteristics (an analog of the de Haas–van Alphen effect). The last is not complex. One of the most striking manifestations of magnetic breakdown is the so-called magnetic breakdown oscillations with a rather large amplitude. These oscillations take place when a noticeable change of the effective probability of the breakdown between large sections of the magnetic breakdown configurations occurs due to the reconstruction of the quantum energy levels as the magnetic field is changed [69].

An approach of the trajectories manifests itself not only in oscillation effects. It can be substantial for the smooth dependence on $H$ of the elements of the conductivity and resistivity tensors. The resulting changes are different for these two tensors. When the approach is caused by the presence of self-intersecting trajectories (an approach of the 1st type), the most important is that at the point of self-intersection the transverse (with respect to $H$) velocity becomes zero (the point of self-intersection is a stopping point). As a result the rotation periods $T$ along the trajectories close to self-intersecting ones are anomalously large. This can be interpreted as an effective decrease of the magnetic field. The smaller the magnetic field, the larger the transverse conductivity. Therefore, an increase of the rotation periods can result in a change of the asymptotic behavior of $\sigma_{xx}$ (or $\sigma_{yy}$) as functions of $\mathbf{H}$. (See the end of the next section. In addition, a similar situation is discussed in Section 5 through the example of the FS of a corrugated cylinder type.)

First of all the influence of magnetic breakdown on the smooth dependence of characteristics of GMP on $H$ stems from the difference between the configuration of electron trajectories for $W_H = 1$ and the configuration for $W_H = 0$. When $W_H = 1$, electrons move in a magnetic field along "new" trajectories, constructed from parts of "the old" trajectories (Fig. 2). Since at the point of magnetic breakdown the velocity does not become zero, taking account of the quantum transition from one electron trajectory to the other in no way contradicts the possibility of describing the motion along the new trajectories with the aid of classical equations. As if due to magnetic breakdown the magnetic field changes the structure and, sometimes also the topology, of the FS. "The new" FS defines "the new" trajectory of conduction electrons [66,67].

When $0 < W_H < 1$ the influence of magnetic breakdown on the smooth dependence of characteristics of GMP on $H$ leads to a change of the role of small angle scattering [68,69]. When electrons

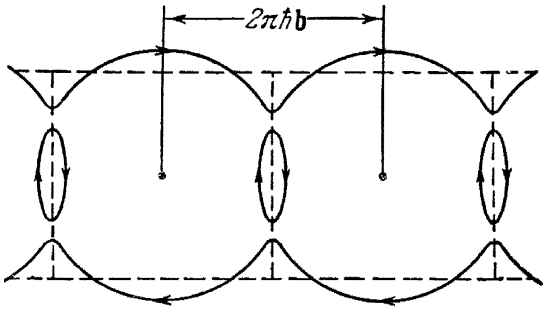



Fig. 2. The trajectories of conduction electrons in momentum space in a magnetic field: in the absence of magnetic breakdown ($W_H = 0$) and for $W_H = 1$.

move along isolated orbits, the transport relaxation time $\tau_{tr}$ enters the expressions for the elements of the tensors $\sigma_{ik}(\mathbf{H})$ and $\rho_{ik}(\mathbf{H})$ describing the classical GMP (oscillation effects depend on the probability of the small angle scattering $\tau_{sa} \ll \tau_{tr}$). In the case of magnetic breakdown due to the small angle collisions the scattering by the magnetic breakdown points becomes a stochastic one. As a result electrons perform random walk along the network of trajectories connected by magnetic breakdown. The quantities $W_H\Omega$ and $(1 - W_H)\Omega$ play the role of the frequency of collisions, and the mean free path falls out of the expression for the dissipative part of the transverse conductivity completely [67,68].

Rather complete information about the theory of magnetic breakdown can be found in the review paper by Stark and Falicov [68], as well as in the subsequent reviews [69,71], where a technique for calculation of physical quantities in the case of the developed magnetic breakdown introduced by Slutzkin [70] was presented.

In the present review we restrict ourselves to a discussion of "the classical" galvanomagnetic phenomena. We find an excuse in the fact that quantum effects of both types (the quantization of the transverse with respect to **H** motion of electrons and the tunneling between classical trajectories situated closely in the **p**-space) were not taken into account in the TGMP. Let us note only, that under the conditions when magnetic breakdown manifests itself clearly, investigations of GMP at different temperatures, that is for different $l$, allow obtaining an additional information about the energy spectrum, the location of the sheets of the FS, and the relaxation properties of the charge carriers. Magnetic breakdown in no way prevents the reconstruction of the FS with the aid of measurements of the magnetoresistivity.

There is a classical effect that can change the nature of electron motion in a magnetic field because of the closeness of the sheets of FS, to each other, if during each collision an electron is sure to jump from one of the sheets to another one. Repeated returns of an electron in the region of the approach of the sheets give rise to this effect [72,73]. It is analogous to the umklapp processes: to describe this effect it is not necessary to take into account the quantization of electron motion in a magnetic field and tunneling. Formally, to calculate these "the umklapp processes", the role of the incoming term of the collision operator has to be analyzed. In the TGMP no such analysis was carried out. Although the effect noted can change the asymptotic behavior of the GMP characteristics and undoubtedly plays a role when finding the temperature dependence of the magnetoresistivity of very pure metals, we do not discuss it here, since a comparatively detailed presentation of all the

aspects of problems relating to "the umklapp processes" from one sheet to another has already been given (see Ref. [73] and the bibliography in the same place).

## 4. GMP in metals with a closed Fermi surface

In the TGMP the main attention was paid to an examination of the asymptotic behavior of galvanomagnetic characteristics of metals in strong magnetic fields, when during the relaxation time the kinematic properties of conduction electrons are able to manifest themselves.

In metals with a closed FS conduction electrons with the Fermi energy perform a periodic motion in the plane perpendicular to the magnetic field, with the period $T=2\pi/\Omega=2\pi m^*c/eH$, since all their orbits in the **p**-space are closed and located inside one primitive cell. The values of the components of the velocity

$$v_x=-\frac{c}{eH}\frac{\partial p_y}{\partial t},\quad v_y=\frac{c}{eH}\frac{\partial p_x}{\partial t}\ , \tag{4.1}$$

vanish when averaged over the period $T$. However, the averaged value of the velocity of an electron along the magnetic field

$$\bar{v}_z=\frac{1}{T}\int_0^T \mathrm{d}t v_z(t,\varepsilon,p_z) \tag{4.2}$$

does not become zero. As a result, when the magnetic field formally tends to infinity, all the elements of the conductivity tensor, except $\sigma_{zz}$ tend to zero. In this case for metals with unequal numbers of electrons ($n_\mathrm{e}$) and holes ($n_\mathrm{h}$) the Hall conductivity does not depend on the scattering mechanism of the charge carriers. It is equal to

$$\sigma_{xy}=\frac{ec(n_\mathrm{e}-n_\mathrm{h})}{H}\ , \tag{4.3}$$

and the Hall "constant"

$$R=\frac{1}{ec(n_\mathrm{e}-n_\mathrm{h})} \tag{4.4}$$

has the same form as in the generalized DLS model.

In a compensated metal ($n_\mathrm{e}=n_\mathrm{h}$) the series expansion of the Hall conductivity in powers of $1/H$ begins with the cubic term, and the Hall "constant" depends on the scattering mechanisms of conduction electrons.

In a strong magnetic field the longitudinal magnetoresistivity ($\mathbf{j}\parallel\mathbf{H}$) reaches saturation, and the asymptotic behavior of the transverse ($\mathbf{j}\perp\mathbf{H}$) magnetoresistivity $\rho_\perp$ is significantly different for compensated and noncompensated metals with closed FS. The magnetoresistivity of a noncompensated metal tends to saturation, and the magnetoresistivity of a compensated metal increases as $H^2$ as the magnetic field is increased (Fig. 3).

In the TGMP a rigorous definition for the numbers of electrons ($n_\mathrm{e}$) and holes ($n_\mathrm{h}$) was given:

$n_\mathrm{e}$ is the number of occupied states with $m^*>0$ in the **p**-space. With an increase of the energy the areas of plane cross-sections $p_H=\mathrm{const}$ of isoenergetic surfaces near the FS and the volume of the FS increase; $n_\mathrm{h}$ is the number of free states with $m^*<0$ inside a sheet of the FS. Its cross-sections and its volume decrease as the energy is increased.

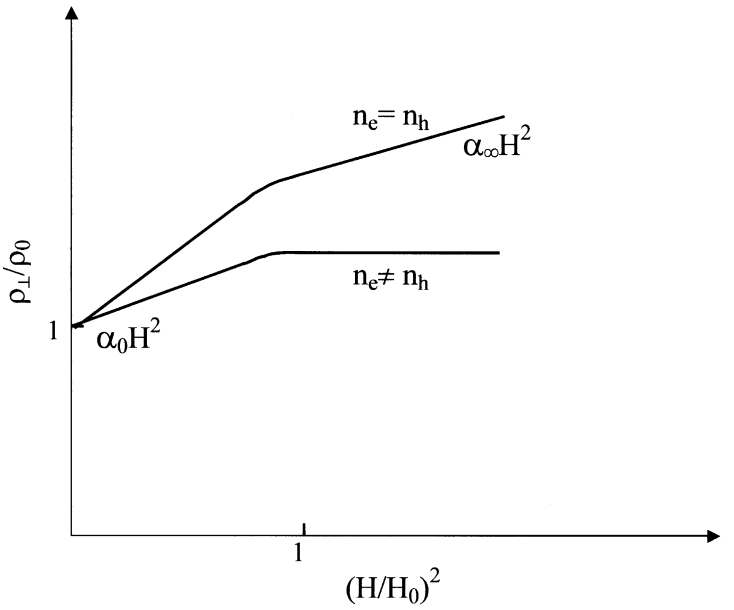



Fig. 3. The dependence of the transverse resistivity ($\mathbf{j} \perp \mathbf{H}$) on the value of the magnetic field in a compensated metal (the upper curve) and in a noncompensated metal with a closed FS.

The directions of going around a trajectory in a magnetic field are different for electrons and holes. Sometimes this difference is attributed to the difference of the signs of their charges, but not of the effective masses, i.e. a positive charge is attributed to the holes.

When $\Omega\tau \gg 1$ and $n_e \neq n_h$, all the characteristics of GMP, except the Hall "constant", depends on the mechanism of scattering of conduction electrons. In anisotropic conductors the Hall field is not equal to zero even when the current flows along a uniform magnetic field.

If the description of the dissipative processes is restricted by the approximation of the relaxation time $\tau$ then, of course, all the asymptotic values of the elements of $\sigma_{ik}(\mathbf{H})$ and $\rho_{ik}(\mathbf{H})$ can be represented as integrals over the FS in terms of the dispersion law of the charge carriers. However, the $\tau$-approximation for the collision integral allows determining the elements of the conductivity and resistivity tensors in order of magnitude only. Their accurate calculation is possible only when solving the relevant integro-differential equations for the functions $\Psi_i$ after the specific definition of the scattering mechanism of conduction electrons. For galvanomagnetic characteristics such an analysis is a separate complex problem. We do not know publications where a complete examination was presented. Nevertheless, some inequalities can be proved under the most general assumptions relating to the form of the collision integral. Let us list them:

1. $ds_i/dH < 0$ – the principle values of the symmetric part of the conductivity tensor are nondecreasing functions of a magnetic field [76].

2. $d^2/\rho dH^2_{H=0} \geqslant 0$ – in the region of weak magnetic fields both the longitudinal and the transverse elements of the resistivity increase as the magnetic field is increased [76].

3. If $n_e \neq n_h$, then $\rho_\infty \geqslant \rho_0$ – the asymptotic value of the resistivity in a strong magnetic field $\rho_\infty$ cannot be smaller than the resistivity $\rho_0$ for $H = 0$ [76].

4. If $n_e = n_h$, then

$$\rho_\perp = \begin{cases} \rho_0 + \alpha_0 H^2, & \Omega\tau \ll 1, \\ \alpha_\infty H^2, & \Omega\tau \gg 1, \end{cases}$$

and $\alpha_\infty < \alpha_0$, – in strong magnetic fields the quadratic growth of the transverse resistivity is less "steep" than in weak magnetic fields [77].

5. The relations comparing the dissipative and the Hall parts of the conductivity tensor, are reminiscent of the Kramers–Kronig relations [78]:

$$s(H) \geqslant \frac{1}{\pi}\left|\int_{-\infty}^{\infty} \frac{a(H')\,\mathrm{d}H'}{H'-H}\right|; \quad |a(H)| \leqslant \frac{1}{\pi}\left|\int_{-\infty}^{\infty} \frac{s(H')\mathrm{d}H'}{H'-H}\right| , \tag{4.5}$$

where $s(H) = [\sigma_{xx}(H) + \sigma_{yy}(H)]/2$ and $a(H) = [\sigma_{yx}(H) - \sigma_{xy}(H)]/2$; $H \equiv H_z$.

All these inequalities follow from the fact that the collision operator $\hat{W}$ is a positive Hermitian operator, and $\partial/\partial t$ is an anti-Hermitian operator:

$$\langle \phi \hat{W} \chi \rangle = \langle \chi \hat{W} \phi \rangle; \quad \langle \phi \hat{W} \phi \rangle > 0 \ ; \tag{4.6}$$

$$\left\langle \phi \frac{\partial \chi}{\partial t} \right\rangle = -\left\langle \chi \frac{\partial \phi}{\partial t} \right\rangle \ . \tag{4.7}$$

The derivation of the inequalities 1–4 becomes clear, if the scalar product of functions defined in the **p**-space is introduced as

$$(\phi, \chi) = \langle \phi \hat{W}^{-1} \chi \rangle \ , \tag{4.8}$$

where $\hat{W}^{-1}$ is the operator that is inverse to the collision operator $\hat{W}$. Then the functions arising in calculations of galvanomagnetic characteristics can be regarded as vectors in a Hilbert space, and well-known mathematical theorems can be used.

Relations 1, 2, and 4 follow from the Parseval inequality, and relation (3) is a consequence of the Schwarz–Bunyakovsky inequality. In fact, the ratio $\rho_{\perp\infty}/\rho_0$ can be written as

$$\frac{\rho_{\perp\infty}}{\rho_0} = \frac{(\hat{W}\Psi_y, \hat{W}\Psi_y)(v_x, v_x)}{(\hat{W}\Psi_y, v_x)^2} \ , \tag{4.9}$$

from which it follows that $\rho_{\perp\infty}/\rho_0 \geqslant 1$, since the product of the norms of two vectors is not smaller than the modulus of their scalar product. The equality is reached, if the vectors $\hat{W}\Psi_y$ and $v_x$ are colinear (it is certainly reached in the $\tau$-approximation in the case of an isotropic energy spectrum and the isotropy of the charge carrier scattering).

Inequalities (5) (see Eqs. (4.5)), comparing the dissipative and the Hall parts of the conductivity tensor, stand by themselves. To derive these inequalities, analytic extensions of all the expressions including the magnetic field $H = H_z$ into the complex plane $H$ has to be examined [78].

According to measurements of the quantum oscillations of the magnetic susceptibility, in alkali metals for different orientations of the magnetic field the areas of the central cross-section of the FS ($\mathbf{pH} = 0$) are indistinguishable within 0.1% i.e. the FS can be approximated by a sphere with the aforementioned accuracy. However, the experimentally measured value of the magnetoresistivity

$\Delta\rho/\rho$ differs from zero significantly, so that even when describing GMP in rather "simple" metals of the type of potassium or sodium, it is found that the $\tau$-approximation for the collision integral is too rough.

The aforementioned inequalities 1–5 can be used as a test of the model chosen for the description of GMP. Their violation (for the given experiment) is evidence that it is necessary to go beyond the limits of the semiclassical TGMP.

In the TGMP serious problems arise when explaining the linear growth of the magnetoresistivity of metals with a closed FS. For polycrystalline samples the linear magnetoresistivity in Kapitza's experiments [5–9] can be explained, although with some "stretching," by an inhomogeneous distribution of current lines. However, as a matter of fact, in the framework of the semiclassical TGMP all attempts to understand the linear dependence on $H$ of the resistivity of a single crystal sample of bismuth failed [28]. This was the reason why the author (E.S. Borovik) was compelled to assume that the linear behavior of $\Delta\rho_{\perp}(H)$ described approximately the passage from one quadratic-law dependence in weak magnetic fields to an other one, or to saturation in strong magnetic fields. The analysis in [79,80] confirmed this assumption in principle.

Alekseevskii et al. [81] experimentally examined in detail the magnetoresistivity of samples of bismuth that contained different impurities of donor and acceptor types. For sufficiently strong magnetic fields they found almost horizontal parts on the curve of $\rho(H)$. They did not see the linear dependence of $\rho$ on $H$ in their experiments. Apparently, in their experiments the tendency to saturation of the resistivity of bismuth in sufficiently strong magnetic fields was related to an imbalance of electrons and holes due to the presence of the impurities actively affecting the electronic structure of the examined samples. When the imbalance of the numbers of electrons and holes is not large, the saturation of the magnetoresistivity has to be found in magnetic fields that are stronger than usual: as was shown in Ref. [80], when $\Omega\tau \gg (n_e + n_h)/|n_e - n_h| \gg 1$. A significant difference of the results obtained in different experimental studies of GMP in bismuth is the evidence of a strong dependence of Bi on the production procedures and the purity of the samples.

According to Shoenberg's model [24], the FS of bismuth consists of three electron and one hole ellipsoids. The calculation of GMP characteristics with the aid of this model did not leave place for hopes to obtain the linear growth of the magnetoresisitivity of single crystalline bismuth samples as $H$ is increased. Later on it was found out that the sheet of the FS relating to the hole states of the charge carriers was slightly bent, in other words it was a slightly nonconvex one (see the preceding section). Consequently, for some orientations of the magnetic field, appearance of the self-intersecting cross-section $p_H = \pm p_c$ of the FS is possible. The period of motion along an orbit that is close to the self-intersecting one, diverges logarithmically when $p_H$ tends to $p_c$, and the condition $\Omega\tau \gg 1$ turns out to be unsatisfiable. As a result, there is a region of magnetic fields for which the growth of the magnetoresistivity is slower than the quadratic one [82]. However, for sufficiently strong magnetic fields the thickness of the layer of electron orbits near $p_H = p_c$ for which $\Omega\tau \leqslant 1$ decreases exponentially as $H$ is increased, and it is not able to compete with other conduction electrons with the Fermi energy. Therefore asymptotically the resistivity increases quadratically when the magnetic field is increased. Most likely in such fields the semiclassical approach is not suitable for the description of GMP.

Possibly, it is the passage to the quantum limit, when all conduction electrons fill the first Landau level [83–85], and $1/\tau$ increases linearly with $H$, that explains the linear dependence of the magnetoresistivity of Bi as $H$ is increased.

## 5. Galvanomagnetic phenomena in strong magnetic fields in metals with open Fermi surfaces

The most important result of the TGMP is the determination of the asymptotic behavior of the magnetoresistivity as a function of a strong magnetic field for different orientations of it with respect to the crystallographic axes of a single crystal sample. It was the TGMP that opened the world of distinctive whimsical structures that are the FS of polyvalent metals. Usually in special books (see, for example, Ref. [86]) the authors depict FS without being concerned about the impression they must make. What concerns them is the possibility of using the drawings for "extracting" from them the dimensions that are necessary for calculations, and so on. Of course, it is easier to picture closed FS than open ones. However, if there is any complication, then because the FS as a rule consists of several sheets, and picturing the FS as a whole, a not so simple question has to be answered: how are these sheets placed in the **p**-space with respect to each other? We would like to remind the reader that in the TGMP this was not important, since the theory of the 1950s did not take into account the possibility for electrons to jump from one sheet of the FS to another one.

An open FS can be pictured as a sum of comparatively simple topological "design elements" that are corrugated cylinders and corrugated planes. With the aid of these elements, an arbitrarily complex open FS can be constructed.

### 5.1. Corrugated cylinder

Let us first examine the galvanomagnetic phenomena in a metal with a FS in the form of a corrugated cylinder with the direction of "openness" along the axis $p_x$ in the magnetic field $\mathbf{H} = (H\cos\theta, 0, \sin\theta)$. When $\theta$ is other than $\pi/2$ all the cross-sections of a weakly corrugated cylinder by the planes $p_H = \mathbf{pH}/H = \text{const}$ are closed. As $\theta$ approaches $\pi/2$, the closed electron orbits in the momentum space become strongly elongated orbits (see Fig. 4). The period of motion along such an orbit is proportion to $1/\cos\theta$. It increases infinite when $\theta \to \pi/2$, and if $\theta = \pi/2$ a strongly elongated orbit "breaks" into two open orbits. In this case the period of an electron's motion changes in a jump-like way. When an electron is moving along the orbit corresponding to the open cross-section of the FS by the plane $p_H = p_z = \text{const}$ the period is

$$T(p_z) = \frac{2\pi m^* c}{eH} = \frac{c}{eH}\int_0^{2\pi\hbar/a} \frac{\mathrm{d}\,p_x}{v_y}\ , \tag{5.1}$$

where $a$ is the size of the primitive cell of the crystal along the axis $x$.

When $\theta = \pi/2$, the value of the velocity $v_y$ averaged over the period does not equal to zero:

$$\bar{v}_y = \frac{1}{T}\int_0^T \mathrm{d}t v_y(t) = \frac{\hbar}{am^*}\ , \tag{5.2}$$

and in the $yz$ plane the charge carriers drift in all possible directions. In this case the elements $\sigma_{yy}$ and $\sigma_{zz}$ of the conductivity tensor in order of magnitude are the same as the conductivity $\sigma_0$ in the absence of a magnetic field. However, since $\bar{v}_x = 0$, in a strong magnetic field the elements $\sigma_{ij}$, for which at least one of the subscripts coincides with $x$, decrease when $H$ is increased.

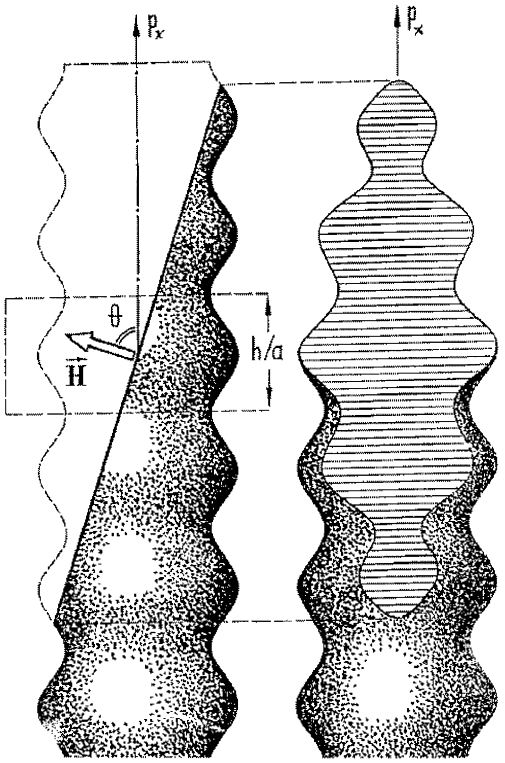



Fig. 4. FS in the form of a corrugated cylinder.

When approaching (with respect to $p_z$) the boundary cross-section $p_z = p_c$ that separates the regions of closed and open cross-sections of the FS, the period of motion $T(p_z)$ increases infinitely. This is because the cross-section $p_z = p_c$ has points of self-intersection, where the open orbits touch each other when $p_z$ tends to $p_c$. An electron spends a long time near the points of self-intersection, since the velocity of its motion in the plane perpendicular to the magnetic field is negligibly small. Near a self-intersecting orbit the condition for the field to be strong, namely, $T(p_z) \ll \tau$, is not fulfilled, and a contribution to $\sigma_{ij}$ of the charge carriers from a small vicinity of the cross-section $p_z = p_c$ can affect the asymptotic behavior of the magnetoresistivity significantly.

To visualize the problem, let us examine the galvanomagnetic effects in a conductor with the simplest model dispersion law of the charge carriers:

$$\varepsilon(\mathbf{p}) = \frac{p_y^2 + p_z^2}{2m} + \eta(v_F\hbar/a)[1 - \cos(a\,p_x/\hbar)] \ , \tag{5.3}$$

where $a$ is the crystalline lattice period along the axis $x$; $\eta$, $m$ and the characteristic Fermi speed $v_F = a\varepsilon_F/2\hbar$ are the same for all the cross-sections of FS by the plane $p_H = \text{const}$, and the Fermi energy is counted from the band edge, i.e. $\varepsilon_{\min} = 0$.

When $\eta > 1$, the FS is a closed one, since the region of the momentum space near $p_x = \pm\pi\hbar/a$ is unattainable for the charge carriers, and all the electron orbits are inside one primitive cell of the

momentum space. When $\eta < 1$, the FS is an open one, and open electron trajectories are possible in a magnetic field orthogonal to the $x$-axis.

Taking advantage of the dispersion law of the charge carriers in form (5.3), with regard to equations of motion of a charge in a magnetic field, the following expression for the matrix $\sigma_{ik}$ is obtained using the $\tau$-approximation for the collision integral:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xx}\dfrac{\gamma\sin\theta}{\gamma^2+\cos^2\theta} & \sigma_{xx}\dfrac{\sin\theta\cos\theta}{\gamma^2+\cos^2\theta} \\ -\sigma_{xx}\dfrac{\gamma\sin\theta}{\gamma^2+\cos^2\theta} & \sigma_{yy} & \sigma_{yy}\dfrac{\cos\theta}{\gamma} \\ \sigma_{xx}\dfrac{\sin\theta\cos\theta}{\gamma^2+\cos^2\theta} & -\sigma_{yy}\dfrac{\cos\theta}{\gamma} & \sigma_0-\sigma_{yy}\dfrac{\cos\theta}{\gamma^2} \end{pmatrix}, \tag{5.4}$$

where $\gamma = mc/eH\tau$, $\sigma_0 = Ne^2\tau/m$ is the conductivity in the plane $yz$ when $H = 0$, and $N$ is the charge carrier density.

The diagonal elements of the conductivity tensor satisfy the relation

$$\sigma_{yy} + \sigma_{xx}\frac{(\gamma\sin\theta)^2}{(\gamma^2+\cos^2\theta)^2} = \sigma_0\frac{\gamma^2}{\gamma^2+\cos^2\theta}\ . \tag{5.5}$$

When $\theta = \pi/2$ the matrix of the elements of the tensor $\sigma_{ij}$ takes the form

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \gamma^{-1}\sigma_{xx} & 0 \\ -\gamma^{-1}\sigma_{xx} & \sigma_0-\gamma^{-2}\sigma_{xx} & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix}, \tag{5.6}$$

and for the matrix of the elements of the resistivity tensor, that is the inverse of matrix (5.6), we have

$$\rho_{ij} = \begin{pmatrix} \sigma_{xx}^{-1}-\sigma_0^{-1}\gamma^{-2} & (\gamma\sigma_0)^{-1} & 0 \\ -(\gamma\sigma_0)^{-1} & \sigma_0^{-1} & 0 \\ 0 & 0 & \sigma_0^{-1} \end{pmatrix}, \tag{5.7}$$

It is easy to see that for arbitrary orientations of the current density and strong magnetic fields, the Hall field is equal to

$$\mathbf{E}_{\mathrm{Hall}} = R[\mathbf{j},\mathbf{H}]\ , \tag{5.8}$$

and the Hall constant $R = 1/Nec$ is inversely proportional to the entire volume confined inside the FS. This universal value of the Hall constant does not depend on the particular shape of the FS in the form of the corrugated cylinder. The point is that when a considerable part of the electrons with the Fermi energy is moving along open trajectories, all the states of the charge carriers remain as electrons (or holes) for all the orientations of the magnetic field, including $\theta = \pi/2$ as well. In this case expression (5.8) for the Hall field follows from the equation of electron motion in electric and magnetic fields in the collisionless limit ($\tau \to \infty$).

Let us examine two limiting cases of a FS in the form of the corrugated cylinder: (i) corrugations are small ($\eta \ll 1$), and (ii) cylinders with "a very slender waist", when $(1 - \eta) \ll 1$ and the number of open cross-sections of the FS is very small.

In the case of a slightly corrugated FS when $\theta = \pi/2$, there is nevertheless a small number of closed orbits of the Fermi electrons. Their contribution to the element $\sigma_{xx}$ of the conductivity,

$$\sigma_{xx}^{\text{close}} = \frac{4e^2\tau}{(2\pi\hbar)^3} \int_{p_c}^{p_0} \mathrm{d}\, p_H 2\pi m^*(p_H) \sum_{n=1}^{\infty} \frac{|v_x^n|^2}{1 + (n\Omega\tau)^2} \ , \tag{5.9}$$

in order of magnitude can even be comparable with the contribution of all the other charge carriers belonging to open cross-sections of the FS. This is because near the point $(0, 0, p_0)$ of the FS, where $p_0^2 = 2m\varepsilon_F(1 + \eta)$, the cyclotron mass $m^*$ is proportional to $\eta^{-1/2}$. It increases on approaching the self-intersecting cross-section $p_H = p_c$, for which $m^*$ tends to infinity as $\ln[p_c/(p_H - p_c)]$. When $\eta^{1/2} \ll \gamma$, during the relaxation time conduction electrons belonging to closed cross-sections of the FS do not manage to accomplish the full revolution along their orbits, and their contribution to $\sigma_{xx}$ has the form [87]

$$\sigma_{xx}^{\text{close}} = \frac{4e^2\tau}{(2\pi\hbar)^3} \int_{p_c}^{p_0} \mathrm{d}\, p_H 2\pi m^*(p_H) v_x^2 \approx \sigma_0 \eta^{5/2} \ . \tag{5.10}$$

When $\eta^{1/2} > \gamma$, in the denominator of formula (5.9) the quantity $\Omega\tau$ no longer can be neglected, since for a considerable part of electrons whose orbits are closed, the frequency of revolution $\Omega$ is greater than the frequency of their collisions. A simple calculation allows to obtain the following interpolational formula [88]:

$$\sigma_{xx}^{\text{close}} = \sigma_0 \eta^{5/2} \gamma_0^2 \int_1^{\infty} \mathrm{d}u \frac{u^3 \exp(-u)}{u^2\gamma_0^2 + \eta} \approx \sigma_0 \frac{\gamma_0^2 \eta^{5/2}}{\gamma_0^2 + \eta} \ . \tag{5.11}$$

When $\eta^{1/2} \ll \gamma$, the main contribution to $\sigma_{xx}$ is from the charge carriers whose trajectories are open. In the immediate proximity of the cross-section $p_H = p_c$ the dependence of the component of the electron velocity $v_y$ on $t$ has a complex form. However, far from this cross-section it can be said that with sufficient accuracy the component of the velocity $v_y(t)$ is equal to $v_y(0)$. Depending on $t$, small corrections can be taken into account using the method of successive approximations with respect to the parameter $\eta$ ($t$ is measured from the central cross-section of the FS $p_x = 0$) to within small terms proportional to $\eta$:

The period of the motion of the charge carriers along an orbit that is far enough from the self-intersecting one, is inversely proportional to $v_y(0)$:

$$T(p_H) = \frac{2\pi}{\Omega} = \frac{2\pi\hbar c}{aeHv_y(0)} = \frac{2\pi v_F}{\Omega_0 v_y(0)} \ , \tag{5.12}$$

and the component $v_x$ of their velocity is a harmonic function of $t$:

$$v_x = \eta v_F \sin(\Omega t) \ . \tag{5.13}$$

In this region of magnetic fields the integration with respect to $p_H$ of the expression for $\sigma_{xx}$ over a small interval $\eta^{1/2}v_F < v_y \ll v_F$ gives the following asymptotic result for $\eta^{1/2} \ll \gamma \ll 1$:

$$\sigma_{xx}^{\text{open}} = \sigma_0 \eta^2 \gamma \ , \tag{5.14}$$

and the magnetoresistivity increases linearly when the magnetic field is increased [89].

Thus, a small part of the conduction electrons with open orbits that move slowly along the $x$-axis give the main contribution to $\sigma_{xx}$, which exceeds significantly the contribution from all the other charge carriers on the FS.

When the magnetic field is increased, the number of conduction electrons for which $T > \tau$ decreases, and the contribution to $\sigma_{xx}$ of the charge carriers with orbits close enough to the cross-section $p_H = p_c$ becomes more and more significant. At the same time, when $\gamma \leqslant \eta^{1/2} \ll 1$, the contributions to $\sigma_{xx}$ of the charge carriers with open and closed trajectories are of the same order of magnitude:

$$\sigma_{xx}^{\text{close}} \approx \sigma_{xx}^{\text{open}} \approx \sigma_0 \eta^{3/2} \gamma_0^2 \ . \tag{5.15}$$

The results given by Eqs. (5.10), (5.14), and (5.15) are valid for an arbitrary FS in the form of a weakly corrugated cylinder [88,89]. This is because when $\eta \ll 1$, the charge carriers near the self-intersecting orbit $p_H = p_c$ give the main contribution to $\sigma_{xx}$. For this orbit the velocity $v_y$ is small, i.e. the charge carriers' energy is weakly dependent on $p_y$. Expanding the energy in a series in powers of $p_y$,

$$\varepsilon(\mathbf{p}) = \sum_{n=1}^{\infty} \varepsilon_n(p_y, p_z) \cos(anp_x/\hbar) \ , \tag{5.16}$$

we obtain the following expression for the period of the charge motion in the magnetic field:

$$T(p_H) = \frac{1}{\eta^{1/2}\Omega_0} \int_0^{\pi} \mathrm{d}\alpha \frac{1}{(\zeta^2 + \sin^2\alpha)^{1/2}} \ , \tag{5.17}$$

where $\zeta^2 = [\varepsilon_0(0, p_c) - \varepsilon_0(0, p_H) + \varepsilon_1(0, p_c) - \varepsilon_1(0, p_H)]/2\varepsilon_1(0, p_H)$, and in order of magnitude $\Omega_0$ is equal to the frequency of revolution of an electron for $\theta = 0$.

When $\zeta \ll 1$ period (5.17) of the charge carriers' motion along the open trajectory diverges logarithmically, and when $\zeta$ is of the order of unity, it is proportional to $\eta^{-1/2}$. When $\eta$ increases, the fraction of the conduction electrons, for which $T(p_H) \geqslant \tau$ at the given value of the magnetic field decreases. When $\eta \approx 1$ their number in a strong magnetic field ($\Omega_0\tau \gg 1$) is exponentially small: it is proportional to $\exp(-\Omega_0\tau)$. Thus, the contribution of such electrons to the elements of the

conductivity tensor is noticeable only in magnetic fields for which $\Omega_0\tau < 10$ (Ref. [82]). If at the same time the parameter $\eta$ is not close to unity, i.e. $(1-\eta)$ is of the order of one too, and there are enough open cross-sections of the FS, the element $\sigma_{xx}$ is inversely proportional to $H^2$ in practically all the region of magnetic fields satisfying the inequality $\gamma_0 = 1/\Omega_0\tau \ll 1$. In this case the asymptotic form of the elements of the conductivity tensor in a strong magnetic field ($\Omega_0\tau \gg 1$) is

$$\sigma_{ij} = \begin{pmatrix} \gamma_0^2 a_{xx} & \gamma_0 a_{xy} & \gamma_0 a_{xz} \\ \gamma_0 a_{yx} & a_{yy} & a_{yz} \\ \gamma_0 a_{zx} & a_{zy} & a_{zz} \end{pmatrix} , \tag{5.18}$$

where the elements of the matrix $a_{ij}$ in order of magnitude are the same as those of the conductivity in the absence of a magnetic field. Due to the symmetry of the problem some of the nondiagonal elements can become zero.

The existence of open electron trajectories results in a strong anisotropy of the transverse magnetoresistivity:

$$\rho = A_1 H^2 \tau \cos^2\alpha + A_2 , \tag{5.19}$$

where in the $\tau$-approximation $A_1$ and $A_2$ depend on the specific form of the dispersion law of the charge carriers only, and $\alpha$ is an angle between the $x$-axis and the direction of the current.

Eqs. (5.18) and (5.19) are also correct for $\Omega_0\tau\eta^{1/2} \gg 1$ and an arbitrary FS of the corrugated cylinder type, if $\eta$ differs from unity significantly. In this case when the electric current is flowing along the $y$-axis, the resistivity, after a small growth with the magnetic field, reaches saturation in strong magnetic fields. When the current deviates from the $y$-axis by an angle that is much greater than $\gamma_0\eta^{-1/2}$, the resistivity increases infinitely proportionally to $H^2$.

If $(1-\eta) = \delta \ll 1$ and $\theta = 0$, the period of motion of the conduction electrons along an open trajectory

$$T(p_H) \approx \frac{1}{\Omega_0} \int_0^{\pi} \frac{d\phi}{[\delta^2 - (p_H/p_F)^2 + \sin^2\phi]^{1/2}} \tag{5.20}$$

increases proportionally to $\ln(1/\delta)$ when $\delta$ tends to zero even for the central cross-section of the FS. Moreover, the number of open orbits is negligibly small. As a result, the resistivity in a magnetic field is of the same order of magnitude as for $H = 0$ so far as $\delta/\ln(1/\delta) \leqslant \gamma_0^2 \ll 1$, and only when $\gamma_0^2 \ll \delta$ does the quadratic growth of the transverse magnetoresistivity in the magnetic field occur.

Thus, in the case of a FS in the form of a corrugated cylinder, a narrow maximum has to be expected for the dependence of the magnetoresistivity on the value and the orientation of the strong magnetic field, when the magnetic field is orthogonal to the axis of the cylinder (Fig. 5). The width of this maximum decreases proportionally to $1/H$ when the magnetic field is increased. When comparing with experiments, it is important to know that under these conditions the resistivity of a polycrystalline wire, whose thickness is one polycrystalline grain, being an average with respect to different orientations of the grains, increases linearly with magnetic field for $\Omega_0\tau \gg 1$, even when for the Fermi electrons the numbers of states with closed and open orbits are of the same order of magnitude.

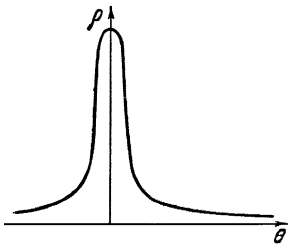



Fig. 5. The dependence of the resistivity on the value of the strong magnetic field for different angles $\theta$ between **H** and the axis of the corrugated cylinder.

It turned out that among common (monatomic) metals none had a FS in the form of a corrugated cylinder. For a long time this model of the FS was of academic interest only. At the present time the class of conductors that are called metals, is extended. The metallic type of conductivity, i.e. an increase of the resistivity as the temperature is increased, became the basic criterion for metals, and GMP were successfully used to reconstruct FS with the aid of experimental data not for intermetallic compounds only, but for synthetic metals of organic origin with a quasi-two-dimensional electron energy spectrum. For several decades the results of these investigations were published in the journal "Synthetic Metals" and in many other prestigious scientific journals.

The model of a FS in the form of a weakly corrugated cylinder is in rather good agreement with experimentally observed galvanomagnetic phenomena in organic "metals" [91–93] and superlattices [94].

### 5.2. *Space net of corrugated cylinders*

A FS consisting of a space net of corrugated cylinders has many more nonclosed cross-sections than a FS in the form of separated corrugated cylinders, since moving in a magnetic field, in **p**-space the charge carriers can pass from one of the cylinders to another one. As a result, the directions of a magnetic field for which open trajectories are possible form a two-dimensional set (Fig. 6). Even for an irrational direction of the magnetic field, when it is impossible to define its orientation with the aid of arbitrarily large Miller indices, the layer of open electron trajectories has an asymptotic common direction for all layers of open cross-sections of the FS [3,90]. Let us assume this direction to be the axis $p_x$ (as before, here and in what follows by the axis $z$ we mean the direction of the magnetic field). The common direction of the open trajectories is situated on one of the crystallographic planes, and is defined by the line of intersection of this plane with the $xy$-plane (Fig. 7).

If the axis $p_x$ coincides with a crystallographic direction in the momentum space, the open electron trajectories are strictly periodic ones. In this case formula (5.19) is valid for the conductivity tensor, and the transverse resistivity of the sample in a strong magnetic field takes the form:

$$\rho(H) = \rho(0)[b_1(H/H_0)^2 D\cos^2\alpha + b_2]\ , \tag{5.21}$$

where $b_1$ and $b_2$ are dimensionless coefficients of the order of unity depending on the explicit form of the dispersion law of the charge carriers, $D$ is the thickness of the layer of the open trajectories related to the size of the primitive cell in momentum space, $\alpha$ is the angle between the direction of

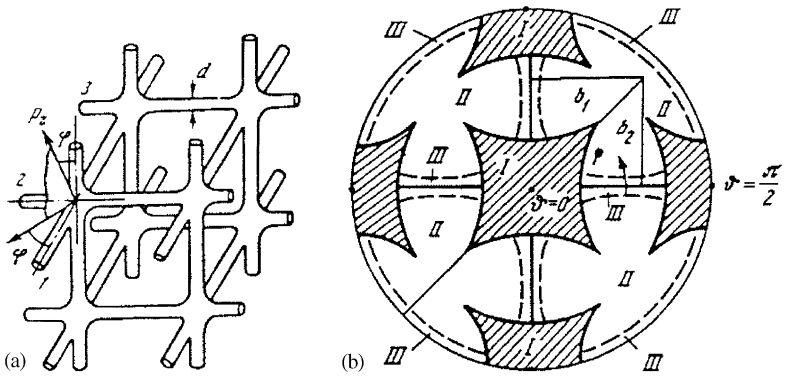



Fig. 6. FS of a hypothetical metal with a simple cubic crystalline lattice (a) and the stereographic projection of the magnetic field directions (the hatched two-dimensional regions I and bold lines) for which open trajectories are possible (b). In the region II there are no open trajectories, and in the region III that is separated by dashed lines, closed orbits are strongly elongated ones.

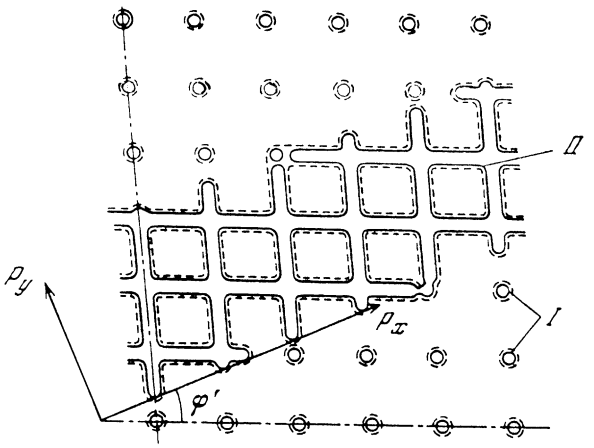



Fig. 7. The plane cross-sections of the isoenergetic surface $\varepsilon(\mathbf{p}) = \varepsilon_F$ presented in Fig. 6 (solid lines) and the surfaces $\varepsilon(\mathbf{p}) = \varepsilon_F + \delta\varepsilon$ containing open electron trajectories. The closed trajectories of the type I enclose the energy regions $\varepsilon < \varepsilon_F$, and the trajectories of the type II enclose the regions $\varepsilon < \varepsilon_F$. In accordance with this, the directions of going around along these trajectories are opposite. Open trajectories separate these two types of lines. The direction of an open trajectory is defined by an angle $\phi'$ for which $\cos\phi' = \cos\phi(1 + \sin^2\phi\tan^2\theta)^{-1/2}$.

the current and the axis $p_x$, and $H_0$ is a characteristic magnetic field for which $\Omega\tau$ is of the order of unity.

Of course, the cross-sections of the FS are not periodic ones when the axis $p_x$ does not coincide with a crystallographic direction. If in this case "the amplitude" of the conduction electron's trajectory along the $x$-axis, i.e. $\Delta x = x_{\max} - x_{\min}$, is much smaller than the mean free path $l$ (let us recall that the trajectory of an electron in the $xy$-plane coincides with the electron trajectory in the momentum space rotated about the angle $\pi/2$ with the similarity ratio $c/eH$), formula (5.21) is valid for the

asymptotics of the transverse resistivity. However, when the magnetic field does not exceed $H_0$ by very much, the growth of the resistivity in the magnetic field can be slower than quadratic. This is because in the case of nonperiodic trajectories and, consequently, a nonperiodic dependence of the electron velocity on the time of motion in a magnetic field, the Fourier coefficients

$$v_x(\omega,\mathbf{p}) = \int_0^\infty \exp(\mathrm{i}\omega t) v_x(t,\mathbf{p})\,\mathrm{d}t \tag{5.22}$$

with small values of $\omega$ can contribute noticeably to the expression for $\sigma_{xx}$,

$$\sigma_{xx} = -\frac{2e^2}{(2\pi\hbar)^3}\int \mathrm{d}^3p\,\frac{\mathrm{d}f_0(\varepsilon)}{\mathrm{d}\varepsilon}\,v_x(\mathbf{p})\frac{1}{\tau}\int_0^\infty \exp(-t/\tau)\mathrm{d}t\int_0^t v_x(t',\mathbf{p})\,\mathrm{d}t' \ . \tag{5.23}$$

As a result a dependence on $H$ of the resistivity relating to the current along the $x$-axis, becomes rather complex. For small intervals of the magnetic field this dependence can be described by an exponential function $H^{q(H)}$. Here the exponent $q$ can differ for different regions of strong magnetic fields and, increasing when the field is increased, it approaches 2 in the region of fairly strong magnetic fields. It is quite possible that in some cases this limiting value of $q$ is outside the semiclassical region of magnetic fields. So, a slower than quadratic growth of the resistivity in the magnetic field, obtained in many experiments, is rather plausible, if the magnetic field was not placed in a symmetry plane of the crystal.

An averaged direction of open cross-sections of the FS changes weakly with a small variation of the orientation of the vector **H** inside the two-dimensional region on the stereographic projection of the magnetic field directions for which there are open electron trajectories.

These intuitive reasons, stated in paper [3], found a rigorous mathematical proof in the works of Novikov and his students that appeared beginning with 1982 (for the list of publications, see the review by Novikov and Mal'cev [1]). A rigorous analysis of topological characteristics of electron energy spectra of metals carried out in these works, allowed determining that nonclosed (open) trajectories on a FS could be subdivided into classes relevant to domains on the unit sphere parametrizing the directions of **H**. Each of the classes was defined by an integer-valued crystallographic plane $\Gamma$, which was a topological characteristic of the given stable group of open trajectories [1]. The domain of directions $\mathbf{n}=\mathbf{H}/H$ for each class of nonclosed trajectories was called a *zone*. In the publications, cited in [1], the structure of the zones was analyzed. The result was compared with the stereographic projections of **n** used in Refs. [3,4] to determine the shapes of FS. The two-dimensional regions of the magnetic field directions leading to the appearance of open cross-sections of FS are the stable zones [1]. Many such zones are possible. Their sizes decrease abruptly when the Miller indices of the plane, where the axis $y$ is placed, increase. The zones do not intersect, although their contact is possible.[4] When approaching the boundary of stability of a zone, the thickness of the layer of open cross-sections of the FS tends to zero in contrast to isolated lines on the stereographic projections (see Fig. 6). Even for an infinitesimal deviation of the magnetic field from these lines the number of open cross-sections of FS vanishes in a jump-like way.

If for a given direction of the magnetic field open cross-sections of isoenergetic surfaces $\varepsilon(\mathbf{p})=\varepsilon$ are possible in a finite interval of the values of the energy $\varepsilon$ for a defined dispersion law of the charge carriers then, as Dynnikov showed, all the nonclosed cross-sections of FS are situated inside

[4] If open trajectories are the result of magnetic breakdown, an intersection of the zones is not prohibited [1].

a band of a finite width (proofs of mathematical theorems and a detailed bibliography can be found in Ref. [1]).

Rather specific is the case of nonergodic electron trajectories, if for a given direction of the magnetic field open electron trajectories are possible only for an isolated value $\varepsilon = \varepsilon_k$ in the energy band. Then the cross-sections of the isoenergetic surface have no common averaged direction, and during a random walk along these trajectories the charge carriers with $\varepsilon_F = \varepsilon_k$ introduce a specific contribution to the conductivity of the sample. In this case the conductivity in the plane perpendicular to the magnetic field, has the form [1]

$$\rho_{xx} = \rho(0)[b_3(H/H_0)^k + b_4], \quad \rho_{yy} = \rho(0)[b_5(H/H_0)^q + b_6] \tag{5.24}$$

where, as before, the coefficients $b_i$ are of the order of unity, and $k + q = 2$. If the directions of the random walk along these trajectories are equiprobable, for $H \gg H_0$ the transverse resistivity of the single crystal sample increases linearly as the magnetic field is increased.

The magnetoresistivity (5.24) can be detected for chosen orientations of the magnetic field only, since the two angular coordinates defining the orientation of the field must comply with two conditions. The first is the narrowing of the energy interval $\Delta\varepsilon$ for which the open electron trajectories are possible to the value $\varepsilon_k$. In addition the coincidence of $\varepsilon_k$ with the Fermi energy is necessary. It can turn out that these conditions are inconsistent.

We do not give here the proofs of these topological theorems that are presented in the review by Novikov and Mal'cev in detail. Let us return to the main theses of the papers [3,4].

In a magnetic field oriented along a second-order symmetry axis of a single crystal, for the charge carriers with the Fermi energy the direction of the drift in the plane perpendicular to the magnetic field can be different in different layers of open cross-sections of the FS. In this case in strong magnetic fields the resistivity saturates for any direction of the current. This type of behavior of the magnetoresistivity relates mainly to conductors with multiply connected FS, for example when isolated corrugated cylinders are inserted into the interstices of a space net of cylinders.

For orientations of the magnetic field along a symmetry axis of higher than second order, only self-intersecting cross-sections of the FS are possible. They separate the domains of states of the electron and hole types. In addition, the same state on a FS in the form of a space net of corrugated cylinders can be of both the electron and the hole type, depending on the orientation of the magnetic field. Therefore, for metals with complex FS (in particular, in the form of a space net of cylinders) a partition of the charge carriers into electrons and holes is rather relative. In a strong magnetic field the asymptotic behavior of the magnetoresistivity and the Hall field is defined by the nature of electron trajectories in **p**-space. For those orientations of a magnetic field, for which all the cross-sections of an open FS by the plane $p_H = \text{const}$ are closed, the galvanomagnetic characteristics have the same form as in the case of a closed FS if, of course, the trajectories do not extend over a large number of primitive cells in momentum space. Again the Hall constant is inversely proportional to the volume confined inside this surface, that is to the number of charge carriers. In the case of a many-sheeted surface it is inversely proportional to the difference in the numbers of electrons $n_e$ and holes $n_h$. However, for orientations of a magnetic field when besides closed cross-sections there are self-intersecting open cross-sections of the FS, the relations between $n_e$ and $n_h$ are significantly different.

In compensated metals, where $n_e = n_h$, for those orientations of a magnetic field for which all the orbits of the charge carriers are closed, the appearance of open cross-sections of the FS leads to

violation of the equality between the numbers of electrons and "holes", and the transverse magnetoresistivity has the form

$$\rho(H) = \rho(0)\frac{D\cos^2\alpha + \gamma_0^2 b_7}{\gamma_0^2[(\Delta V/V + b_8 D)^2 + b_9 D] + \gamma_0^4 b_{10}} \ , \tag{5.25}$$

where $\Delta V = V_1 - V_2$ is the difference between the electron and the hole volumes of the FS without taking account of open cross-sections, and $D$ is the relative thickness of the layer of the open cross-sections. As before, the values of $b_i$ are of the order of unity, $V = V_1 + V_2$, and $\alpha$ is the angle between the $x$-axis and the direction of the current. Eq. (5.25) describes the behavior of the magnetoresistivity in the case of a strong magnetic field being near the boundary of the two-dimensional region on the stereographic projection of the normals where open cross-sections of FS appear. When $D$ tends to zero, $\Delta V$ also tends to zero.

When the magnetic field deviates by a small angle $\theta$ from a symmetry axis of an order higher than the second, the number of open cross-sections of the FS is also very small. However, the electron and the hole volumes are essentially decompensated and Eq. (5.21) for the magnetoresistivity is valid after the replacement of $D$ by the angle of deviation $\theta$.

Also we would like to note that for a polycrystalline wire made from a compensated metal, whose thickness is of the order of the size of a grain, and the grains have different crystallographic orientations, the TGMP predicts a close to quadratic growth of the magnetoresistivity as the magnetic field is increased, since even in the absence of open cross-sections of the FS the transverse resistivity is proportional to $H^2$ when $\Omega\tau \gg 1$. However, in the case of decompensated metals with small zones of open electron trajectories, for a polycrystalline wire, whose thickness is of the order of the size of a grain, the resistivity averaged over different orientations of the grains,

$$\bar{\rho}(H) = \rho(0)[A\tau(H/H_0)^2 + B(H/H_0) + C] \ , \tag{5.26}$$

in a certain region of strong magnetic fields ($H_0/A\tau \gg H \gg H_0$) increases linearly as the magnetic field increases. In Eq. (5.26) the coefficient $A$ is proportional to the size of the two-dimensional stable zone of normals on the stereographic projection; $B$ and $C$ are of the order of unity. Of course, for a massive bulk polycrystalline sample the resistivity, averaged over the orientations of the grains, is not reduced to a simple sum of the resistivities of the individual grains. The problem concerning the current flowing through a massive sample of a metal with an open FS results in a rather specific dependence of the resistivity on the value of the strong magnetic field [96,97], namely, $\bar{\rho}(H)$ is proportional to $H^{4/3}$.

In metals with a crystalline lattice of cubic symmetry, open FS can be assumed in the form of a space net of corrugated cylinders only. In Fig. 8 a FS in the form of a space net of corrugated cylinders directed along the space diagonals of a cubic cell of the crystal is shown, and in Fig. 9 we present the stereographic projections of the magnetic field directions for which open electron trajectories are possible. In Fig. 10 diagrams of the rotation of the field $\mathbf{H}$ for a fixed orientation of the current are presented. On these diagrams the passage of the transverse magnetoresistivity ($\mathbf{H} \perp \mathbf{j}$) from saturation to an unlimited increase in the magnetic field, locates definitively the boundary of the region of the directions $\mathbf{H}/H$, when open cross-sections of the FS by the plane $p_H = \text{const}$ appear.

A strong anisotropy of the transverse magnetoresistivity indicating the existence of open cross-sections of the FS, is observed for almost all the metals except indium, the group of alkali metals,

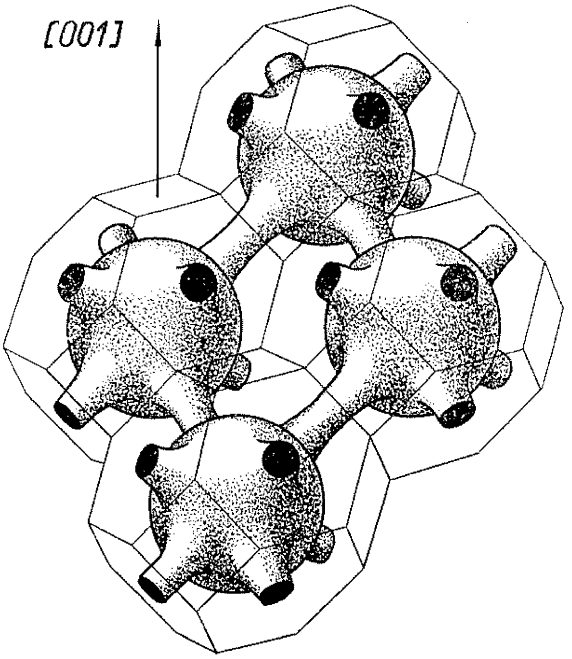



Fig. 8. FS of metals with a faced-centered cubic lattice in the form of a space net of corrugated cylinders directed along the space diagonals of the primitive cell of the momentum space.

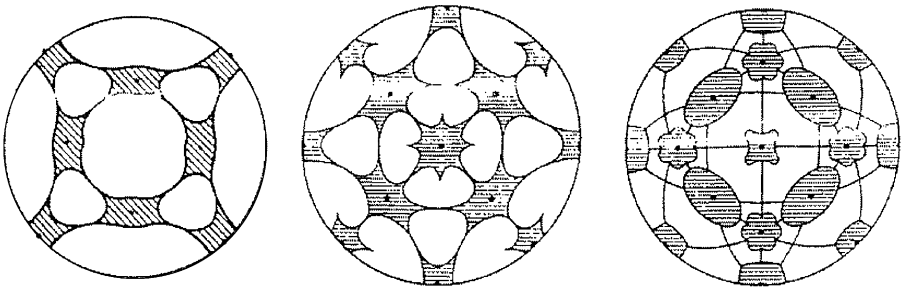


Fig. 9. A schematic form of the stereographic projections of the magnetic field directions resulting in the appearance of open cross-sections of the surface shown in Fig. 8, when the thickness of the cross connections increases (from (a) to (c)).

and semimetals of the bismuth type (Bi, As, Sb). An analysis of rotation diagrams of the magnetic field for different orientations of the current density obtained in experiments, allowed determining the region of the directions of the magnetic field for which open cross-sections of the FS were possible. The stereographic projection of such directions, constructed according to Gaidukov's data on the anisotropy of the magnetoresistivity of gold [95] (Fig. 11), is structurally close to one of the stereographic projections in Fig. 9. Thus, apparently, the FS of noble metals (gold, silver) and copper has the form of a space net of corrugated cylinders directed along the space diagonals of a cubic cell. For these metals that have the same crystallographic structure, the experimental study

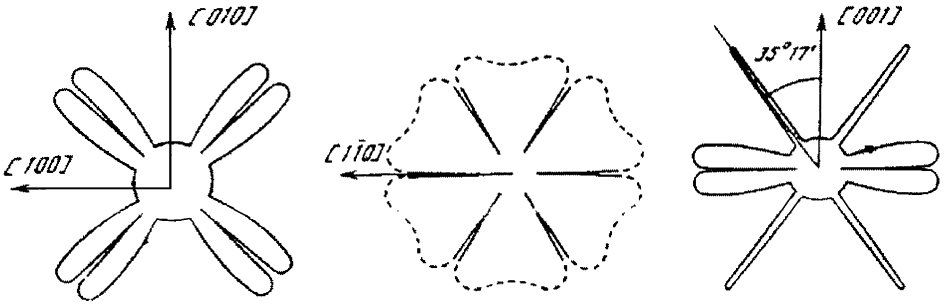



Fig. 10. Some examples of the angular dependences of the resistivity (the rotation diagrams $\mathbf{H} \perp \mathbf{j}$ for a fixed direction of the current density) for metals with a FS consisting of corrugated cylinders directed along the space diagonals of the cubic cell. The electric current is directed (a) along the axis [001]; (b) along the axis [111]; (c) along the axis [110].

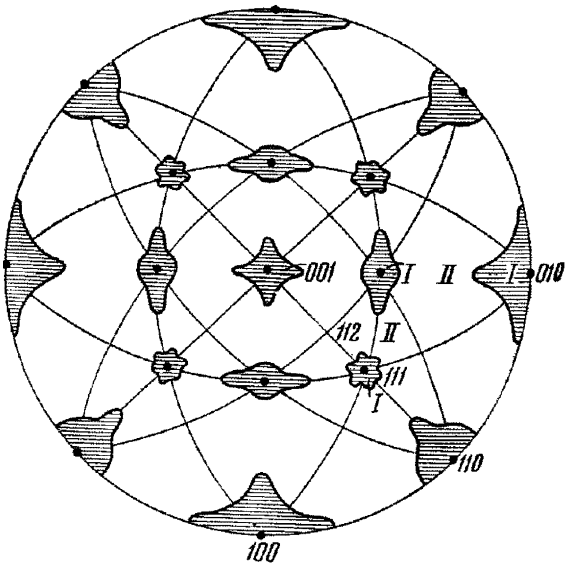
of GMP in strong magnetic fields, performed by Alexeevskii and Gaidukov, allowed determining the topology of the FS completely. The difference between their FS is in the thickness of the corrugated cylinders only: the maximal thickness corresponds to copper and the minimal one to silver.

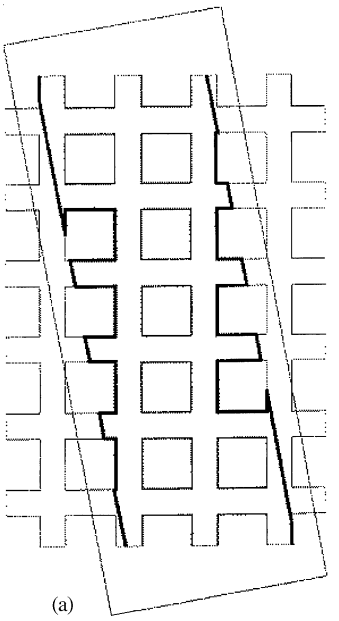
If in the case of noble metals the resistivity reaches saturation rather often, on the rotation diagrams of the magnetic field ($\mathbf{H} \perp \mathbf{j}$), for some metals of cubic symmetry, for example for lead, saturation in strong magnetic fields is observed very rarely. This is a group of compensated metals where in the absence of open and self-intersecting cross-sections of the FS $V_1 = V_2$. The resistivity of such metals is described by Eqs. (5.21) and (5.25).

It must be noted that compensation of electron and hole volumes $V_1$ and $V_2$ contained inside the FS within one primitive cell of $\mathbf{p}$-space with energies $\varepsilon < \varepsilon_F$ and $\varepsilon > \varepsilon_F$, just as in the case of an open FS, is not a low-probability event, but a natural phenomenon met mainly in metals with an even valency.

### 5.3. Plane net of corrugated cylinders

Open FS in the form of a plane net of corrugated cylinders are possible in degenerate semiconductors with a tetragonal crystallographic structure. Of course, open cross-sections of such FS are situated in the same $p_1 p_2$ plane. The sizes of the stable zone—a two-dimensional region of the magnetic field directions near the $x_3$-axis for which there are open electron trajectories—strongly depend on the minimal thickness of the cylinders (see Fig. 12). As in the case of the space net of the cylinders, the greater the thickness of the cylinders, the greater the sizes of the stable zone. If the $x_3$-axis coincides with a symmetry axis of the crystal of an order higher than the second, open trajectories appear only if the magnetic field deviates from the $x_3$-axis. The thickness $\Delta$ of the layer of open trajectories increases as the angle of the magnetic field deviation from the $x_3$-axis increases. However, when approaching the boundary of the stable zone, $\Delta$ tends to zero again, except in directions of the magnetic field that are in a crystallographic plane with small Miller indices. Near these directions outside the stable zone of the normals (bold lines on the stereographic projections, see Fig. 12b) the electron trajectories are closed. They are strongly elongated, as in the case of a FS in the form of an isolated corrugated cylinder. The contribution of the charge carriers from the



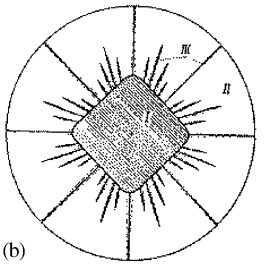




Fig. 11. The stereographic projection of the magnetic field directions for which there are open cross-sections of the FS of gold constructed according to the data on the anisotropy of the magnetoresistivity (Gaidukov [95]).

Fig. 12. FS in the form of a plane net of rectangular cylinders (a) and the stereographic projection of the normals to the open plane cross-sections (b).

strongly elongated orbits to the elements of the conductivity tensor, depending on the angle of the deviation of the magnetic field from one of these directions $\theta$, has a sharp maximum with a width of the order of $\gamma_0$.

A collection of the magnetic field directions resulting in the appearance of open cross-sections of a FS in the form of a plane net of corrugated cylinders, differs from the one shown in Fig. 12b by the shape of the stable zone near the $x_3$-axis. When the boundary of this zone is known, the minimal thickness of the cylinders and some other details of the shape of FS can be determined. As the minimal thickness of the cylinders increases, the size of the stable zone of the normals increases as well, and the one-dimensional set of the magnetic field orientations resulting in the appearance of open cross-sections of the FS becomes more and more branching. When the thickness is large enough, the plane set of such cylinders looks like corrugated planes joined by cross connections. This is the form of the open FS of tin (Fig. 13, according to Alexeevskii, Gaidukov, Lifshitz, Peschansky [98]).

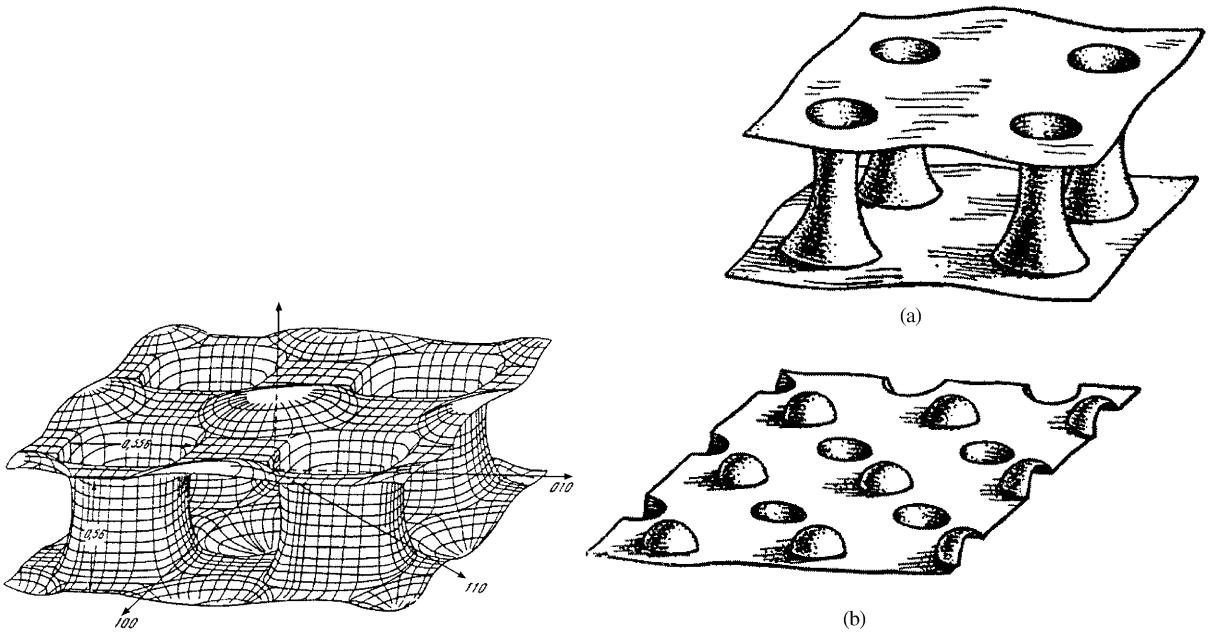



Fig. 13. The topological structure of the FS of tin [98].

Fig. 14. FS in the form of corrugated planes joined by cross connections (a) and without the cross connections (b).

### 5.4. Corrugated plane

The corrugated sheets of an isoenergetic surface are mutually conjugated due to the requirement of the central symmetry of the charge carriers' dispersion law. If these planes are isolated from each other (Fig. 14a), open electron trajectories in **p**-space are possible for any orientation of the magnetic field, except for the single one that is orthogonal to the osculating plane and coincides with a symmetry axis of a order higher than the second. If the corrugation is not small, or it is not sufficiently small, when describing GMP the semiclassical approach can be used even in the region of very low temperatures.

The existence of cross connections joining the adjacent corrugated planes in pairs, contracts the region of the magnetic field directions for which open plane cross-sections of the isoenergetic surface are possible (Fig. 15). On such cross connections the nature of the charge carriers is different in principle for the orientations of the magnetic field along the $x_3$-axis and in the plane orthogonal to this axis. Therefore, the Hall field increasing mainly according to a linear law as the magnetic field is increased, nevertheless does not contain information about the full number of the charge carriers. If the corrugated planes have a preferred direction of corrugation (Fig. 16), the increase of the resistivity in a strong magnetic field can be changed to saturation in very high magnetic fields.

## 6. Conclusions

Of course, during the years after the TGMP was developed, theoretical studies improving and refining the theory continued. To reconstruct the topological structure of the electron energy spectrum

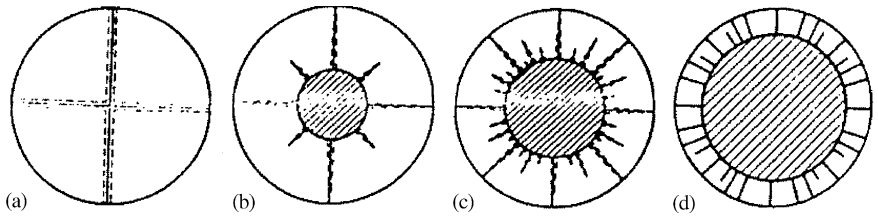



Fig. 15. A schematic form of the stereographic projections of the magnetic field directions for which there are open cross-sections $p_H = \text{const}$ of FS in the form of corrugated planes joined by cross connections (Fig. 14a) for different thicknesses of the cross connections.

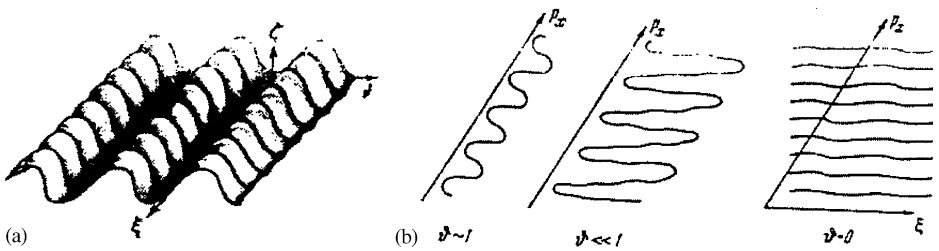



Fig. 16. An isoenergetic surface of the type of "the corrugated plane" with a preferred direction of the corrugation (a) and its intersections with the plane $p_H = \text{const}$ (b).

of degenerate conductors with the aid of measurements of galvanomagnetic characteristics, sufficiently strong magnetic fields are necessary, providing the characteristic radius of electron trajectories to be much smaller than the electron mean free path. With that end in view, the efficiency of the magnetic field can be raised by an increase of the charge carriers' mean free path. In connection with this it was necessary to have extremely pure perfect samples in which at liquid helium temperatures the mean free path of the conduction electrons was of the order of millimeters, and not rarely it is comparable with the thickness of the samples. In this case mechanisms of dissipation of electron fluxes related to the scattering of conduction electrons by the boundaries of the sample can no longer be ignored. This was the reason to start examination of size galvanomagnetic effects. At the present time the study of size effects is stimulated both by technical needs (miniaturization of devices of different kinds), and by the increasing interest in the surface of the sample as an independent physical object. The state of the surface can be determined accurately with the aid of GMP experiments, when in a strong magnetic field the permanent current is concentrated near the surface of the metal (static skin effect) [99–102], or by measuring the amplitudes of the Sondheimer oscillation of the magnetoresistivity in thin plates [103]. When measuring the magnetoresistivity of thin whiskers of zinc, cadmium, antimony and bismuth, Gaidukov and his collaborators [104] examined in detail the nature of the interaction of the charge carriers with the surface of the metallic sample. In particular, it was ascertained experimentally that the probability of specular reflection by the surface of the sample depended on the angle at which an electron impinged on the surface.

The spectroscopic approach to the properties of normal metals "put aside" the detailed investigation of dissipative processes. For practically all metals and intermetallic compounds we have detailed information about their FS, however the knowledge of dissipative processes related to the bulk

scattering of the charge carriers remains at "the prior fermilogical" level. On the other hand, when developing fermiology, it was a hope that after the energy spectrum of normal metals would be found, i.e. after the reconstruction of the FS and the determination of electron velocities, a regular investigation of dissipative processes in metals would start. In our "dreams" the FS of a metal, playing the role of "a landscape map", was visible. The dissipative characteristics of electrons were plotted on this "map" side by side with the electron velocities. Recognizing that the dissipative characteristics depended not on the temperature only, but on the state of the crystal also, the need for some standardization of real objects was understood. It was necessary to know to which object the experimentally determined dissipative characteristics, such as the relaxation times, the transport mean free paths, and so on, were relevant. The dream remains a dream: apparently, there are no such works, or their number is very few.

However, the TGMP developed in different, but much more perspective, directions: analysis of GMP in multilayered structures, superlattices and other rather complex objects.

During the last decades great attention was paid to the examination of the magnetoresistivity and the Hall field in low-dimensional conductors. To a considerable extent the interest to conductors with strong anisotropy of the conductivity was stimulated by Little's idea [105] concerning the possibility of high-temperature superconductivity in quasi-one-dimensional conductors consisting of molecular polymeric chains. Due to the efforts of physicists and chemists enormous number of low-dimensional conductors were synthesized. It was found that among organic conductors more perspective for this purpose were ion-radical complexes of charge transport, which had a layered structure (in this case the temperature of the transition to the superconducting state was an order of magnitude greater than in quasi-one-dimensional conductors). A considerable part of the layered conductors of organic origin has a conductivity of the metallic type not along the layers only, but across the layers as well. In magnetic fields of the order of 10 T a strongly pronounced Shubnikow–de Haas effect is observed, since the quasi-two-dimensional nature of the electron energy spectrum is conducive to drawing a large number of the charge carriers with the Fermi energy into the formation of quantum oscillation effects. To describe electron processes in such conductors the concept of quasi-particles (conduction electrons carrying the charge) well justified in the theory of ordinary metals, can be used. For these systems the theory of GMP was based on the same principles as the TGMP in ordinary metals [2–4]. Here it is fully in place to set up the inverse problem, that is the problem of the reconstruction of the electron energy spectrum on the basis of data of galvanomagnetic and other experiments in strong magnetic fields that are highly sensitive to the spectrum structure.

In low-dimensional conductors GMP have their own features in the region of not so strong magnetic fields ($\mu H \ll \eta\varepsilon_F$) also, when the semiclassical approach to the description of electron processes is still valid. In particular, the GMP characteristics are rather sensitive to the orientation of the magnetic field with respect to the layers [88,89]. In a sufficiently strong magnetic field, when $\mu H \geqslant \eta\varepsilon_F$, for two-dimensional conductors (the parameter of quasi-two-dimensionality $\eta = 0$) only the quantum examination of GMP is correct.

We earlier limited ourselves to the description of GMP in the framework of the semiclassical approach only, but it is impossible not to mention the discovery of the quantum Hall effect in two-dimensional conductors by Von Klitzing and his collaborators. The quantization of the Hall resistivity,

$$\rho_{xy} = \frac{2\pi\hbar}{ve^2}; \quad v = p/q; \quad p \text{ and } q \text{ are integers},$$

is observed at low temperatures in a sufficiently strong magnetic field in two-dimensional inversion layers of *n*- and *p*-types, in silicon MDS-structures, in heterostructures based on GaAs, InP, InAs, GaSb, and so on. The ordinary quantum Hall effect ($v$ is an integer) was discovered first in silicon MDS-structures [106], and the fractional Hall effect was discovered in $Al_xGa_{1-x}AS$–GaAs heterostructures [107]. Many reviews are devoted to this unique phenomenon, and even a special monograph [108] exists.

There is important progress in the development of computational methods for investigations of the electronic energy spectrum. At the same time GMP and other phenomena highly sensitive to the structure of the spectrum in strong magnetic fields, do not exhaust their significance. In the future they will still be used repeatedly, in particular for testing computational models.

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