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Spectra of infinite graphs with tails

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We compute explicitly (modulo solutions of certain algebraic equations) the spectra of infinite graphs obtained by attaching one or several infinite paths to some vertices of given finite graphs. The main result concerns a canonical form for the adjacency matrix of such infinite graphs, and the algorithm of its calculation. The argument relies upon the spectral theory of eventually free Jacobi matrices. We also study some other couplings of infinite graphs (stars and Bethe–Caley trees).

Keywords: infinite graphs; adjacency operator; spectrum; Jacobi matrices of finite rank; Jost function

AMS Subject Classifications: Primary: 05C63; Secondary: 05C76; 47B36; 47B15; 47A10

1. Introduction and main results

We begin with rudiments of the graph theory. For the sake of simplicity, we restrict ourselves with simple, connected, undirected, finite or infinite (countable) weighted graphs, although the main result holds for weighted multigraphs and graphs with loops as well. We will label the vertex set $\mathcal{V}(\Gamma)$ by positive integers $\mathbb{N} = \{1, 2, \dots\}$, $\{v\}_{v\in\mathcal{V}} = \{j\}_{j=1}^{\omega}$, $\omega \le \infty$. The symbol $i \sim j$ means that the vertices i and j are incident, i.e. $\{i, j\}$ belongs to the edge set $\mathcal{E}(\Gamma)$. A graph Γ is weighted if a positive number d_{ij} (a weight) is assigned to each edge $\{i, j\} \in \mathcal{E}(\Gamma)$. In case $d_{ij} = 1$ for all i, j, the graph is unweighted.

The degree (valency) of a vertex $v \in \mathcal{V}(\Gamma)$ is a number $\gamma(v)$ of edges emanating from v. A graph Γ is said to be locally finite, if $\gamma(v) < \infty$ for all $v \in \mathcal{V}(\Gamma)$, and uniformly locally finite, if $\sup_{\mathcal{V}} \gamma(v) < \infty$.

The spectral graph theory deals with the study of spectra and spectral properties of certain matrices related to graphs (more precisely, operators generated by such matrices in the standard basis $\{e_k\}_{k\in\mathbb{N}}$ and acting on the corresponding Hilbert spaces $\mathbb{C}^n$ or $\ell^2 = \ell^2(\mathbb{N})$). One of the most notable among them is the *adjacency matrix* $A(\Gamma)$

$$A(\Gamma) = [a_{ij}]_{i,j=1}^{\omega}, \quad a_{ij} = \begin{cases} d_{ij}, & \{i, j\} \in \mathcal{E}(\Gamma); \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

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The corresponding adjacency operator will be denoted by the same symbol. It acts as

$$A(\Gamma)\, e_k = \sum_{j\sim k} a_{jk}\, e_j, \quad k \in \mathbb{N}. \tag{1.2}$$

Clearly, $A(\Gamma)$ is a symmetric, densely defined linear operator, whose domain contains the set of all finite linear combinations of the basis vectors. The operator $A(\Gamma)$ is bounded and self-adjoint in ℓ^2, as long as the graph Γ is uniformly locally finite.

Whereas the spectral theory of finite graphs is very well established (see, e.g. [1–4]), the corresponding theory for infinite graphs is in its infancy. We refer to [5–7] for the basics of this theory. In contrast to the general consideration in [6], our goal is to carry out a complete spectral analysis (canonical models for the adjacency operators and computation of the spectrum) for a class of infinite graphs which loosely speaking can be called 'finite graphs with tails attached to them'. To make the notion precise, we define first an operation of coupling well known for finite graphs (see, e.g. [4, Theorem 2.12]).

Definition 1.1 Let Γ_k, $k = 1, 2$, be two weighted graphs with no common vertices, with the vertex sets and edge sets $\mathcal{V}(\Gamma_k)$ and $\mathcal{E}(\Gamma_k)$, respectively, and let $v_k \in \mathcal{V}(\Gamma_k)$. A weighted graph $\Gamma = \Gamma_1 + \Gamma_2$ will be called a *coupling by means of the bridge* $\{v_1, v_2\}$ *of weight* $d > 0$ if

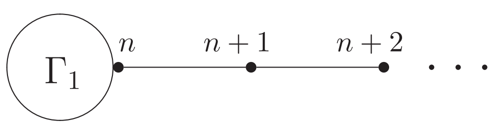
$$\mathcal{V}(\Gamma) = \mathcal{V}(\Gamma_1) \cup \mathcal{V}(\Gamma_2), \qquad \mathcal{E}(\Gamma) = \mathcal{E}(\Gamma_1) \cup \mathcal{E}(\Gamma_2) \cup \{v_1, v_2\}. \tag{1.3}$$

So we join Γ_2 to Γ_1 by the new edge of weight d between v_2 and v_1.

If the graph Γ_1 is finite, $V(\Gamma_1) = \{1, 2, \ldots, n\}$, and $V(\Gamma_2) = \{j\}_{j=1}^{\omega}$, we can with no loss of generality put $v_1 = n$, $v_2 = 1$, so the adjacency matrix $A(\Gamma)$ is written as a block matrix

$$A(\Gamma) = \begin{bmatrix} A(\Gamma_1) & E_d \\ E_d^* & A(\Gamma_2) \end{bmatrix}, \qquad E_d = \begin{bmatrix} 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \ldots \\ d & 0 & 0 & \ldots \end{bmatrix}. \tag{1.4}$$

If $\Gamma_2 = \mathbb{P}_\infty(\{a_j\})$, the one-sided weighted infinite path, we can view the coupling $\Gamma = \Gamma_1 + \mathbb{P}_\infty(\{a_j\})$ as a finite graph with the tail. This is exactly the class of graphs we will be dealing with in the paper. Each such graph has a finite number of essential ramification nodes (a vertex v is an essential ramification node if $\gamma(v) \geq 3$, see [8]).



A special class of infinite matrices will play a crucial role in what follows.

Under *Jacobi or tridiagonal matrices*, we mean here one-sided infinite matrices of the form

$$J = J(\{b_j\},\{a_j\}) = \begin{bmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & \ddots & \\ & & \ddots & \ddots & \end{bmatrix}, \quad b_j \in \mathbb{R}, \quad a_j > 0. \tag{1.5}$$

They generate linear operators called the Jacobi operators on the Hilbert space $\ell^2(\mathbb{N})$. The matrix

$$J_0 := \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{1.6}$$

called a *discrete Laplacian* or a *free Jacobi matrix* is of particular interest in the sequel.

Given two Jacobi matrices $J_k = J(\{\beta_j^{(k)}\},\{\alpha_j^{(k)}\})$, $k = 1, 2$, the matrix J_2 is called a *truncation* of J_1 (and J_1 is an *extension* of J_2) if

$$\beta_j^{(2)} = \beta_{j+q}^{(1)}, \quad \alpha_j^{(2)} = \alpha_{j+q}^{(1)}, \quad j \in \mathbb{N},$$

for some $q \in \mathbb{N}$. In other words, J_2 is obtained from J_1 by deleting the first q rows and columns. If $J_2 = J_0$, J_1 is said to be a *Jacobi matrix of finite rank* or an *eventually free Jacobi matrix*.

The Jacobi matrices arise in the spectral graph theory, thanks to the relation for the adjacency matrix $A(\mathbb{P}_\infty(\{a_j\}))$ of the weighted path

$$A(\mathbb{P}_\infty(\{a_j\})) := J(\{0\},\{a_j\}). \tag{1.7}$$

In case of the unweighted path, that is, $a_j \equiv 1$, we have

$$A(\mathbb{P}_\infty) = J_0. \tag{1.8}$$

Note that the adjacency matrix for the finite (unweighted) path $\mathbb{P}_m$ with m vertices is the finite Jacobi matrix $J(\{0\},\{1\})$ of order m. The spectrum of this matrix is well known [2, p.9]

$$\sigma(\mathbb{P}_m) = \left\{2\cos\frac{\pi j}{m+1}\right\}_{j=1}^{m}. \tag{1.9}$$

It follows from (1.4) that for an arbitrary finite weighted graph G

$$A(G + \mathbb{P}_\infty(\{a_j\})) = \begin{bmatrix} A(G) & E_d \\ E_d^* & J(\{0\},\{a_j\}) \end{bmatrix}. \tag{1.10}$$

We suggest here a 'canonical' form for such block matrices and the algorithm of their reducing to this form.

THEOREM 1.2 *Let A be a block matrix in ℓ^2,*

$$A = \begin{bmatrix} \mathcal{A} & E_d \\ E_d^* & J \end{bmatrix}, \quad E_d = \begin{bmatrix} 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots \\ d & 0 & 0 & \dots \end{bmatrix}, \quad d > 0, \tag{1.11}$$

where $\mathcal{A} = [a_{ij}]_{i,j=1}^n$ *is a real symmetric matrix of order* n, $J = J(\{\beta_j\}, \{\alpha_j\})$ *is a Jacobi matrix. Then,* A *can be reduced to the block diagonal form*

$$A \simeq \begin{bmatrix} \widehat{\mathcal{A}} & \\ & \widehat{J} \end{bmatrix} \tag{1.12}$$

where $\widehat{\mathcal{A}}$ *is a real symmetric matrix of order* $n-k$, *for some* $1 \le k \le n$, *and the Jacobi matrix* $\widehat{J}$ *is an extension of* J. *In other words, there is a unitary operator* U *on* ℓ^2 *such that*

$$U^{-1}AU = \widehat{\mathcal{A}} \oplus \widehat{J}. \tag{1.13}$$

Corollary 1.3 *Given a finite weighted graph* G, *the adjacency operator of the coupling* $\Gamma = G + \mathbb{P}_\infty(\{a_j\})$ *is unitarily equivalent to the orthogonal sum*

$$U^{-1}A(\Gamma)U = F(\Gamma) \oplus J(\Gamma) \tag{1.14}$$

of a finite-dimensional operator $F(\Gamma)$ *and a Jacobi operator* $J(\Gamma)$ *which is an extension of* $J(\{0\}, \{a_j\})$.

If $\mathbb{P}_\infty$ is unweighted, then, by definition, $J(\Gamma)$ is of finite rank.

We call $F(\Gamma)$ a *finite-dimensional component* of the coupling Γ, and $J(\Gamma)$ its *Jacobi component*. In particular, the spectrum of $\Gamma = G + \mathbb{P}_\infty$ is

$$\sigma(\Gamma) = [-2, 2] \cup \sigma_d(\Gamma), \quad \sigma_d(\Gamma) = \{\lambda_j\}_{j=1}^{\omega}, \ \omega < \infty, \tag{1.15}$$

the discrete spectrum of Γ, is a union of the eigenvalues of $F(\Gamma)$ and $J(\Gamma)$, see Section 2.

We explain the algorithm of finding the finite-dimensional and Jacobi components in Section 3, and illustrate it on a number of examples in the further sections. The spectral analysis of eventually free Jacobi matrices is discussed in Section 2.

To get some flavour of the results on the structure of spectra for certain graphs with tails, consider the following example. Let $\mathbb{C}_m$ be a cycle of order m, $\Gamma := \mathbb{C}_m + \mathbb{P}_\infty$. This graph can be viewed as a 'kite' with infinite tail.

Proposition 1.4 *The spectrum of the graph* $\Gamma = \mathbb{C}_m + \mathbb{P}_\infty$ *is*

$$\sigma(\mathbb{C}_m + \mathbb{P}_\infty) = [-2, 2] \cup \sigma_d(\Gamma)$$

with the discrete component

$$\sigma_d(\Gamma) = \left\{2\cos\frac{2\pi k}{m}\right\}_{k=1}^{l} \bigcup \{\lambda_1(m), \lambda_2(m)\}, \ l = \left[\frac{m-1}{2}\right], \tag{1.16}$$

where

$$\lambda_j(m) = x_j(m) + \frac{1}{x_j(m)}, \quad j = 1, 2, \quad -1 < x_2(m) < 0 < x_1(m) < 1$$

are real roots of the polynomial $p(x) = x^m + 2x^2 - 1$.

We observe here a number of eigenvalues lying on the essential (absolutely continuous) spectrum $[-2, 2]$ (the hidden spectrum in the terminology of [9]), and two eigenvalues off $[-2, 2]$.

Remark 1.5 Given a finite graph G, one can attach $p \geq 1$ copies of the infinite path $\mathbb{P}_\infty$ to *some* vertex $v \in \mathcal{V}(G)$. Although the graph Γ thus obtained is not exactly the coupling in the sense of Definition 1.1, its adjacency operator acts similarly to one for the coupling. Indeed, it is not hard to see that

$$A(\Gamma) = \begin{bmatrix} A(G) & E_{\sqrt{p}} \\ E^*_{\sqrt{p}} & J_0 \end{bmatrix} \oplus \left(\bigoplus_{i=1}^{p-1} J_0 \right). \tag{1.17}$$

Hence, Theorem 1.2 applies, and the spectral analysis of such graph can be accomplished.

Surprisingly enough, the case when $p \geq 1$ infinite rays are attached to *each* vertex of a finite graph G is easy to work out, and the spectrum of such graph can be found explicitly in terms of the spectrum of G. Denote such graph by $\Gamma = G + \mathbb{P}_\infty(p)$.

Theorem 1.6 *Given a finite graph G with vertices* $1, 2, \ldots, n$ *and the spectrum* $\sigma(G) = \{\lambda_j\}_{j=1}^n$, *let* $\Gamma = G + \mathbb{P}_\infty(p)$, $p \in \mathbb{N}$. *Then, the adjacency operator* $A(\Gamma)$ *is unitarily equivalent to the orthogonal sum*

$$A(\Gamma) \simeq \bigoplus_{j=1}^{n} J(\lambda_j, \sqrt{p}) \oplus \left(\bigoplus_{i=1}^{(p-1)n} J_0 \right),$$
$$J(\lambda_j, \sqrt{p}) := J(\{\lambda_j, 0, 0, \ldots\}, \{\sqrt{p}, 1, 1, \ldots\}). \tag{1.18}$$

The spectrum of Γ *is*

$$\sigma(\Gamma) = [-2, 2] \bigcup \bigcup_{j=1}^{n} \sigma_d(J(\lambda_j, \sqrt{p})). \tag{1.19}$$

For $p = 1$, this result is proved in [10].

The spectral theory of infinite graphs with one or several rays attached to certain finite graphs was initiated in [10–13], wherein several particular examples of unweighted (background) graphs are examined. The spectral analysis of similar graphs appeared earlier in the study of thermodynamical states on complex networks.[9] We argue in the spirit of [14–16] and supplement to the list of examples. The general canonical form for the adjacency matrices of such graphs and the algorithm of their reducing to this form suggested in the paper apply to a wide class of couplings (not only graphs with tails, see Section 6), and also to Laplacians on graphs of such type.

2. Jacobi matrices of finite rank and their spectra

For the class of eventually free Jacobi matrices, the complete spectral analysis is available at the moment, see [17,18]. A basic object known as the *perturbation determinant* [19] is a key ingredient of perturbation theory.

Given bounded linear operators T_0 and T on the Hilbert space such that $T - T_0$ is a nuclear operator, the perturbation determinant is defined as

$$L(\lambda; T, T_0) := \det(I + (T - T_0)R(\lambda, T_0)), \quad R(\lambda, T_0) := (T_0 - \lambda)^{-1} \tag{2.1}$$

is the resolvent of operator T_0, an analytic operator function on the resolvent set $\rho(T_0)$.

The perturbation determinant is designed for the spectral analysis of the perturbed operator T, once the spectral analysis for T_0 is available. In particular, the essential spectra of T and T_0 agree, and the discrete spectrum of T is exactly the zero set of the analytic function L on $\rho(T_0)$, at least if the latter is a domain, i.e. a connected, open set in the complex plane.

In the simplest case, $\operatorname{rank}(T - T_0) < \infty$ the perturbation determinant is the standard finite-dimensional determinant. Indeed, now

$$(T - T_0)h = \sum_{k=1}^{p} \langle h, \varphi_k\rangle\, \psi_k, \quad (T - T_0)R(\lambda, T_0)h = \sum_{k=1}^{p} \langle h, R^*(\lambda, T_0)\varphi_k\rangle\, \psi_k,$$

so L can be computed by the formula (see, e.g. [19, Section IV.1.3])

$$L(\lambda; T, T_0) = \det[\delta_{ij} + \langle R(\lambda, T_0)\psi_i, \varphi_j\rangle]_{i,j=1}^{p}. \tag{2.2}$$

Our particular concern is $T_0 = J_0$, the free Jacobi matrix. Its resolvent matrix in the standard basis in ℓ^2 is given by (see, e.g. [20])

$$R\Big(z + \frac{1}{z},\, J_0\Big) = [r_{ij}(z)]_{i,j=1}^{\infty}, \quad r_{ij}(z) = \frac{z^{|i-j|} - z^{i+j}}{z - z^{-1}}, \quad z \in \mathbb{D}. \tag{2.3}$$

If $T = J$ is a Jacobi matrix of finite rank p, we end up with the computation of the ordinary determinant (2.2) of order p.

It is instructive for the further usage to compute two simplest perturbation determinants for rank $(J - J_0) = 1$ and 2.

Example 2.1 Let

$$J = J(\{b_j\}, \{1\}): \quad b_j = 0, \quad j \neq q,$$

so $J - J_0 = \langle \cdot, e_q\rangle\, b_q e_q$. By (2.3) and (2.2),

$$\widehat{L}(z) := L\Big(z + \frac{1}{z};\, J, J_0\Big) = 1 + b_q r_{qq}(z) = 1 - b_q z\, \frac{z^{2q} - 1}{z^2 - 1}. \tag{2.4}$$

Similarly, let

$$J = J(\{0\}, \{a_j\}): \quad a_j = 1, \quad j \neq q,$$

so $J - J_0 = \langle \cdot, e_q\rangle\, (a_q - 1)\, e_{q+1} + \langle \cdot, e_{q+1}\rangle\, (a_q - 1)\, e_q$, and again

$$\begin{aligned}\widehat{L}(z) &= \begin{vmatrix} 1 + (a_q - 1)\, r_{q,q+1}(z) & (a_q - 1)\, r_{qq}(z) \\ (a_q - 1)\, r_{q+1,q+1}(z) & 1 + (a_q - 1)\, r_{q+1,q}(z) \end{vmatrix} \\ &= 1 + (1 - a_q^2) z^2\, \frac{z^{2q} - 1}{z^2 - 1}\,. \end{aligned} \tag{2.5}$$

In the Jacobi matrices setting, there is yet another way of computing perturbation determinants based on the so-called Jost solution and Jost function (see, e.g. [7, Section 3.7]).

Consider the basic recurrence relation for the Jacobi matrix J

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \Big(z + \frac{1}{z}\Big)\, y_n, \quad z \in \mathbb{D}, \quad n \in \mathbb{N}, \tag{2.6}$$

where we put $a_0 = 1$. Its solution $y_n = u_n(z)$ is called the *Jost solution* if

$$\lim_{n\to\infty} z^{-n} u_n(z) = 1, \quad z \in \mathbb{D}. \tag{2.7}$$

In this case, the function $u = u_0$ is called the *Jost function*.

The Jost solution certainly exists for finite rank Jacobi matrices. The Jost function is now an algebraic polynomial, called the *Jost polynomial*. Indeed, let

$$a_{q+1} = a_{q+2} = \ldots = 1, \qquad b_{q+1} = b_{q+2} = \ldots = 0.$$

One can put $u_k(z) = z^k$, $k = q+1, q+2, \ldots$ and then determine $u_q, u_{q-1}, \ldots, u_0$ consecutively from (2.6). So,

$$\begin{aligned} a_q\, u_q(z) &= z^q, \\ a_{q-1}a_q\, u_{q-1}(z) &= \alpha_q\, z^{q+1} - b_q z^q + z^{q-1}, \quad \alpha_q := 1 - a_q^2, \end{aligned} \tag{2.8}$$

etc., and in general,

$$u_{q-k}(z) = \sum_{j=-k}^{k} \beta_{q,j} z^{q+j}, \quad k = 0, 1, \ldots, q, \quad \beta_{q,j} \in \mathbb{R}, \;\; \beta_{q,-k} = 1.$$

In particular, for $q = 1$

$$a_1\, u(z) = \alpha_1\, z^2 - b_1 z + 1, \tag{2.9}$$

and for $q = 2$

$$a_1 a_2\, u(z) = \alpha_2\, z^4 - (b_2 + b_1\alpha_2)\, z^3 + (\alpha_1 + \alpha_2 + b_1 b_2)\, z^2 - (b_1 + b_2)\, z + 1. \tag{2.10}$$

The relation between the perturbation determinant and the Jost function is given by

$$u(z) = \prod_{j=1}^{\infty} a_j^{-1} \cdot \widehat{L}(z), \tag{2.11}$$

see, e.g. [20], and such recursive way of computing perturbation determinants is sometimes far easier than computing ordinary determinants (2.2), especially for large enough ranks of perturbation.

Example 2.2 Let $J = J(\{b_j\}, \{a_j\})$ be a Jacobi matrix such that

$$b_j = 0, \quad j \neq 1, \qquad a_j = 1, \quad j \neq q.$$

We have $u_{q+j}(z) = z^{q+j}$, $j = 1, 2 \ldots$,

$$\begin{aligned} a_q u_q(z) &= z^q, \quad a_q u_{q-1}(z) = \alpha_q\, z^{q+1} + z^{q-1}, \\ a_q u_{q-2} &= \alpha_q\, (z^{q+2} + z^q) + z^{q-2}, \end{aligned}$$

and, by induction,

$$a_q u_{q-k}(z) = \alpha_q\, z^{q-k+2}\, \frac{z^{2k} - 1}{z^2 - 1}, \quad k = 1, 2, \ldots, q-1. \tag{2.12}$$

Next, for $q = 1$, we have exactly (2.9), so let $q \geq 2$. The recurrence relation (2.6) with $n = 1$ gives

$$a_q u(z) + b_1 a_q u_1(z) + a_q u_2(z) = \left(z + \frac{1}{z}\right) a_q u_1(z),$$

and so we come to the following expression for the Jost polynomial

$$a_q u(z)(z^2 - 1) = \alpha_q\,(z - b_1)\,z^{2q+1} - b_1 a_q^2 z^3 + a_q^2 z^2 + b_1 z - 1. \tag{2.13}$$

Similarly, for the Jacobi matrix $J = J(\{b_j\}, \{a_j\})$ with

$$b_j = 0, \quad j \neq q; \qquad a_j = 1, \qquad j \neq 1$$

one has

$$a_1 u(z) = -b_q \frac{z^{2q+1} + \alpha_1 z^{2q-1} - \alpha_1 z^3 - z}{z^2 - 1} + \alpha_1 z^2 + 1. \tag{2.14}$$

For the Jacobi matrix $J = J(\{0\}, \{a_j\})$ with $a_j = 1$, $j \neq 1, q$, the Jost polynomial is given by

$$a_1 a_q u(z) = \alpha_q \frac{z^{2q+2} + \alpha_1 z^{2q} - \alpha_1 z^4 - z^2}{z^2 - 1} + \alpha_1 z^2 + 1. \tag{2.15}$$

The spectral theorem for finite rank Jacobi matrices due to Damanik and Simon [17] provides a complete description of the spectral measure of such matrices.

THEOREM DS *Let $J = J(\{b_j\}, \{a_j\})$ be a Jacobi matrix of finite rank*

$$a_{q+1} = a_{q+2} = \ldots = 1, \qquad b_{q+1} = b_{q+2} = \ldots = 0,$$

and $u = u_0(J)$ be its Jost polynomial. Then,

- *u is a real polynomial of degree $\deg u \leq 2q$, $\deg u = 2q$ if and only if $a_q \neq 1$.*
- *All roots of u in the unit disk $\mathbb{D}$ are real and simple, $u(0) \neq 0$. A number λ_j is an eigenvalue of J if and only if*

$$\lambda_j = z_j + \frac{1}{z_j}, \qquad z_j \in (-1, 1), \quad u(z_j) = 0. \tag{2.16}$$

- *The spectral measure $\sigma(J)$ is of the form*

$$\sigma(J, \mathrm{d}x) = \sigma_{ac}(J, \mathrm{d}x) + \sigma_d(J, \mathrm{d}x) = w(x)\,\mathrm{d}x + \sum_{j=1}^{N} \sigma_j \delta(\lambda_j), \tag{2.17}$$

where

$$w(x) := \frac{\sqrt{4 - x^2}}{2\pi |u(e^{it})|^2}, \quad x = 2\cos t, \quad \sigma_j = \frac{z_j (1 - z_j^{-2})^2}{u'(z_j) u(1/z_j)}.$$

Note that $|u(e^{it})|^2 = Q(x)$, $x = 2\cos t$, Q is a real polynomial of the same degree as the Jost polynomial u.

Sometimes, two-sided Jacobi matrices

$$J = J(\{b_j\}_{j\in\mathbb{Z}}, \{a_j\}_{j\in\mathbb{Z}}) = \begin{bmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & b_0 & a_0 & & & \\ & & a_0 & b_1 & a_1 & & \\ & & & a_1 & b_2 & a_2 & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2.18}$$

$b_j \in \mathbb{R}$, $a_j > 0$, which generate linear operators on the two-sided $\ell^2(\mathbb{Z})$, arise in the canonical models for certain infinite graphs. Such matrices are said to have a finite rank if there are integers $N^{\pm}$, $N^- < N^+$ so that

$$b_n = 0, \quad n \notin [N^-, N^+], \quad a_n = 1, \quad n \notin [N^-, N^+ - 1].$$

In this case, one has a pair of Jost solutions $\{u_n^{\pm}\}$ of the basic recurrence relation (2.6) (with $n \in \mathbb{Z}$) which satisfies

$$u_n^+(z) = z^n, \quad n \geq N^+, \quad u_n^-(z) = z^{-n}, \quad n \leq N^-. \tag{2.19}$$

The Wronskian of two solutions $\{f_n(z)\}_{n\in\mathbb{Z}}$ and $\{g_n(z)\}_{n\in\mathbb{Z}}$ of (2.6) are defined as

$$[f, g] := a_n\big(f_n(z)g_{n+1}(z) - f_{n+1}(z)g_n(z)\big) \tag{2.20}$$

(the right-hand side does not actually depend on n). We denote by

$$w(z) := [u^+(z), u^-(z)], \qquad z \in \mathbb{D} \tag{2.21}$$

the Wronskian of two Jost solutions of (2.6).

It is well known (see, e.g. [21, Theorem 10.4]) that the spectrum of a two-sided Jacobi matrix J of the finite rank is

$$\sigma(J) = [-2, 2] \cup \sigma_d(J), \qquad \sigma_d(J) = \{\lambda_j\}_{j=1}^N,$$

and for the eigenvalues λ_j, the relation

$$\exists\, h \in \ell^2(\mathbb{Z}) :\ Jh = \Big(\zeta + \frac{1}{\zeta}\Big)h \iff w(\zeta) = 0, \quad \zeta \in \mathbb{D}, \tag{2.22}$$

holds. So the eigenvalues of J are exactly the Zhukovsky images of zeros of Wronskian w (2.21) in the unit disk.

Example 2.3 Let

$$J = J(\{0\}_{j\in\mathbb{Z}}, \{a_j\}_{j\in\mathbb{Z}}), \quad a_j = 1, \quad j \neq 0. \tag{2.23}$$

We now have $N^- = 0$, $N^+ = 1$. A simple computation gives

$$w(z) = \frac{1}{a_0 z} - a_0 z,$$

and so

$$\sigma_d(J) = \pm\Big(a_0 + \frac{1}{a_0}\Big), \qquad a_0 > 1,$$

and $\sigma_d(J)$ is empty for $0 < a_0 \leq 1$.

Example 2.4 Let

$$J = J(\{0\}_{j\in\mathbb{Z}}, \{a_j\}_{j\in\mathbb{Z}}), \quad a_j = 1, \quad j \neq \pm 1. \tag{2.24}$$

We now have $N^- = -1$, $N^+ = 2$. A simple computation gives

$$w(z) = \alpha_1\alpha_{-1}z^3 + \left(\frac{\alpha_1}{a_{-1}} + \frac{\alpha_{-1}}{a_1} - \frac{1}{a_1a_{-1}}\right)z + \frac{1}{a_1a_{-1}z},$$
$$\alpha_k := \frac{1}{a_k} - a_k, \quad k = \pm 1,$$

and so to determine the discrete spectrum $\sigma_d(J)$, one has to analyse the roots of the quadratic equation

$$Q(y) = Ay^2 - By + 1 = 0, \quad A := (a_{-1}^2 - 1)(a_1^2 - 1), \quad B := a_1^2 + a_{-1}^2 - 1. \tag{2.25}$$

Precisely, each root y_0 of this equation in (0, 1) generates two symmetric eigenvalues

$$\lambda_\pm := \pm\left(\sqrt{y_0} + \frac{1}{\sqrt{y_0}}\right).$$

The algebraic equations which we encounter later on cannot in general be solved explicitly. By means of the following well-known result,[22, p.41] we can determine how many roots (if any) they have in $(-1, 1)$.

THEOREM (Descarte's rule) *Let $a(x) = a_0x^n + \ldots + a_n$ be a real polynomial. Denote by $\mu(a)$ the number of its positive roots, and $\nu(a)$ the number of the sign changes in the sequence $\{a_0, \ldots, a_n\}$ of its coefficients (the zero coefficients are not taken into account). Then, $\nu(a) - \mu(a)$ is a nonnegative even number.*

3. Canonical form for adjacency matrix

Before starting the proof, we note that the algorithm of constructing a new basis suggested below is even more important than the result itself. To find the spectrum of graphs in the next sections, we will have to carry it out by hand. The result of Theorem 1.2 only guarantees that the procedure can be accomplished with enough paper and patience.

The algorithm applies in a number of situations beyond graphs with one tail (for instance, for the chain of cycles, the ladder with missing rungs, etc.). We will elaborate on this topic in our forthcoming papers.

Proof of Theorem 1.2 One of the versions of the Spectral Theorem for self-adjoint linear operator on a separable Hilbert space claims that each such operator A is unitarily equivalent to an infinite orthogonal sum of Jacobi operators (operators with the simple spectrum), see, e.g. [23, Section VII.2, Lemma 2]. As a matter of fact, the proof is just a consecutive application of the Gram–Schmidt orthogonalization process to the sequence $\{A^n h_k\}_{n\geq 1}$ with an appropriate sequence of vectors $\{h_k\}_{k\geq 1}$. Our algorithm is actually based on this method.

We construct an orthogonal basis $\{\widetilde{e}_k\}_{k\geq 1}$ in ℓ^2 so that the matrix of A in this basis has the canonical form (1.12). We take

$$\widetilde{e}_k = e_k, \quad k = n, n+1, \dots, \qquad \widetilde{e}_k = \sum_{i=1}^{n-1} b_{ik} e_i, \quad k = 1, 2, \dots, n-1, \tag{3.1}$$

where an orthogonal matrix $B = [b_{ij}]_{i,j=1}^{n-1}$ (more precisely, the matrix with orthogonal columns) is to be determined. It is clear that

$$\begin{aligned} A\widetilde{e}_{n+k} &= \alpha_{k-1}\widetilde{e}_{n+k-1} + \beta_k \widetilde{e}_{n+k} + \alpha_k \widetilde{e}_{n+k+1}, \quad k = 2, 3, \dots \\ A\widetilde{e}_{n+1} &= d\widetilde{e}_n + \beta_1 \widetilde{e}_{n+1} + \alpha_1 \widetilde{e}_{n+2}. \end{aligned} \tag{3.2}$$

The following notation will be convenient throughout the proof

$$w_{jk} := \sum_{i=1}^{n-1} b_{ik} a_{ij}, \qquad j = 1, 2, \dots, n, \quad k = 1, 2, \dots, n-1. \tag{3.3}$$

(1) Put $b_{i,n-1} := a_{in}$, $i = 1, 2, \dots, n-1$, so

$$A\widetilde{e}_n = Ae_n = \sum_{i=1}^{n-1} a_{in} e_i + a_{nn}\widetilde{e}_n + d\widetilde{e}_{n+1} = \widetilde{e}_{n-1} + a_{nn}\widetilde{e}_n + d\widetilde{e}_{n+1}. \tag{3.4}$$

Next,

$$\begin{aligned} A\widetilde{e}_{n-1} &= \sum_{i=1}^{n-1} a_{in} Ae_i = \sum_{i=1}^{n-1} a_{in} \sum_{j=1}^{n} a_{ij} e_j = \sum_{j=1}^{n} w_{j,n-1} e_j \\ &= \sum_{j=1}^{n-1} w_{j,n-1} e_j + \sum_{i=1}^{n-1} a_{in}^2 \cdot \widetilde{e}_n. \end{aligned}$$

Note that $\sum_i a_{in}^2 = \sum_i b_{i,n-1}^2 > 0$, for, otherwise, the original matrix already has the canonical form. Define

$$y_1 := \frac{\sum_{j=1}^{n-1} w_{j,n-1} b_{j,n-1}}{\sum_{j=1}^{n-1} b_{j,n-1}^2}, \qquad b_{j,n-2} := w_{j,n-1} - y_1 b_{j,n-1}. \tag{3.5}$$

Then,

$$A\widetilde{e}_{n-1} = \widetilde{e}_{n-2} + y_1\widetilde{e}_{n-1} + z_1\widetilde{e}_n, \quad z_1 := \sum_{i=1}^{n-1} b_{i,n-1}^2,$$

and orthogonality of the columns

$$\sum_{i=1}^{n-1} b_{i,n-2} b_{i,n-1} = 0 \tag{3.6}$$

follows directly from the choice of y_1 (3.5).

If $\widetilde{e}_{n-2} = 0$, that is, $b_{i,n-2} = 0$ for all $i = 1, \dots, n-1$, the algorithm terminates since

$$A\widetilde{e}_{n-1} = y_1\widetilde{e}_{n-1} + z_1\widetilde{e}_n,$$

the subspace $\mathcal{L}_{n-1} := \text{span}\{\widetilde{e}_j\}_{j\ge n-1}$ is invariant for A and so is its orthogonal complement $\mathcal{L}_{n-1}^{\perp} = \ell^2 \ominus \mathcal{L}_{n-1}$, which is finite dimensional. Thus, we can extend the basis $\{\widetilde{e}_j\}_{j\ge n-1}$ in $\mathcal{L}_{n-1}$ in an arbitrary way to the basis in the whole ℓ^2. The latter is exactly the basis we are looking for.

If $\widetilde{e}_{n-2} \ne 0$, we proceed to the next step.

(2) We have

$$A\widetilde{e}_{n-2} = \sum_{i=1}^{n-1} b_{i,n-2} Ae_i = \sum_{i=1}^{n-1} b_{i,n-2} \sum_{j=1}^{n} a_{ij} e_j = \sum_{j=1}^{n} w_{j,n-2} e_j$$
$$= \sum_{j=1}^{n-1} w_{j,n-1} e_j + w_{n,n-2} e_n = \sum_{j=1}^{n-1} w_{j,n-1} e_j$$

($w_{n,n-2} = 0$ by (3.6)). Put

$$y_2 := \frac{\sum_{j=1}^{n-1} w_{j,n-2} b_{j,n-2}}{\sum_{j=1}^{n-1} b_{j,n-2}^2}, \quad z_2 := \frac{\sum_{j=1}^{n-1} w_{j,n-2} b_{j,n-1}}{\sum_{j=1}^{n-1} b_{j,n-1}^2} \tag{3.7}$$

and define

$$b_{j,n-3} := w_{j,n-2} - y_2 b_{j,n-2} - z_2 b_{j,n-1}, \quad j = 1, 2, \ldots, n-1, \tag{3.8}$$

which leads to

$$A\widetilde{e}_{n-2} = \widetilde{e}_{n-3} + y_2 \widetilde{e}_{n-2} + z_2 \widetilde{e}_{n-1}. \tag{3.9}$$

The orthogonality relations

$$\sum_{i=1}^{n-1} b_{i,n-3} b_{i,n-2} = \sum_{i=1}^{n-1} b_{i,n-3} b_{i,n-1} = 0 \tag{3.10}$$

stem directly from the definition of y_2,z_2 and (3.6).

Again, if $\widetilde{e}_{n-3} = 0$, we are done. If not, we proceed to the next step.

In such a way, we determine consecutively pairwise orthogonal columns $\{b_{i,n-1}\}_{i=1}^{n-1}, \ldots$ $\{b_{i,n-k}\}_{i=1}^{n-1}$, a pair of numbers (y_k, z_k) and put

$$b_{j,n-k-1} := w_{j,n-k} - y_k b_{j,n-k} - z_k b_{j,n-k+1}, \quad j = 1, 2, \ldots, n-1 \tag{3.11}$$

with the new column orthogonal to all previous ones, and

$$A\widetilde{e}_{n-k} = \widetilde{e}_{n-k-1} + y_k \widetilde{e}_{n-k} + z_k \widetilde{e}_{n-k+1}. \tag{3.12}$$

If $\widetilde{e}_{n-k-1} = 0$, then $A\widetilde{e}_{n-k} = y_k \widetilde{e}_{n-k} + z_k \widetilde{e}_{n-k+1}$, $\mathcal{L}_{n-k} := \text{span}\{\widetilde{e}_j\}_{j\ge n-k}$ is A-invariant, and so is its orthogonal complement which is finite dimensional. Otherwise, the algorithm can be extended to yet another step.

When the algorithm continues till $\widetilde{e}_1$, we come to the orthogonal set $\{\widetilde{e}_j\}_{j=1}^{n-1}$ in span $\{e_j\}_{1\le j\le n-1}$, so necessarily $\widetilde{e}_0 = 0$. In this case, the finite-dimensional component is missing. The operator A in the normalized basis $\{\widehat{e}_k\}_{k\ge 1}$, $\widehat{e}_k = \|\widetilde{e}_k\|^{-1}\, \widetilde{e}_k$ is given by the Jacobi matrix which agrees with the original matrix J from some point on. The proof is complete.

4. Spectra of trees with tails

As we have already mentioned in the Introduction, the adjacency matrix $A(\Gamma)$ for the coupling $\Gamma = G + \mathbb{P}_\infty(\{a_j\})$ is (1.11). The algorithm suggested in Theorem 1.2 provides a way to implement the spectral analysis of the adjacency operator $A(\Gamma)$ which consists of two stages. First, one has to apply the above algorithm by hand to obtain the canonical form (1.12). Second, the solutions of two algebraic equations (the characteristic equation for the finite-dimensional component $F(\Gamma)$ and the Jost equation for the Jacobi component $J(\Gamma)$) provide the spectrum of Γ.

Example 4.1 'A weighted star'.

We begin with the coupling $\Gamma = S_n(w) + \mathbb{P}_\infty$, where $S_n(w)$ is a simple weighted star graph of order $n+1$, $n \ge 2$, with vertices $1, \dots, n$ of degree 1 and the weight of the edge $(k, n+1)$ is w_k, $1 \le k \le n$. The canonical basis $\{\widehat{e}_k\}_{k\in\mathbb{N}}$ looks as follows. We put

$$\widehat{e}_j := e_j, \quad j \ge n+1 \implies A(\Gamma)\,\widehat{e}_j = \widehat{e}_{j-1} + \widehat{e}_{j+1}, \quad j \ge n+2.$$

Next, let $w := (w_1, w_2, \dots, w_n)$, $\|w\| = \sqrt{w_1^2 + \dots + w_n^2}$, and let

$$\widehat{e}_n := \frac{1}{\|w\|} \sum_{j=1}^{n} w_k\, e_j\,.$$

Then,

$$A(\Gamma)\,\widehat{e}_{n+1} = \|w\|\,\widehat{e}_n + \widehat{e}_{n+2}, \qquad A(\Gamma)\,\widehat{e}_n = \|w\|\,\widehat{e}_{n+1}.$$

So the Jacobi subspace and Jacobi component of Γ are

$$\mathcal{L}_J = \operatorname{span}\{\widehat{e}_j\}_{j\ge n}, \quad J(\Gamma) = J\big(\{0\}, \{\|w\|, 1, 1, \dots\}\big). \tag{4.1}$$

To find the finite-dimensional component, we construct an orthonormal basis in $\mathbb{C}^n$ by means of a unitary matrix $\xi = [\xi_{kj}]_{k,j=1}^n$ with the specified last column

$$f_j := \sum_{k=1}^{n} \xi_{kj} e_k(n), \qquad \xi_{kn} = \frac{w_k}{\|w\|}, \quad k = 1, \dots, n, \tag{4.2}$$

where $\{e_k(n)\}_{k=1}^n$ is the standard basis in $\mathbb{C}^n$. Put

$$\widehat{e}_j := \{f_j, 0, 0, \dots\}, \quad j = 1, \dots, n. \tag{4.3}$$

The orthogonality relations $\langle \widehat{e}_k, \widehat{e}_n \rangle = 0$, $1 \le k \le n-1$, give

$$A(\Gamma)\,\widehat{e}_k = \sum_{j=1}^{n} \xi_{kj}\xi_{kn} \cdot \widehat{e}_{n+1} = 0. \tag{4.4}$$

Hence, the finite-dimensional component $F(\Gamma) = \mathbb{O}_{n-1}$ on the subspace $\operatorname{span}\{\widehat{e}_j\}_{j=1}^{n-1}$. So the canonical form is

$$A(\Gamma) \simeq \mathbb{O}_{n-1} \oplus J\big(\{0\}, \{\|w\|, 1, 1, \dots\}\big). \tag{4.5}$$

The Jost polynomial is now given by (2.9)

$$\|w\|\, u(z) = (1 - \|w\|^2) z^2 + 1.$$

Clearly, $u > 0$ for $\|w\| \le 1$, and it has zeros inside $(-1, 1)$ if and only if $\|w\| > \sqrt{2}$. In this case, the spectrum is $\sigma(\Gamma) = [-2, 2] \cup \sigma_d(\Gamma)$ with the discrete component

$$\sigma_d(S_n(w) + \mathbb{P}_\infty) = \Big\{0^{(n-1)}, \pm\Big(\sqrt{\|w\|^2 - 1} + \frac{1}{\sqrt{\|w\|^2 - 1}}\Big)\Big\}. \tag{4.6}$$

For the unweighted star S_n, we have

$$\sigma_d(S_n + \mathbb{P}_\infty) = \Big\{0^{(n-1)}, \pm\Big(\sqrt{n-1} + \frac{1}{\sqrt{n-1}}\Big)\Big\}, \quad n \ge 3,$$
$$\sigma_d(S_2 + \mathbb{P}_\infty) = \{0^{(1)}\}. \tag{4.7}$$

The spectrum $\sigma(S_n + \mathbb{P}_\infty)$ was found in [13].

Note that S_n is a complete bipartite graph, $S_n = K_{1,n}$. For the general complete bipartite graph $K_{p,n+1-p}$, see Example 5.6 below.

Remark 4.2 Although an explicit form of the matrix $\xi = [\xi_{kj}]_{k,j=1}^n$ in (4.2) is immaterial, it is worth noting that in the unweighted case, $\xi_{kn} = n^{-1/2}$, $1 \le k \le n$, and one can take

$$\xi = \mathcal{F}_n := \frac{1}{\sqrt{n}}\,[\varepsilon_n^{kj}]_{k,j=1}^n, \qquad \varepsilon_n := e^{\frac{2\pi i}{n}}, \tag{4.8}$$

which is known as the *Fourier matrix*. Clearly, there are lots of options for ξ to be a real orthogonal matrix (rotation in $\mathbb{R}^n$ with appropriate Euler's angles, orthogonal polynomials, etc.).

Example 4.3 'A multiple star'.

Consider an unweighted star-like graph $S_{n,p}$ with n rays, $n \ge 2$, each of which contains $p+1$ vertices, $p \ge 2$. The vertices are numbered as

$$\{1, n+1, \ldots, (p-1)n+1\}, \quad \{2, n+2, \ldots, (p-1)n+2\}, \quad \ldots \quad \{n, 2n, \ldots, pn\},$$

and the root is $pn+1$, so $S_{n,1} = S_n$. Let $\Gamma = S_{n,p} + \mathbb{P}_\infty$ with the path attached to the root. As above, we put $\widehat{e}_j := e_j$, $j = pn+1, \ldots$, and

$$\widehat{e}_{p(n-1)+i} := \frac{1}{\sqrt{n}} \sum_{q=1}^{n} e_{(i-1)n+q}, \quad i = 1, 2, \ldots, p. \tag{4.9}$$

We have $A(\Gamma)\,\widehat{e}_j = \widehat{e}_{j-1} + \widehat{e}_{j+1}$, $j = pn+2\ldots$,

$$A(\Gamma)\widehat{e}_{pn+1} = \sqrt{n}\widehat{e}_{pn} + \widehat{e}_{pn+2}, \quad A(\Gamma)\,\widehat{e}_{pn} = \widehat{e}_{pn-1} + \sqrt{n}\,\widehat{e}_{pn+1},$$
$$A(\Gamma)\,\widehat{e}_{pn} = \widehat{e}_{pn-1} + \sqrt{n}\,\widehat{e}_{pn+1},$$
$$A(\Gamma)\,\widehat{e}_{p(n-1)+i} = \widehat{e}_{p(n-1)+i-1} + \widehat{e}_{p(n-1)+i+1}, \quad i = 2, \ldots, p-1,$$
$$A(\Gamma)\,\widehat{e}_{p(n-1)+1} = \widehat{e}_{p(n-1)+2},$$

so the Jacobi subspace and Jacobi component of Γ are

$$\mathcal{L}_J = \operatorname{span}\{\widehat{e}_j\}_{j\ge p(n-1)+1},$$
$$J(\Gamma) = J(\{0\},\{a_j\}), \quad a_j = \begin{cases} \sqrt{n}, & j = p; \\ 1, & j \ne p. \end{cases} \tag{4.10}$$

To find the finite-dimensional component note that, by the construction, $\widehat{e}_{p(n-1)+k} \in \operatorname{span}\{e_{(k-1)n+1}, \dots, e_{kn}\}$. As in the above example, we supplement each $\widehat{e}_{p(n-1)+i}$ to the basis in this subspace by means of the Fourier matrix (4.8)

$$f_j^{(k)} := \sum_{q=1}^{n} \xi_{qj} e_{(k-1)n+q}, \qquad f_n^{(k)} := \sum_{q=1}^{n} \xi_{qn} e_{(k-1)n+q} = \widehat{e}_{p(n-1)+k}$$

for $1 \le j \le n-1$, $1 \le k \le p$. As in (4.4) we have

$$A(\Gamma) f_j^{(1)} = f_j^{(2)}, \quad A(\Gamma) f_j^{(2)} = f_j^{(1)} + f_j^{(3)}, \dots,$$
$$A(\Gamma) f_j^{(p-1)} = f_j^{(p-2)} + f_j^{(p)}, \quad A(\Gamma) f_j^{(p)} = f_j^{(p-1)}. \tag{4.11}$$

Relations (4.11) mean that the subspace $\mathcal{H}_j := \operatorname{span}\{f_j^{(1)}, \dots, f_j^{(p)}\}$ is $A(\Gamma)$-invariant, and $A(\Gamma)|\mathcal{H}_j = A(\mathbb{P}_p)$. There are exactly $n-1$ such subspaces for $j = 1, \dots, n-1$. Finally, we come to the following canonical form for the adjacency matrix

$$A(\Gamma) \simeq F(\Gamma) \oplus J(\Gamma), \qquad F(\Gamma) = \bigoplus_{j=1}^{n-1} A(\mathbb{P}_p). \tag{4.12}$$

The Jost polynomial is computed in (2.5)

$$-\sqrt{n} u(z) = (n-1) z^2 \frac{z^{2p}-1}{z^2-1} - 1 = \frac{(n-1) z^{2p+2} - n z^2 + 1}{z^2 - 1}.$$

It is easy to see that the polynomial $q(x) = (n-1)x^{p+1} - nx + 1$ has exactly two positive roots $0 < x_1(p,n) < x_1'(p,n) = 1$, so the first one has the spectral meaning. Hence, $\sigma(\Gamma) = [-2,2] \cup \sigma_d(\Gamma)$ with the discrete component

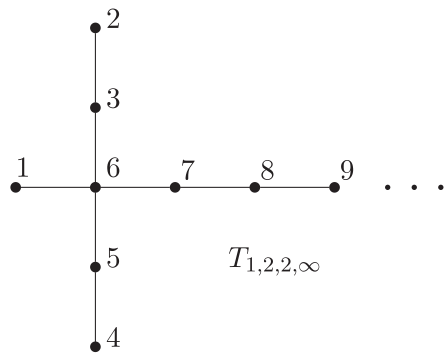
$$\sigma_d(\Gamma) = \left\{2\cos\frac{\pi j}{p+1}\right\}_{j=1}^{p} \bigcup \left\{\pm\left(\sqrt{x_1(p,n)} + \frac{1}{\sqrt{x_1(p,n)}}\right)\right\}, \tag{4.13}$$

so we have p eigenvalues, each of multiplicity $n-1$, on the absolutely continuous spectrum $[-2,2]$ and two simple eigenvalues off $[-2,2]$.

The spectrum of the multiple star was studied in [12], but no explicit formulae were provided.

The problem becomes harder (in the sense of computation) if the original finite star-like graph is nonsymmetric (the rays are different).

Example 4.4 'A sword'.



This graph (also known as $T(1, 2, 2, \infty)$) can be viewed as a coupling $\Gamma = T(1, 2, 2) + \mathbb{P}_\infty$. We put $\widehat{e_k} = e_k$, $k \ge 6$, so $A(\Gamma)\,\widehat{e_k} = \widehat{e}_{k-1} + \widehat{e}_{k+1}$ for $k \ge 7$. Next, take

$$\widehat{e_5} := \frac{e_1 + e_3 + e_5}{\sqrt{3}}, \quad \widehat{e_4} := \frac{e_2 + e_4}{\sqrt{2}}, \quad \widehat{e_3} := \frac{-2e_1 + e_3 + e_5}{\sqrt{6}}.$$

A direct computation leads to the Jacobi subspace and Jacobi component of Γ

$$\mathcal{L}_J = \operatorname{span}\{\widehat{e}_j\}_{j\ge 3}, \quad J(\Gamma) = J\Big(\{0\}, \Big\{\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, \sqrt{3}, 1, 1 \ldots\Big\}\Big). \tag{4.14}$$

To compute the finite-dimensional component, we put

$$\widehat{e_2} := \frac{e_3 - e_5}{\sqrt{2}}, \quad \widehat{e_1} := \frac{e_2 - e_4}{\sqrt{2}},$$

so $A(\Gamma)\,\widehat{e_2} = \widehat{e_1}$, $A(\Gamma)\,\widehat{e_1} = \widehat{e_2}$, and $\{\widehat{e}_j\}_{j\ge 1}$ turns out to be the canonical basis in ℓ^2 for $A(\Gamma)$. The adjacency operator is unitarily equivalent to

$$A(\Gamma) \simeq F(\Gamma) \oplus J(\Gamma), \qquad F(\Gamma) = A(\mathbb{P}_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.15}$$

The Jost polynomial can be computed directly from relations (2.6)

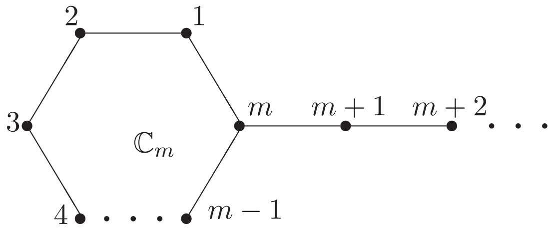
$$u(z) = \sqrt{\frac{3}{2}}\,(1 - z^2 - 3z^4 - 2z^6).$$

It is not hard to see that the cubic polynomial $p(x) = 2x^3 + 3x^2 + x - 1$ has the only real root x_1, $0 < x_1 < 1$. Hence,

$$\sigma(\Gamma) = [-2, 2] \cup \sigma_d(\Gamma), \quad \sigma_d(\Gamma) = \Big\{\pm 1, \pm\Big(\sqrt{x_1} + \frac{1}{\sqrt{x_1}}\Big)\Big\}. \tag{4.16}$$

5. Spectra of graphs with cycles and tails

Proof of Proposition 1.4 Consider the coupling of m-cycle and the infinite path $\Gamma = \mathbb{C}_m + \mathbb{P}_\infty$. We call it the kite graph with infinite tail.



The construction depends heavily on the parity of m. We provide the detailed proof for the case of odd m (the case of even m goes through in the same fashion).

So, let $m = 2n+1$, $n \geq 1$. With $\widehat{e_k} := e_k$, $k \geq 2n+1$, we put

$$\widehat{e}_{n+j} := \frac{e_{n-j+1} + e_{n+j}}{\sqrt{2}}, \quad \widehat{e}_j := \frac{e_{n-j+1} - e_{n+j}}{\sqrt{2}}, \quad j = 1, 2, \ldots, n, \tag{5.1}$$

so $A(\Gamma)\,\widehat{e}_{n+1} = \widehat{e}_{n+1} + \widehat{e}_{n+2}$,

$$\begin{aligned} A(\Gamma)\,\widehat{e}_j &= \widehat{e}_{j-1} + \widehat{e}_{j+1}, \quad j = n+2, \ldots, 2n-1, 2n+2, 2n+3, \ldots \\ A(\Gamma)\,\widehat{e}_{2n+1} &= \sqrt{2}\,\widehat{e}_{2n} + \widehat{e}_{2n+2}, \quad A(\Gamma)\,\widehat{e}_{2n} = \widehat{e}_{2n-1} + \sqrt{2}\,\widehat{e}_{2n+1}, \\ A(\Gamma)\,\widehat{e}_j &= \widehat{e}_{j-1} + e_{j+1}, \quad j = n+2, \ldots, 2n-1, \end{aligned}$$

(with the obvious modification for $n = 1, 2$). The Jacobi subspace and Jacobi components are now $\mathcal{L}_J = \operatorname{span}\{\widehat{e}_j\}_{j \geq n+1}$,

$$J(\Gamma) := J(\{1, 0, 0, \ldots\}, \{a_j\}), \quad a_j = \begin{cases} 1, & j \neq n; \\ \sqrt{2}, & j = n. \end{cases} \tag{5.2}$$

The finite-dimensional component is $\mathcal{L}_J = \operatorname{span}\{\widehat{e}_j\}_{1 \leq j \leq n}$, and relations

$$A(\Gamma)\,\widehat{e}_1 = -\widehat{e}_1 + \widehat{e}_2, \quad A(\Gamma)\,\widehat{e}_j = \widehat{e}_{j-1} + e_{j+1}, \quad j = 2, \ldots, n-1, \quad A(\Gamma)\,\widehat{e}_n = \widehat{e}_{n-1},$$

provide the canonical form

$$A(\Gamma) \simeq F(\Gamma) \oplus J(\Gamma), \quad F(\Gamma) := J(\{-1, 0, \ldots, 0\}, \{1\}) \tag{5.3}$$

is a Jacobi matrix of order n.

The Jost polynomial is given in (2.13)

$$-\sqrt{2}(z+1)u(z) = z^{2n+1} + 2z^2 - 1.$$

The Descarte's rule applied to the polynomial $p_{2n+1}(x) = x^{2n+1} + 2x^2 - 1$ shows that there are two real roots such that $-1 < x_2(n) < 0 < x_1(n) < 1$. Both of them contribute to the discrete spectrum of the Jacobi component

$$\sigma_d(J(\Gamma)) = \left\{ x_1(n) + \frac{1}{x_1(n)}, x_2(n) + \frac{1}{x_2(n)} \right\}. \tag{5.4}$$

To find the spectrum of $F(\Gamma)$ in (5.3), we expand the characteristic determinant over the first row

$$\det|F(\Gamma) + x| = \det|A(\mathbb{P}_n) + x| - \det|A(\mathbb{P}_{n-1} + x| = U_n\Big(\frac{x}{2}\Big) - U_{n-1}\Big(\frac{x}{2}\Big),$$

where U_n is the Chebyshev polynomial of the second kind,

$$U_n(\cos t) := \frac{\sin(n+1)t}{\sin t}, \quad U_n(\cos t) - U_{n-1}(\cos t) = \frac{\cos \frac{2n+1}{2} t}{\cos \frac{t}{2}}.$$

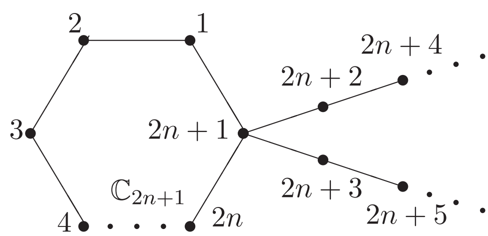
So the spectrum of $F(\Gamma)$ is

$$\sigma(F(\Gamma)) = \left\{2\cos\frac{2\pi j}{2n+1}\right\}_{j=1}^{n} \tag{5.5}$$

and $\sigma(\Gamma) = [-2, 2] \cup \sigma_d(J(\Gamma)) \cup \sigma(F(\Gamma))$. The proof is complete.

Example 5.1 'A cycle with a double tail'.

Given the cycle $\mathbb{C}_{2n+1}$, we attach two infinite paths to the last vertex (see Remark 1.5). Denote this graph by $\Gamma = \mathbb{C}_{2n+1} + 2\mathbb{P}_\infty$.



We proceed as in the case of the kite graph above. As in (5.1), put

$$\widehat{e}_{n+j} := \frac{e_{n-j+1} + e_{n+j}}{\sqrt{2}}, \quad \widehat{e}_j := \frac{e_{n-j+1} - e_{n+j}}{\sqrt{2}}, \quad j = 1, 2, \ldots, n,$$

and

$$\widehat{e}_{2n+1} = e_{2n+1}, \quad \widehat{e}_{2n+1+k} = \frac{e_{2n+2k} + e_{2n+2k+1}}{\sqrt{2}}, \quad k = 1, 2, \ldots.$$

Clearly, the subspace $\widehat{\mathcal{L}} = \text{span}\{\widehat{e}_j\}_{j\geq 1}$ is $A(\Gamma)$-invariant and

$$A(\Gamma)\,|\widehat{\mathcal{L}} = F(\Gamma) \oplus J(\Gamma),$$

where $F(\Gamma)$ is given in (5.3), and

$$J(\Gamma) := J(\{1, 0, 0, \ldots\}, \{a_j\}), \quad a_j = \begin{cases} 1, & j \neq n, n+1; \\ \sqrt{2}, & j = n, n+1. \end{cases}$$

The orthogonal complement $\ell^2 \ominus \widehat{\mathcal{L}}$ is $A(\Gamma)$-invariant, spanned by the system of vectors

$$\widehat{f}_k = \frac{e_{2n+2k} - e_{2n+2k+1}}{\sqrt{2}}, \quad k = 1, 2, \ldots,$$

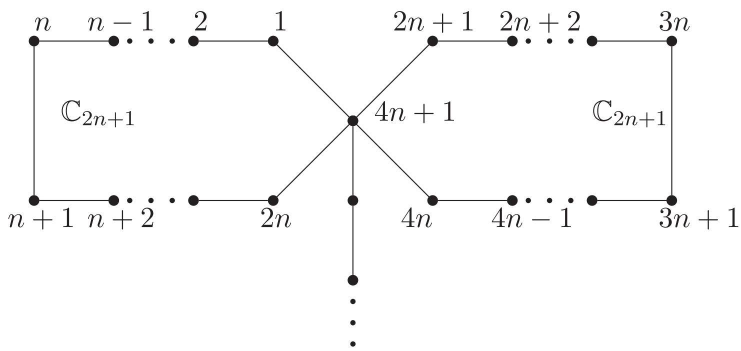
and $A(\Gamma)\,|\ell^2 \ominus \widehat{\mathcal{L}} = J_0$. Finally,

$$A(\Gamma) \simeq F(\Gamma) \oplus J(\Gamma) \oplus J_0. \tag{5.6}$$

We omit the lengthy computation of the Jost polynomial for $J(\Gamma)$ and the discrete spectrum of $J(\Gamma)$ based on the general Theorem DS.

In exactly the same way, we can compute the spectrum of a cycle with multiple tail, i.e. $p \geq 2$ tails attached to a vertex of the cycle.

Example 5.2 'A propeller with equal blades'.



The graph Γ is composed of two equal cycles $\mathbb{C}_{2n+1}$ having one common vertex, with the infinite path attached to it. We put $\widehat{e}_k = e_k$, $k \geq 4n+1$,

$$\widehat{e}_{4n-j} := \frac{e_{j+1} + e_{2n-j} + e_{2n+j+1} + e_{4n-j}}{2}, \qquad j = 0, 1, \dots, n-1.$$

Then, $A(\Gamma)\,\widehat{e}_{3n+1} = \widehat{e}_{3n+1} + \widehat{e}_{3n+2}$,

$$A(\Gamma)\,\widehat{e}_{4n-j} = \widehat{e}_{4n-j-1} + \widehat{e}_{4n-j+1}, \quad j = 1, 2, \dots, n-2,$$
$$A(\Gamma)\,\widehat{e}_{4n} = \widehat{e}_{4n-1} + 2\widehat{e}_{4n+1}, \quad A(\Gamma)\,\widehat{e}_{4n+1} = 2\widehat{e}_{4n} + \widehat{e}_{4n+2}.$$

So the Jacobi subspace and Jacobi components are now $\mathcal{L}_J = \text{span}\{\widehat{e}_j\}_{j \geq 3n+1}$,

$$J(\Gamma) := J(\{1, 0, 0, \dots\}, \{a_j\}), \quad a_j = \begin{cases} 1, & j \neq n; \\ 2, & j = n. \end{cases} \tag{5.7}$$

The finite-dimensional component arises by putting

$$f_j := \frac{e_{n-j+1} - e_{n+j}}{\sqrt{2}}, \quad g_j := \frac{e_{3n-j+1} - e_{3n+j}}{\sqrt{2}},$$
$$h_j := \frac{e_{n-j+1} + e_{n+j} - e_{3n-j+1} - e_{3n+j}}{2}, \quad j = 1, \dots, n.$$

Each of the sequences $\{f_j\}$, $\{g_j\}$, $\{h_j\}$ satisfies

$$A(\Gamma)\,y_1 = -y_1 + y_2, \quad A(\Gamma)\,y_k = y_{k-1} + y_{k+1}, \; k = 2, \dots, n-1, \quad A(\Gamma)\,y_n = y_{n-1},$$

so each of the linear spans is n-dimensional and $A(\Gamma)$-invariant subspace, and $A(\Gamma)$ acts as (cf. (5.3))

$$F(\Gamma) = \bigoplus_{k=1}^{3} J(\{-1, 0, \dots, 0\}, \{1\}).$$

The Jost polynomial

$$-2(z+1)u(z) = 3z^{2n+1} + 4z^2 - 1,$$

has two real roots such that $-1 < x_2(n) < 0 < x_1(n) < 1$. Both of them contribute to the discrete spectrum of the Jacobi component

$$\sigma_d(J(\Gamma)) = \left\{x_1(n) + \frac{1}{x_1(n)}, x_2(n) + \frac{1}{x_2(n)}\right\}.$$

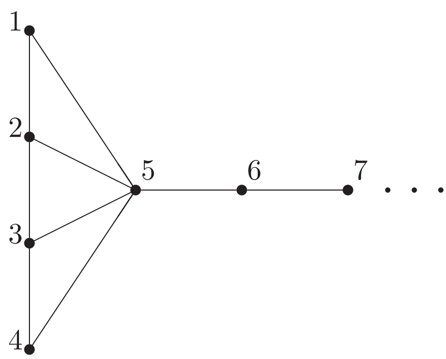
The spectrum of $F(\Gamma)$ is given in (5.5), but now each eigenvalue has multiplicity 3.

For spectral properties of finite propeller graphs, see [24].

Remark 5.3 If the blades are $\mathbb{C}_{2n}$, the argument goes through in exactly the same way. Moreover, the algorithm applies equally well to propellers with p *equal* blades for $p \geq 2$ with the infinite path attached to their common vertex (flowers).

For the rest three examples, the argument is close to one above, so we provide only the answer. For detailed computation, see [25].

Example 5.4 'An umbrella'.



The canonical from is

$$A(\Gamma) \simeq F(\Gamma) \oplus J(\Gamma), \tag{5.8}$$

where

$$F(\Gamma) := \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad J(\Gamma) := J\left(\left\{-\frac{1}{2}, \frac{3}{2}, 0, 0, \dots\right\}, \left\{\frac{1}{2}, 2, 1, 1, \dots\right\}\right).$$

The Jost polynomial is $u(z) = -3z^4 - 3z^3 - 3z^2 - z + 1$, having two real roots $-1 < x_2 < 0 < x_1 < 1$ with the spectral meaning. The spectrum is

$$\sigma(\Gamma) = [-2, 2] \bigcup \sigma_d(\Gamma), \quad \sigma_d(\Gamma) = \left\{\frac{-1 \pm \sqrt{5}}{2}, x_1 + \frac{1}{x_1}, x_2 + \frac{1}{x_2}\right\}. \tag{5.9}$$

Example 5.5 'A wheel'.

The wheel W_n is a graph consisting of a cycle on n vertices, $n \geq 3$, and a vertex $n + 1$, adjacent to each of $1, 2, \dots, n$ (cf. [1, p. 49]). Consider the coupling $\Gamma = W_n + \mathbb{P}_\infty$ with the path attached to the root $n + 1$.

The canonical form is

$$A(\Gamma) \simeq F(\Gamma) \oplus J(\Gamma), \tag{5.10}$$

where

$$F(\Gamma) = \mathrm{diag}\left\{2\cos\frac{2\pi}{n}, \ldots, 2\cos\frac{2\pi(n-1)}{n}\right\},$$
$$J(\Gamma) = J(\{2, 0, 0, \ldots\}, \{\sqrt{n}, 1, 1, \ldots\}).$$

The Jost polynomial is computed in (2.9): $-\sqrt{n}u(z) = (n-1)z^2 + 2z - 1$. It has two roots

$$x_2(n) = -\frac{\sqrt{n}+1}{n-1} < 0 < x_1(n) = \frac{\sqrt{n}-1}{n-1},$$

such that $x_1(n) \in (0,1)$ for all $n \geq 3$. For $x_2(n)$, we have

$$x_2(n) \leq -1, \quad n = 3, 4; \qquad x_2(n) > -1, \quad n \geq 5.$$

Hence, $\sigma(\Gamma) = [-2, 2] \cup \sigma_d(\Gamma)$, where the discrete spectrum is either

$$\sigma_d(\Gamma) = \left\{2\cos\frac{2k\pi}{n}\right\}_{k=1}^{n-1} \bigcup \left\{x_1(n) + \frac{1}{x_1(n)}\right\}, \quad n = 3, 4,$$

or

$$\sigma_d(\Gamma) = \left\{2\cos\frac{2k\pi}{n}\right\}_{k=1}^{n-1} \bigcup \left\{x_1(n) + \frac{1}{x_1(n)}, x_2(n) + \frac{1}{x_2(n)}\right\}, \quad n \geq 5.$$

Example 5.6 ‘The complete bipartite graph’.

Let $G = K_{p,n+1-p}$ be the complete bipartite graph of order $n+1$ [4, p.16], $\Gamma = K_{p,n+1-p} + \mathbb{P}_\infty$. The canonical form is

$$A(\Gamma) \simeq F(\Gamma) \oplus J(\Gamma), \qquad F(\Gamma) := \mathbb{O}_{n-2} \tag{5.11}$$

and

$$J(\Gamma) := J(\{0\}, \{\sqrt{p(n-p)}, \sqrt{p}, 1, 1, \ldots\}).$$

The Jost polynomial is given in (2.10)

$$-p\sqrt{n-p}\,u(z) = (p-1)z^4 + (p(n-p) + p - 2)z^2 - 1 = q(z^2).$$

It is easy to see that the quadratic polynomial q has the only root $x_1(p,n)$ in $(-1,1)$, $0 < x_1(p,n) < 1$, so $\sigma(\Gamma) = [-2,2] \cup \sigma_d(\Gamma)$,

$$\sigma_d(K_{p,n+1-p} + \mathbb{P}_\infty) = \{0^{(n-2)}\} \bigcup \left\{\pm\left(\sqrt{x_1(p,n)} + \frac{1}{\sqrt{x_1(p,n)}}\right)\right\}. \tag{5.12}$$

For $p = 1$, we have Example 4.1.

The situation when an equal number of tails are attached to *each* vertex of a given finite Graph, G turns out to be relatively simple.

Proof of Theorem 1.6 We label the vertices along the rays (off G) as

$$R_i = \{m_j + i\}_{j=1}^{\infty}, \quad m_j := n + (j-1)pn, \quad i = 1, 2, \ldots, pn.$$

Put

$$g_j(k) := \frac{1}{\sqrt{p}} \sum_{l=1}^{p} e_{m_j+(k-1)p+l}, \quad k = 1, 2, \ldots, n, \quad j \in \mathbb{N},$$

so $\{g_j(k)\}_{j,k}$ is an orthonormal sequence in ℓ^2. It is easy to see from the way of labelling of the vertices that the adjacency operator $A(\Gamma)$ acts on these vectors as

$$\begin{aligned} A(\Gamma)\, g_j(k) &= g_{j-1}(k) + g_{j+1}(k), \quad j = 2, 3, \ldots \\ A(\Gamma)\, g_1(k) &= \sqrt{p}\, e_k + g_2(k). \end{aligned} \tag{5.13}$$

Next, by the definition,

$$A(\Gamma)\; e_k = \{A(G)e_k^{(n)}, 0, 0, \ldots\} + \sqrt{p}\, g_1(k), \quad k = 1, 2, \ldots, n, \tag{5.14}$$

where $\{e_k^{(n)}\}_{k=1}^n$ is the standard basis in $\mathbb{C}^n$.

To construct the canonical basis, we invoke the eigenvectors of $A(G)$

$$f_k^{(n)} := \sum_{q=1}^{n} \zeta_{qk}\, e_q^{(n)}, \qquad \zeta = [\zeta_{qk}]_{q,k=1}^n$$

which is a unitary matrix, so $A(G) f_k^{(n)} = \lambda_k f_k^{(n)}$, $k = 1, \ldots, n$. With such a notation at hand, we put

$$\begin{aligned} \widehat{e}_0(k) &:= \sum_{q=1}^{n} \zeta_{qk}\, e_q = (f_k^{(n)}, 0, 0 \ldots), \\ \widehat{e}_j(k) &:= \sum_{q=1}^{n} \zeta_{qk}\, g_j(q), \quad j \in \mathbb{N}, \quad k = 1, \ldots, n. \end{aligned}$$

It is clear from (5.13) that $A(\Gamma)\; \widehat{e}_j(k) = \widehat{e}_{j-1}(k) + \widehat{e}_{j+1}(k)$, $j \ge 2$,

$$A(\Gamma)\, \widehat{e}_1(k) = \sum_{q=1}^{n} \zeta_{qk}\, (\sqrt{p}\, e_q + g_2(q)) = \sqrt{p}\, \widehat{e}_0(k) + \widehat{e}_2(k),$$

and

$$\begin{aligned} A(\Gamma)\, \widehat{e}_0(k) &= \sum_{q=1}^{n} \zeta_{qk}\, A(\Gamma)\; e_q = \left(\sum_{q=1}^{n} \zeta_{qk}\, A(G) e_q^{(n)}, 0, 0, \ldots \right) + \sqrt{p}\, \widehat{e}_1(k) \\ &= \lambda_k \widehat{e}_0(k) + \sqrt{p}\, \widehat{e}_1(k). \end{aligned}$$

Hence, the subspace $\widehat{\mathcal{L}} := \mathrm{span}\{\widehat{e}_j(k)\}_{1 \le k \le n,\, j=0,1\ldots}$ is $A(\Gamma)$-invariant and

$$A(\Gamma)\, |\widehat{\mathcal{L}} \simeq \bigoplus_{k=1}^{n} J(\lambda_k, \sqrt{p}).$$

It is not hard to see that on the orthogonal complement

$$A(\Gamma)\, |\ell^2 \ominus \widehat{\mathcal{L}} \simeq \bigoplus_{k=1}^{(p-1)n} J_0.$$

The proof is complete.

To find the discrete spectrum of $J(\lambda_k, \sqrt{p})$, one has to solve the Jost Equation (2.9)

$$(p-1)x^2 + \lambda_k x - 1 = 0, \tag{5.15}$$

pick its spectral roots, that is the roots in $(-1, 1)$, and then take their Zhukovsky images. The simplest case is $p = 1$, when each λ_k with $|\lambda_k| > 1$ generates one eigenvalue $\lambda_k + \lambda_k^{-1}$ of $A(\Gamma)$. For $p \geq 3$, an elementary analysis of Equation (5.15) shows that there are two spectral roots for $|\lambda_k| < \frac{p(p-2)}{p-1}$, and there is one such root otherwise.

The case $p = 2$ is particularly simple. Equation (5.15) has the roots

$$x_\pm = \frac{\lambda_k \pm \sqrt{\lambda_k^2 + 4}}{2}, \quad x_- < 0 < x_+.$$

It is easy to see that there is exactly one spectral root for $\lambda_k \neq 0$, and no such roots for $\lambda_k = 0$. The discrete spectrum now looks as follows

$$\sigma_d(\Gamma) = \bigcup_{\lambda_k \neq 0} \operatorname{sign}\lambda_k \sqrt{\lambda_k^2 + 4}.$$

6. Stars and Bethe lattices

Consider two examples wherein two-sided Jacobi matrices arise naturally.

Example 6.1 ‘A double infinite star graph’.

Denote by $S_{k,\infty}$ the infinite star-like graph with $k \geq 2$ infinite rays emanating from the common root (cf. Example 4.3). The main object under consideration is the coupling of two such graphs with the bridge connecting their roots, so $\Gamma = S_{p,\infty} + S_{q,\infty}$.

We label the vertices of the ‘right’ star $S_{p,\infty}$ along the rays by positive integers as

$$\{1, 2, p+2, \ldots\},\ \{1, 3, p+3, \ldots\}, \ldots,\ \{1, p+1, 2p+1, \ldots\},$$

so the root is 1. Similarly, we number the vertices of the ‘left’ graph $S_{q,\infty}$ by negative integers, so the root is -1 and the vertices along the rays are

$$\{-1, -2, -q-2, \ldots\},\ \{-1, -3, -q-3, \ldots\}, \ldots,\ \{-1, -q-1, -2q-1, \ldots\}.$$

The underlying ℓ^2 space is $\ell^2(\mathbb{Z}_0)$, $\mathbb{Z}_0 := \mathbb{Z}\backslash\{0\}$, and the standard basis is $\{e_j\}_{j\in\mathbb{Z}_0}$.

We construct an orthonormal system $\{\widehat{e}_j\}_{j\in\mathbb{Z}_0}$ by

$$\widehat{e}_1 := e_1, \qquad \widehat{e}_k := \frac{1}{\sqrt{p}} \sum_{i=2}^{p+1} e_{(k-2)p+i}, \quad k \geq 2;$$

$$\widehat{e}_{-1} := e_{-1}, \qquad \widehat{e}_j := \frac{1}{\sqrt{q}} \sum_{i=2}^{q+1} e_{(j+2)q-i}, \quad j \leq -2.$$

It is easy to see that $A(\Gamma)$ acts as

$$A(\Gamma)\widehat{e}_1 = \widehat{e}_{-1} + \sqrt{p}\,\widehat{e}_2, \qquad A(\Gamma)\widehat{e}_{-1} = \sqrt{q}\,\widehat{e}_{-2} + \widehat{e}_1,$$
$$A(\Gamma)\widehat{e}_2 = \sqrt{p}\,\widehat{e}_1 + \widehat{e}_3, \qquad A(\Gamma)\widehat{e}_{-2} = \widehat{e}_{-3} + \sqrt{q}\,\widehat{e}_{-1},$$

$$A(\Gamma)\,\widehat{e}_k = \widehat{e}_{k-1} + \widehat{e}_{k+1}, \quad k \ge 3, \qquad A(\Gamma)\,\widehat{e}_{-j} = \widehat{e}_{-j-1} + \widehat{e}_{-j+1}, \quad j \ge 3.$$

Hence, the subspace $\widehat{\mathcal{L}} = \text{span}\{\widehat{e}_j\}_{j\in\mathbb{Z}_0}$ is $A(\Gamma)$-invariant and

$$A(\Gamma)\,|\widehat{\mathcal{L}} = J(p,q) = J(\{0\}, \{a_j\}_{j\in\mathbb{Z}}), \quad a_j = \begin{cases} \sqrt{p}, & j=1; \\ \sqrt{q}, & j=-1; \\ 1, & j \ne \pm 1. \end{cases} \tag{6.1}$$

To supplement the system $\{\widehat{e}_j\}_{j\in\mathbb{Z}_0}$ to the orthonormal basis, we proceed in a standard way

$$\widehat{e}_k^{(l)} := \sum_{i=2}^{p+1} \xi_{i-1,l}\, e_{(k-1)p+i}, \quad l = 1,\dots,p, \quad k = 1,2,\dots, \qquad [\xi_{rs}] = \mathcal{F}_p;$$

$$\widehat{e}_j^{(l)} := \sum_{i=2}^{q+1} \eta_{i-1,l}\, e_{(j+1)q-i}, \quad l = 1,\dots,q, \quad j = -1,-2,\dots, \quad [\eta_{rs}] = \mathcal{F}_q.$$

Finally,

$$A(\Gamma) \simeq J(p,q) \oplus \left(\bigoplus_{i=1}^{p+q-2} J_0 \right). \tag{6.2}$$

To find the discrete spectrum, we apply Example 2.4 with $a_{-1} = \sqrt{q}$, $a_1 = \sqrt{p}$, so characteristic Equation (2.25) looks

$$Q(y) = (p-1)(q-1)y^2 - (p+q-1)y + 1 = 0.$$

It is easy to see that this equation always has two positive roots $0 < y_- < y_+$. In the cases $\min(p,q) = 2$ and $p = q = 3$, we have $y_+ \ge 1$, so only the first root has the spectral meaning. Otherwise, $0 < y_- < y_+ < 1$, so both of them are the spectral roots. Hence,

$$\sigma_d(J(p,q)) = \pm\left(\sqrt{y_-} + \frac{1}{\sqrt{y_-}}\right), \qquad \min(p,q) = 2 \ \text{ or } \ p = q = 3,$$

or

$$\sigma_d(J(p,q)) = \left\{ \pm\left(\sqrt{y_-} + \frac{1}{\sqrt{y_-}}\right), \pm\left(\sqrt{y_+} + \frac{1}{\sqrt{y_+}}\right) \right\}$$

for the rest of the values $p, q \ge 2$. Here,

$$y_\pm = \frac{p+q-1 \pm \sqrt{(p-q)^2 + 2(p+q) - 3}}{2(p-1)(q-1)}.$$

Note that if the bridge has weight d, the above argument ends up with the Jacobi matrix

$$J(p,q,d) = J(\{0\}, \{a_j\}_{j\in\mathbb{Z}}), \quad a_j = \begin{cases} \sqrt{p}, & j=1; \\ \sqrt{q}, & j=-1; \\ d, & j=0, \end{cases} \qquad a_j = 1, \quad j \ne 0, \pm 1,$$

for which the spectrum can be found explicitly (see Introduction).

The example was studied in [10] using a different model based on one-sided Jacobi matrices (1.5), with no explicit expressions provided.

Example 6.2 'Infinite regular trees'.

The Bethe–Cayley trees $\mathbb{B}_d$ of degree $d \in \mathbb{N}$ (or Bethe lattices in physical literature) are among the most notable infinite trees. For the introduction to the spectral theory on such trees, see [7, Chapter 10]. We consider here the coupling of two copies of $\mathbb{B}_d$ with the bridge connecting their roots, $\Gamma = \mathbb{B}_d + \mathbb{B}_d$. Note that Γ is an infinite regular tree of degree $d+1$. We number the vertices of Γ as in the previous example (the vertices of the 'right' copy by positive integers, and of the 'left' copy by negative integers). So, the vertices of k's generation in the right copy (the root forms the 1st generation) are labelled with

$$\{l_{k-1}+1, l_{k-1}+2, \dots, l_k\}, \qquad l_k := \sum_{j=1}^{k-1} d^j = \frac{d^k-1}{d-1}, \quad k \in \mathbb{N}.$$

Again, the underlying Hilbert space is $\ell^2(\mathbb{Z}_0)$ with the basis $\{e_j\}_{j\in\mathbb{Z}_0}$.

Put $\widehat{e}_{\pm 1} := e_{\pm 1}$,

$$\widehat{e}_k := \frac{1}{\sqrt{d^{k-1}}} \sum_{i=l_{k-1}+1}^{l_k} e_i, \qquad \widehat{e}_{-k} := \frac{1}{\sqrt{d^{k-1}}} \sum_{j=l_{k-1}+1}^{l_k} e_{-j}, \quad k = 2, 3, \dots.$$

The operator $A(\Gamma)$ acts as

$$A(\Gamma)\, \widehat{e}_1 = \widehat{e}_{-1} + \sqrt{d}\,\widehat{e}_2, \qquad A(\Gamma)\, \widehat{e}_{-1} = \sqrt{d}\,\widehat{e}_{-2} + \widehat{e}_1,$$
$$A(\Gamma)\, \widehat{e}_k = \sqrt{d}\,\widehat{e}_{k-1} + \sqrt{d}\,\widehat{e}_{k+1}, \qquad |k| \ge 2.$$

So the subspace $\widehat{\mathcal{L}} = \operatorname{span}\{\widehat{e}_j\}_{j\in\mathbb{Z}_0}$ is $A(\Gamma)$-invariant and

$$A(\Gamma)\,\lceil\rceil\mathcal{L} = J(\Gamma) = \sqrt{d}\, J(\{0\}, \{a_j\}_{j\in\mathbb{Z}}), \quad a_j = \begin{cases} 1/\sqrt{d}, & j = 0; \\ 1, & j \ne 0 \end{cases} \tag{6.3}$$

The structure of operator $A(\Gamma)$ on the orthogonal complement is the same as in the case of $\mathbb{B}_d$ (see [14], [7, Theorem 10.2.2]), so finally

$$A(\Gamma) \simeq J(\Gamma) \oplus \left(\bigoplus_{i=1}^{\omega_d} \sqrt{d}\, J_0 \right), \qquad \omega_d = \begin{cases} \infty, & d \ge 2; \\ 0, & d = 1. \end{cases} \tag{6.4}$$

To find the discrete spectrum of $A(\Gamma)$, note that $J(\Gamma)$ (6.3) is the subject of Example 2.3. It follows that the discrete spectrum of $J(\Gamma)$ is empty, and so

$$\sigma(\Gamma) = [-2\sqrt{d}, 2\sqrt{d}] \tag{6.5}$$

and is pure absolutely continuous of infinite multiplicity.

Remark 6.3 Since $\mathbb{B}_1 = \mathbb{P}_\infty$, the question arises naturally whether it is possible to find spectra of the couplings $\Gamma = G + \mathbb{B}_d$. The above algorithm works equally well in this situation and leads to the canonical form

$$A(G + \mathbb{B}_d) \simeq A(G + \mathbb{P}_\infty) \oplus \left(\bigoplus_{i=1}^{\omega_d} \sqrt{d}\, J_0 \right).$$

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