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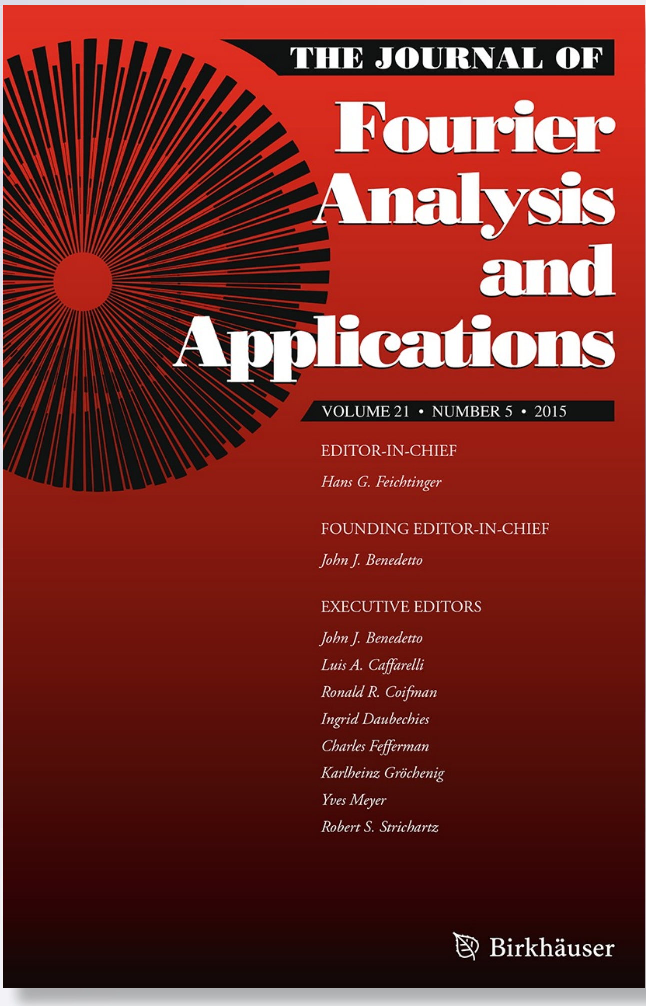

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Schoenberg Matrices of Radial Positive Definite Functions and Riesz Sequences of Translates in $L^2(\mathbb{R}^n)$

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Abstract Given a function f on the positive half-line $\mathbb{R}_+$ and a sequence (finite or infinite) of points $X = \{x_k\}_{k=1}^{\omega}$ in $\mathbb{R}^n$, we define and study matrices $\mathcal{S}_X(f) = [f(\|x_i - x_j\|)]_{i,j=1}^{\omega}$ called Schoenberg's matrices. We are primarily interested in those matrices which generate bounded and invertible linear operators $S_X(f)$ on $\ell^2(\mathbb{N})$. We provide conditions on X and f for the latter to hold. If f is an ℓ^2-positive definite function, such conditions are given in terms of the Schoenberg measure σ_f. Examples of Schoenberg's operators with various spectral properties are presented. We also approach Schoenberg's matrices from the viewpoint of harmonic analysis on $\mathbb{R}^n$, wherein the notion of the strong X-positive definiteness plays a key role. In particular, we prove that *each radial ℓ^2-positive definite function is strongly X-positive definite* whenever X is a separated set. We also implement a "grammization" procedure for certain positive definite Schoenberg's matrices. This leads to Riesz–Fischer and Riesz sequences (Riesz bases in their linear span) of the form $\mathcal{F}_X(g) = \{g(\cdot - x_j)\}_{x_j \in X}$ for certain radial functions $g \in L^2(\mathbb{R}^n)$.

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1 Introduction

Positive definite functions have a long history, entering as an important chapter in all treatments of harmonic analysis. They can be traced back to papers of Carathéodory, Herglotz, Bernstein, culminating in Bochner's celebrated theorem from 1932 to 1933. See definitions in Sect. 2.1.1.

In this paper we will be dealing primarily with radial positive definite functions (RPDF). RPDF's have significant applications in probability theory, Fourier analysis, and approximation theory, where they occur as characteristic functions or Fourier transforms of spherically symmetric probability distributions [18,20,34,36], covariance functions of stationary and isotropic random fields [29], radial basis functions in scattered data interpolation [37], data assimilation in geodesy [23].

We stick to the standard notation for the inner product $\langle u, v\rangle_n = \langle u, v\rangle = u_1 v_1 + \ldots + u_n v_n$ of two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, and $\|u\|_n = \|u\| = \sqrt{\langle u, u\rangle}$ for the Euclidean norm of u. Let us emphasize from the outset that throughout the whole paper n is an arbitrary and fixed positive integer.

Definition 1.1 A real-valued and continuous function f on $\mathbb{R}_+ = [0, \infty)$ is called a *radial positive definite function on* $\mathbb{R}^n$, if for an arbitrary finite set $\{x_1, \ldots, x_m\}$ of points $x_k \in \mathbb{R}^n$, and an arbitrary finite set $\{\xi_1, \ldots, \xi_m\}$ of complex numbers $\xi_k \in \mathbb{C}$

$$\sum_{k,j=1}^{m} f(\|x_k - x_j\|)\xi_j \overline{\xi}_k \geq 0. \tag{1.1}$$

We denote this class by Φ_n.

The characterization of RPDF's is a fundamental result of I. Schoenberg [30,31] (see, e.g., [3, Theorem 5.4.2]).

Theorem 1.2 *A function* $f \in \Phi_n$, $f(0) = 1$, *if and only if there exists a probability measure* ν_f *on* $\mathbb{R}_+$ *such that*

$$f(r) = \int_0^{\infty} \Omega_n(rt)\, \nu_f(dt), \qquad r \in \mathbb{R}_+, \tag{1.2}$$

where

$$\Omega_n(s) := \Gamma(q+1)\left(\frac{2}{s}\right)^q J_q(s) = \sum_{j=0}^{\infty} \frac{\Gamma(q+1)}{j!\,\Gamma(j+q+1)}\left(-\frac{s^2}{4}\right)^j, \quad q := \frac{n}{2} - 1, \tag{1.3}$$

J_q is the Bessel function of the first kind and order q. Moreover,

$$\Omega_n(\|x\|) = \int_{S^{n-1}} e^{i\langle u,x\rangle} s_n(du), \qquad x \in \mathbb{R}^n, \tag{1.4}$$

where s_n is the normalized surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

The first three functions Ω_n, $n = 1, 2, 3$, can be computed as

$$\Omega_1(s) = \cos s, \quad \Omega_2(s) = J_0(s), \quad \Omega_3(s) = \frac{\sin s}{s}. \tag{1.5}$$

The main object under consideration in this paper arises from the definition of RPDF's.

Definition 1.3 Let $X = \{x_k\}_{k=1}^{\omega} \subset \mathbb{R}^n$ be a (finite or infinite) set of distinct points in $\mathbb{R}^n$, and let f be a real-valued function defined on the right half-line $\mathbb{R}_+$. A matrix (finite or infinite)

$$\mathcal{S}_X(f) := [f(\|x_i - x_j\|)]_{i,j=1}^{\omega}, \qquad \omega \le \infty, \tag{1.6}$$

will be called a *Schoenberg matrix* generated by the set X and the function f. This function is referred to as the *Schoenberg symbol*.

It is clear that $\mathcal{S}_X(f)$ is a Hermitian (real symmetric) matrix. By the definition, a function $f \in \Phi_n$ if for each finite set $X \subset \mathbb{R}^n$ the Schoenberg matrix $\mathcal{S}_X(f)$ is nonnegative definite, $\mathcal{S}_X(f) \ge 0$.

We undertake a detailed study of Schoenberg's matrices from two different points of view. The first one, considered in Sect. 3, comes from operator theory.

If the columns of $\mathcal{S}_X(f)$ are in $\ell^2 := \ell^2(\mathbb{N})$, then one can associate a closed symmetric operator $S_X(f)$ with $\mathcal{S}_X(f)$ in a natural way (see Sect. 3.1). We call it a *Schoenberg operator*. If $S_X(f)$ appears to be bounded, a matrix $\mathcal{S}_X(f)$ (admitting some abuse of language) will be called bounded. The *first main goal of the paper is to find necessary and sufficient conditions on X and f, which ensure that the matrix $\mathcal{S}_X(f)$ is bounded*. We also suggest conditions on X and f for $S_X(f)$ to be invertible, i.e., to have a bounded inverse.

Denote by $\mathcal{L}(X)$ the linear span of X, a subspace in $\mathbb{R}^n$ of dimension $d = \dim \mathcal{L}(X) \le n$. Throughout the paper we assume that X is a *separated set*, i.e.,

$$d_* = d_*(X) := \inf_{i \ne j} \|x_i - x_j\| > 0, \tag{1.7}$$

(the term *uniformly discrete* is also in common usage). We denote by $\mathcal{X}_d$ the class of all separated sets $X \subset \mathbb{R}^n$ with $\dim \mathcal{L}(X) = d$. With no loss of generality we can assume that $x_1 = 0$.

Next, denote by $\mathcal{M}_+$ the class of nonnegative and monotone decreasing functions

$$f \in \mathcal{M}_+ : \ \ f \ge 0, \qquad f \downarrow, \qquad f(0) = 1. \tag{1.8}$$

With this preparation our main result on boundedness of $\mathcal{S}_X(f)$ reads as follows.

Theorem 1.4 *Let $f \in \mathcal{M}_+$, $X \in \mathcal{X}_d$, $d \le n$.*

(i) *If $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$, then the Schoenberg operator $S_X(f)$ is bounded on ℓ^2 and*

$$\|S_X(f)\| \le 1 + d^2 \left(\frac{5}{d_*(X)} \right)^d \int_0^\infty t^{d-1} f(t)\, dt. \tag{1.9}$$

(ii) *Moreover, $S_X(f)$ has a bounded inverse whenever, in addition,*

$$d_*(X) > 5 d^{2/d} \, \|t^{d-1} f\|_{L^1(\mathbb{R}_+)}^{1/d}. \tag{1.10}$$

(iii) *Conversely, let $S_Y(f)$ be bounded for at least one δ-regular set Y. Then $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$.*

Concerning regular sets see Definition 3.3. For instance, $X = \delta \mathbb{Z}^n$ is δ-regular.

In particular, Theorem 1.4 completely describes bounded operators $S_X(f)$ with symbols f from the classes $\Phi_\infty(\alpha)$ defined below in Sect. 2.1.3.

Our second viewpoint on Schoenberg's matrices is related to harmonic analysis on $\mathbb{R}^n$.

It was proved in [33] (see also [16, Theorem 3.6]) that for each function $f \in \Phi_n$ and each finite set of distinct points $X \subset \mathbb{R}^n$ the Schoenberg matrix $\mathcal{S}_X(f)$ is positive definite and non-singular, that is, the minimal eigenvalue $\lambda_{min}(\mathcal{S}_X(f)) > 0$. This fact has been heavily exploited in [16] for investigation of certain spectral properties of $2D$ and $3D$ Schrödinger operator with a *finite number* of point interactions.

The situation is much more delicate for infinite test sets X.

Definition 1.5 Let $f \in \Phi_n$ and $X = \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$. We say that f is *strongly X-positive definite* if for any set $\xi = \{\xi_1, \ldots, \xi_m\} \subset \mathbb{C}$ of complex numbers, not identically zeros, and any finite set $\{x_j\}_{j=1}^m$ of distinct points $x_j \in X$ there exists a constant $c(X) > 0$, independent of ξ and m and such that

$$\sum_{k,j=1}^m f(\|x_k - x_j\|) \xi_j \overline{\xi_k} > c(X) \sum_{k=1}^m |\xi_k|^2. \tag{1.11}$$

The same definition with obvious changes applies to general (not necessarily radial) positive definite functions.

We show that if $f \in \Phi_n$ is strongly X-positive definite, then X is necessarily *separated* (see Proposition 3.21).

Our *second main goal* is to find properties of $f \in \Phi_n$ which ensure f to be strongly X-positive definite for *each* $X \in \mathcal{X}_n$. In this direction we obtain the following result.

Theorem 1.6 *Let* (const $\ne$) $f \in \Phi_n$, $n \ge 2$, *with the representing measure ν_f from* (1.2). *If ν_f is equivalent to the Lebesgue measure on $\mathbb{R}_+$, then f is strongly X-positive definite for each $X \in \mathcal{X}_n$.*

It might be worth noting that for each $f \in \Phi_n$ there exists a separated set $X(f)$ so that f is strongly $X(f)$-positive definite [21, Corollary 2.19].

Actually, the most complete result on the strong X-positive definiteness and the boundedness of $S_X(f)$ is obtained for the class $\Phi_\infty := \bigcap_{n\in\mathbb{N}} \Phi_n$ and its subclasses $\Phi_\infty(\alpha)$, $\alpha \in (0, 2]$ defined in the next section. It looks as follows.

Theorem 1.7 *Let $f \in \Phi_\infty(\alpha)$, $0 < \alpha \leq 2$. Then*

(i) *f is strongly X-positive definite for each $X \in \mathcal{X}_d$, $d \in \mathbb{N}$. In particular, if $\mathcal{S}_X(f)$ generates an operator $S_X(f)$ on ℓ^2, then it is positive definite and so invertible.*
(ii) *If the Schoenberg measure σ_f in (2.6) satisfies*

$$\int_0^\infty s^{-\frac{d}{\alpha}} \sigma_f(ds) < \infty, \qquad d \in \mathbb{N}, \tag{1.12}$$

then the Schoenberg operator $S_X(f)$ is bounded and invertible for each $X \in \mathcal{X}_d$.
(iii) *Conversely, if $S_Y(f)$ is bounded for at least one δ-regular set $Y \in \mathcal{X}_d$, then (1.12) holds.*

The notion of the strong X-positive definiteness makes sense for any $f \in \Phi_n$ regardless of whether the Schoenberg operator $S_X(f)$ is well defined or not. In the former case the strong X-positive definiteness of f is identical to positive definiteness of $S_X(f)$, i.e., validity of the inequality

$$\langle S_X(f)h, h\rangle \geq \varepsilon \|h\|^2, \qquad h \in \operatorname{dom} S_X(f) \subset \ell^2 \tag{1.13}$$

with some $\varepsilon > 0$ independent of h. So Definition 1.5 merely extends property (1.13) of $S_X(f)$, when the latter exists, to the case of an arbitrary Schoenberg matrix $\mathcal{S}_X(f)$, not necessarily generating an operator in ℓ^2.

A concept of "grammization" plays a key role in the rest of the Sect. 4.

It is a common knowledge that every positive matrix is a Gram matrix of a certain system of vectors

$$\mathcal{A} = [a_{ij}]_{i,j\in\mathbb{N}} \geq 0 \Leftrightarrow \mathcal{A} = [\langle \varphi_i, \varphi_j\rangle]_{i,j\in\mathbb{N}} =: Gr(\{\varphi_k\}_{k\in\mathbb{N}}, \mathcal{H}), \tag{1.14}$$

$\{\varphi_k\}_{k\in\mathbb{N}}$ are vectors in a Hilbert space $\mathcal{H}$. The main applications of Theorems 1.4 and 1.6 are based on the grammization procedure for certain Schoenberg's matrices and concern Riesz–Fischer and Riesz sequences of translates $\mathcal{F}_X(g) = \{g(\cdot - x_j)\}_{j\in\mathbb{N}}$, $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$, of radial functions $g \in L^2(\mathbb{R}^n)$.

Let us recall some basic notions from harmonic analysis on the Hilbert spaces ([25, Sect. C.3.3], [40]).

Definition 1.8 Let $\mathcal{F} = \{h_k\}_{k\in\mathbb{N}}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$.

(i) $\mathcal{F}$ is called a *Riesz–Fischer sequence* if for all $\{\xi_1, \cdots, \xi_m\} \subset \mathbb{C}$ and $m \in \mathbb{N}$ there is a constant $c > 0$ such that

$$\left\| \sum_{j=1}^{m} \xi_j h_j \right\|_{\mathcal{H}}^2 \geq c^2 \sum_{j=1}^{m} |\xi_j|^2. \tag{1.15}$$

(ii) $\mathcal{F}$ is called a *Bessel sequence* if for all $\{\xi_1, \cdots, \xi_m\} \subset \mathbb{C}$ and $m \in \mathbb{N}$ there is a constant $C < \infty$ such that

$$\left\| \sum_{j=1}^{m} \xi_j h_j \right\|_{\mathcal{H}}^2 \leq C^2 \sum_{j=1}^{m} |\xi_j|^2. \tag{1.16}$$

(iii) $\mathcal{F}$ is called a *Riesz sequence* (or a Riesz basis in its linear span) if $\mathcal{F}$ is both Riesz–Fischer and Bessel sequence. If $\mathcal{F}$ is complete in $\mathcal{H}$, $\mathcal{F}$ is referred to as a *Riesz basis*.

Theorem 1.9 *Let $g \in L^2(\mathbb{R}^n)$, $n \geq 2$, be a real-valued and radial function such that its Fourier transform*

$$\widehat{g}(t) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(x) e^{-i\langle t,x\rangle}\, dx \neq 0 \tag{1.17}$$

a.e., and let $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$. Then the following statements are equivalent.

(i) *$\mathcal{F}_X(g) = \{g(\cdot - x_j)\}_{j\in\mathbb{N}}$ forms a Riesz–Fischer sequence in $L^2(\mathbb{R}^n)$;*
(ii) *$\mathcal{F}_X(g)$ is uniformly minimal in $L^2(\mathbb{R}^n)$;*
(iii) *X is a separated set, i.e., $d_*(X) > 0$.*

Theorem 1.10 *Let $g \in L^2(\mathbb{R}^n)$, $n \geq 2$, be a real-valued and radial function such that its Fourier transform $\widehat{g} \neq 0$ a.e., and let $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$. Assume that*

$$v(\xi) := \sup_{\|t\|\geq\|\xi\|} |f(t)| \in L^1(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n, \quad f(t) := \int_{\mathbb{R}^n} g(t+y) g(y)\, dy. \tag{1.18}$$

Then the following statements are equivalent.

(i) *$\mathcal{F}_X(g)$ forms a Riesz sequence in $L^2(\mathbb{R}^n)$;*
(ii) *$\mathcal{F}_X(g)$ forms a basis in its linear span;*
(iii) *$\mathcal{F}_X(g)$ is uniformly minimal in $L^2(\mathbb{R}^n)$;*
(iv) *X is a separated set, i.e., $d_*(X) > 0$.*

For minimal and uniformly minimal sequences of vectors see Definition 4.20.

Corollary 1.11 *Let $g \in L^2(\mathbb{R}^n)$, $n \geq 2$, be a real-valued and radial function with compact support, $g \not\equiv 0$, and let $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$. The sequence $\mathcal{F}_X(g)$ forms a Riesz sequence in $L^2(\mathbb{R}^n)$ if and only if X is a separated set.*

Remark 1.12 It is worth mentioning the following result which is a special case of [12, Theorem 1.2]: no sequence $\mathcal{F}_X(g)$ of translates can form a Riesz *basis* for $L^2(\mathbb{R}^n)$. In particular, for a radial function $g \in L^2(\mathbb{R}^n)$ with compact support the system $\mathcal{F}_X(g)$ can be complete in $L^2(\mathbb{R}^n)$ only for sets X with $d_*(X) = 0$.

The idea of the proof relies upon the fact that the system $\mathcal{F}_X(g)$ performs the grammization of a certain Schoenberg's matrix. Indeed, it is clear that the inner product of two translates $g_\xi(\cdot) := g(\cdot - \xi)$ and $g_\eta(\cdot) := g(\cdot - \eta)$, $\xi, \eta \in \mathbb{R}^n$ is

$$\langle g_\xi, g_\eta\rangle_{L^2(\mathbb{R}^n)} = f(\eta - \xi), \qquad f(x) = \int_{\mathbb{R}^n} g(x+y)\overline{g(y)}\,dy. \tag{1.19}$$

Moreover, the function f is positive definite. In fact, the class of functions f (1.19) is exactly the class of all positive definite functions with absolutely continuous Bochner's measures (the latter result is known as the Wiener–Khinchin criterion). Clearly, f is a radial function as long as g is, so $f \in \Phi_n$. Thus we come to the following principal equality

$$Gr(\mathcal{F}_X(g), L^2(\mathbb{R}^n)) = \mathcal{S}_X(f). \tag{1.20}$$

So once we show that the Schoenberg operator $S_X(f)$ is bounded and invertible on ℓ^2, the result of Theorem 1.10 is immediate from the classical theorem of Bari (see [15, Theorem 6.2.1], [25, Sect. 3.3.1, (iv)]). The latter claims that the property of the Gram matrix $Gr\{\varphi_k\}_{k\in\mathbb{N}}$ to generate a bounded and invertible operator on ℓ^2 amounts to the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ to be a Riesz sequence in the corresponding Hilbert space. Thereby we make up a bridge between Riesz sequences and Gram matrices on the one hand and Schoenberg's matrices and operators on the other hand.

We consider a number of examples which satisfy the assumptions of Proposition 1.9 and Theorem 1.10. Among them

$$g(x) = g_a(x) = e^{-a\|x\|^2}, \qquad g(x) = g_{a,\mu}(x) = \left(\frac{a}{\|x\|}\right)^{\mu} K_\mu(a\|x\|), \tag{1.21}$$

where K_μ is the modified Bessel function of the second kind and order μ, $0 \le \mu < n/4$.

Let us emphasize, that our choice of the second system in (1.21) is also motivated by applications to certain elliptic operators with point interactions, since the functions $g_{a,\mu}(\cdot - x_j)$ occur naturally in the spectral theory of such operators for certain other values of μ. We hope to continue the study of this subject in our forthcoming papers.

It is worth stressing that in the abstract setting the uniform minimality is much weaker than the Riesz sequence property. Nonetheless the equivalence of these properties is well known for certain classical systems such as

(i) Exponential system $\{e^{i\lambda_k x}\}_{\lambda_k\in\mathbb{C}}$ in $L^2[0,a)$, $a \le \infty$, provided that $\inf_k(\operatorname{Im}\lambda_k) > -\infty$.

(ii) The system of rational functions

$$\{(1-|\lambda_k|^2)^{1/2}(1-\lambda_k z)^{-1}\}_{\lambda_k\in\mathbb{D}} \in L^2(\mathbb{T}), \quad \mathbb{D}:=\{z : |z| < 1\}, \quad \mathbb{T}:=\{z : |z|=1\}.$$

From the very starting point we were influenced by the paper [21], wherein a tight relation between the spectral theory of 3D Schrödinger operators with *infinitely many* point interactions and RPDF's in $\mathbb{R}^3$ was discovered and exploited in both directions. In particular, a special case of Theorem 1.7 (for $n = d = 3$ and $\alpha = 1$) was proved in [21] by applying machinery of the spectral theory and the grammization of the Schoenberg–Bernstein matrix $\mathcal{S}_X(e^{-as})$, which is achieved for $n = 3$ by the system

$$g_{a,1/2}(x - x_j) = \sqrt{\frac{a}{\|x - x_j\|}}\, K_{1/2}(a\|x - x_j\|) = \sqrt{\frac{\pi}{2}}\, \frac{e^{-a\|x-x_j\|}}{\|x - x_j\|}, \quad j \in \mathbb{N},$$

(see 4.28). However the spectral methods applied in [21] *cannot be extended to either* $n \ge 4$ *or* $\alpha \ne 1$. Our reasoning is based on the harmonic analysis on $\mathbb{R}^n$ and works for an arbitrary dimension $n \ge 2$.

2 Preliminaries

2.1 Positive Definite Functions

Recall some basic facts and notions related to positive definite functions [3,7,34,37].

Definition 2.1 A function $h : \mathbb{R}^n \to \mathbb{C}$ is called *positive definite on* $\mathbb{R}^n$ if h is continuous at the origin, and for arbitrary finite sets $\{x_1, \dots, x_m\}$, $x_k \in \mathbb{R}^n$ and $\{\xi_1, \dots, \xi_m\} \subset \mathbb{C}$ we have

$$\sum_{k,j=1}^{m} h(x_k - x_j)\xi_j \overline{\xi}_k \ge 0. \tag{2.1}$$

The set of positive definite function on $\mathbb{R}^n$ is denoted by $\Phi(\mathbb{R}^n)$. Clearly, a function $h \in \Phi(\mathbb{R}^n)$ if and only if it is continuous at the origin, and the matrix $\mathcal{B}_X(h) := [h(x_k - x_j)]_{k,j=1}^m$ is nonnegative definite, $\mathcal{B}_X(h) \ge 0$, for all finite subsets $X = \{x_j\}_{j=1}^m$ in $\mathbb{R}^n$.

A celebrated theorem of Bochner [10] gives a description of the class $\Phi(\mathbb{R}^n)$.

Theorem 2.2 *A function h is positive definite on $\mathbb{R}^n$ if and only if there exists a finite positive Borel measure μ_h on $\mathbb{R}^n$ such that*

$$h(x) = \int_{\mathbb{R}^n} e^{i\langle u,x\rangle} \mu_h(du), \qquad x \in \mathbb{R}^n. \tag{2.2}$$

When h is a radial function, $h(\cdot) = f(\|\cdot\|)$, $f \in \Phi_n$, the representing measure ν_f in (1.2) is related to the Bochner measure μ_h by $\nu_f\{[0,r]\} = \mu_h\{\|x\| \le r\}$ (cf. [3, Sect. V.4.2]).

2.1.1 Class Φ_∞ of ℓ^2-Radial Positive Definite Functions

Going over to the classes Φ_n of PRDF's, note that the sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ is known to be nested, i.e., $\Phi_{n+1} \subset \Phi_n$, and inclusion is proper (see [30], [34, Sect. 6.3]). So the intersection $\Phi_\infty = \bigcap_{n\in\mathbb{N}} \Phi_n$ comes in naturally. The class Φ_∞ is the case of study in the pioneering paper of I. Schoenberg [30]. According to the Schoenberg theorem (see, e.g., [3, Theorem 5.4.3]), $f \in \Phi_\infty$, $f(0) = 1$, if and only if it admits an integral representation

$$f(t) = \int_0^\infty e^{-st^2}\, \sigma_f(ds), \qquad t \geq 0, \tag{2.3}$$

with σ_f being a probability measure on $\mathbb{R}_+$. The measure σ_f, which is called a *Schoenberg measure* of $f \in \Phi_\infty$, is then uniquely determined by f.

Another characterization of the class Φ_∞ is $\Phi_\infty = \Phi(\ell^2)$, where the latter is the class of radial positive definite functions on the real Hilbert space ℓ^2 (see, e.g., [34, p. 283]). Indeed, since $\mathbb{R}^n$ is embedded in ℓ^2 for each $n \in \mathbb{N}$, we have $\Phi(\ell^2) \subset \Phi_\infty$. Conversely, let $f \in \Phi_\infty$ and $Y = \{y_k\}_{k=1}^m \subset \ell^2$, $y_k = (y_{k1}, y_{k2}, \ldots)$. Define truncations $y_k^{(n)} := (y_{k1}, y_{k2}, \ldots, y_{kn}, 0, 0, \ldots) \in \mathbb{R}^n$. Then for each n

$$[f(\|y_i^{(n)} - y_j^{(n)}\|)]_{i,j=1}^m \geq 0.$$

As $\lim_{n\to\infty} \|y_i^{(n)} - y_j^{(n)}\| = \|y_i - y_j\|$ and f is continuous, the matrix $[f(\|y_i - y_j\|)]_{i,j=1}^m$ is also positive definite, as claimed.

2.1.2 Bernstein Class $CM(\mathbb{R}_+)$ of Completely Monotone Functions

Definition 2.3 A function $f \in C^\infty(\mathbb{R}_+)$ is called *completely monotone* if

$$(-1)^k f^{(k)}(t) \geq 0, \qquad t > 0, \quad k = 0, 1, 2, \ldots. \tag{2.4}$$

The set of such functions is denoted by $CM(\mathbb{R}_+)$. A function f belongs to the subclass $CM_0(\mathbb{R}_+)$ of $CM(\mathbb{R}_+)$ if $f \in CM(\mathbb{R}_+)$ and $f(+0) = 1$.

A fundamental theorem of Bernstein–Widder ([8,39], see also [3, p. 204]) claims that $f \in CM(\mathbb{R}_+)$ if and only if there exists a positive Borel measure τ_f on $\mathbb{R}_+$ such that

$$f(t) = \int_0^\infty e^{-st}\, \tau_f(ds), \qquad t > 0. \tag{2.5}$$

The measure τ_f, which is called a *Bernstein measure* of $f \in CM(\mathbb{R}_+)$, is then uniquely determined by f. τ_f is a probability measure if and only if $f \in CM_0(\mathbb{R}_+)$.

2.1.3 Subclasses $\Phi_\infty(\alpha)$ of Radial Positive Definite Functions

By definition, the class $\Phi_\infty(\alpha)$ consists of functions which admit an integral representation

$$f(t)=\int_0^\infty e^{-st^\alpha}\,\sigma_f(ds),\qquad t\ge 0,\quad 0<\alpha\le 2, \tag{2.6}$$

σ_f is a probability measure on $\mathbb{R}_+$. So, $\Phi_\infty(2)=\Phi_\infty$, $\Phi_\infty(1)=CM_0(\mathbb{R}_+)$. We call the functions $f\in\Phi_\infty(\alpha)$ *α-stable*.

The classes $\Phi_\infty(\alpha)$ are known to admit the following characterization [9]: $f\in\Phi_\infty(\alpha)$, $0<\alpha\le\ 2$, if and only if the function $f(\|x\|_\alpha)$ is positive definite, where

$$x=(x_1,x_2,\ldots),\qquad \|x\|_\alpha:=\left(\sum_{n=1}^\infty |x_j|^\alpha\right)^{\frac{1}{\alpha}}.$$

Note that the family $\{\Phi_\infty(\alpha)\}_{0<\alpha\le 2}$ is nested, i.e.,

$$\Phi_\infty(\alpha_1)\subset\Phi_\infty(\alpha_2),\qquad 0<\alpha_1<\alpha_2\le 2, \tag{2.7}$$

and the inclusion is proper (see, e.g., [9,13]). Indeed, (2.7) is equivalent to

$$\Phi_\infty(\alpha)\subset\Phi_\infty(1)=CM_0(\mathbb{R}_+),\qquad 0<\alpha<1, \tag{2.8}$$

(a simple change of variables under the integral sign). Next, it is known (and can be easily verified by induction, using Leibniz chain rule) that the function $f=e^{-g}\in CM(\mathbb{R}_+)$ provided $g'\in CM(\mathbb{R}_+)$. Hence

$$\exp(-sx^\alpha)\in CM_0(\mathbb{R}_+),\qquad 0<\alpha\le 1,$$

so (2.4) holds for this function. Differentiation under the integral sign shows that the same is true for each $f\in\Phi_\infty(\alpha)$ and (2.8) follows. The same argument implies $\exp(-sx^\beta)\notin\Phi_\infty(\alpha)$ for $\beta>\alpha$.

Functions of the class $\Phi_\infty(\alpha)$ arise naturally in connection with isometric embedding of certain metric (Banach, finite dimensional) spaces into L^p spaces. For instance, a normed space E admits an isometric embedding into L^p, $0<p\le 2$, if and only if the function $\exp(-\|x\|^p)$ is positive definite on E [9] (see also [36, Chap. 2.7], [20, Chap. 6], and references therein). A criterion for an isometric embedding is obtained in [2, Theorem 6.1].

2.2 Infinite Matrices and Schur Test

We say that an infinite matrix $\mathcal{A}=[a_{kj}]_{k,j\in\mathbb{N}}$ with complex entries a_{kj} generates a bounded linear operator A on the Hilbert space ℓ^2 (or simply that an infinite matrix is a bounded operator on ℓ^2) if there exists a bounded linear operator A such that

$$\langle Ax, y\rangle = \sum_{k,j=1}^{\infty} a_{kj} x_k \overline{y_j}, \qquad x = (x_k)_{k\in\mathbb{N}}, \quad y = (y_k)_{k\in\mathbb{N}}, \quad x, y \in \ell^2. \tag{2.9}$$

Clearly, if $\mathcal{A}$ defines a bounded operator A, then A is uniquely determined by equalities (2.9).

The following result known as the *Schur test* (due in substance to I. Schur) provides certain general conditions for an infinite matrix $\mathcal{A} = [a_{ij}]_{i,j\in\mathbb{N}}$ to define a bounded linear operator A on ℓ^2 (see, e.g., [24, Theorem 5.2.1]). One of the simplest versions can be stated as follows.

Lemma 2.4 *Let $\mathcal{A} = [a_{ij}]_{i,j\in\mathbb{N}}$ be an infinite Hermitian matrix which satisfies*

$$C := \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty. \tag{2.10}$$

Then $\mathcal{A}$ defines a bounded self-adjoint operator A on ℓ^2 with $\|A\| \le C$.

Note that the Schur test applies to general (not necessarily Hermitian) matrices with two independent conditions for their rows and columns

$$C_1 := \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty, \quad C_2 := \sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty,$$

and the bound for the norm is $\|A\|^2 \le C_1 C_2$.

The condition for compactness of A is similar.

Lemma 2.5 *Suppose that*

$$\delta_p := \sup_{j\ge p} \sum_{k\ge p} |a_{jk}| < \infty, \quad \forall p \in \mathbb{N}, \quad \text{and} \quad \lim_{p\to\infty} \delta_p = 0. \tag{2.11}$$

Then the Hermitian matrix $\mathcal{A} = [a_{kj}]_{k,j\in\mathbb{N}}$ generates a compact self-adjoint operator on ℓ^2.

For the proof see, e.g., [21, Lemma 2.23].

3 Schoenberg Matrices from Operator Theory Viewpoint

3.1 Bounded Schoenberg Operators

Sometimes an infinite Schoenberg matrix generates a linear operator $S_X(f)$ on ℓ^2. We call $S_X(f)$ a *Schoenberg operator*. The main problem we address here concerns conditions on the test set $X \subset \mathbb{R}^n$ and the Schoenberg symbol f for $S_X(f)$ to be bounded.

We will be dealing primarily with separated sets X,

$$d_* = d_*(X) := \inf_{i \neq j} \|x_i - x_j\| > 0.$$

Recall the notation $\mathcal{X}_n$ for the class of all separated sets in $\mathbb{R}^n$ and $\mathcal{L} = \mathcal{L}(X)$ for the linear span of X, $d = \dim \mathcal{L} \le n$.

The result below gives an upper bound for a number of points of a separated set X in a spherical layer

$$U_r(p, q, a, X) := \{y \in \mathcal{L}(X) : \ pr \le \|y - a\| < qr\}, \quad q > p \ge 0,$$

centered at $a \in \mathcal{L}(X)$.

Lemma 3.1 *Let $X = \{x_k\}_{k \in \mathbb{N}} \in \mathcal{X}_n$, $d_*(X) = \varepsilon > 0$, and let $a \in \mathcal{L}(X)$. Then for the number $N_m(X)$ of points $\{x_k\}$ contained in $U_\varepsilon(m, m+1, a, X)$, $m = 0, 1, \ldots$, the inequality*

$$N_m(X) = \operatorname{card}\big(X \bigcap U_\varepsilon(m, m+1, a, X)\big) \le (2m+3)^d - (2m-1)^d < d\, 5^d\, m^{d-1} \tag{3.1}$$

holds.

Proof Take $x_j \in X \cap U_\varepsilon(m, m+1, a, X)$ and consider the balls $B_{\varepsilon/2}(x_j) = \{x \in \mathcal{L} : \|x - x_j\| < \varepsilon/2\}$, centered at x_j. They are contained in the spherical layer $U_\varepsilon(m - 1/2, m + 3/2, a, X)$ and pairwise disjoint. Since the volume of this layer is

$$\operatorname{Vol}(U_\varepsilon(m-1/2, m+3/2, a, X)) = \kappa_d\Big[\big((m+3/2)\varepsilon\big)^d - \big((m-1/2)\varepsilon\big)^d\Big],$$
$$\kappa_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$$

is the volume of the unit ball in $\mathbb{R}^d$, and the volume of the ball $\operatorname{Vol}(B_{\varepsilon/2}(x_j)) = \kappa_d(\varepsilon/2)^d$, the number $N_m(X)$ satisfies (3.1), as claimed. □

As far as the Schoenberg symbol f in the definition of Schoenberg's matrices goes, we assume here that it is a nonnegative, monotone decreasing function on $\mathbb{R}_+$, and $f(0) = 1$, i.e., $f \in \mathcal{M}_+$ (cf. 1.8). Further assumptions on the behavior of f at infinity will vary.

We proceed with a simple technical result.

Lemma 3.2 *Let $h \in \mathcal{M}_+$ and $d \in \mathbb{N}$. Then*

$$\sum_{m=1}^{\infty} m^{d-1}\, h(m) < \infty \iff \int_0^{\infty} t^{d-1}\, h(t)\, dt < \infty. \tag{3.2}$$

More precisely, for all $p \in \mathbb{N}$

$$2^{-d+1} \int_p^\infty t^{d-1}\, h(t)\, dt \le \sum_{m=p}^\infty m^{d-1}\, h(m) \le d \int_{p-1}^\infty t^{d-1}\, h(t)\, dt. \tag{3.3}$$

Proof An elementary inequality

$$\frac{m^{d-1}}{d} \le \frac{m^d - (m-1)^d}{d} \le m^{d-1}, \qquad m \in \mathbb{N},$$

gives for $h \in \mathcal{M}_+$

$$\int_{m-1}^m t^{d-1}\, h(t)\, dt \ge h(m) \int_{m-1}^m t^{d-1}\, dt = h(m)\, \frac{m^d - (m-1)^d}{d} \ge \frac{m^{d-1} h(m)}{d},$$

so summation over m leads to the right inequality in (3.3). Similarly,

$$\int_m^{m+1} t^{d-1}\, h(t)\, dt \le h(m) \int_m^{m+1} t^{d-1}\, dt = h(m)\, \frac{(m+1)^d - m^d}{d} \le (m+1)^{d-1} h(m),$$

and hence

$$\sum_{m=p}^\infty (m+1)^{d-1}\, h(m) \ge \int_p^\infty t^{d-1}\, h(t)\, dt.$$

It remains only to note that $m+1 \le 2m$ for $m \in \mathbb{N}$. □

For a one-dimensional X, i.e., $d(X) = 1$, condition (3.2) is just $f \in L^1(\mathbb{R}_+)$.

The following notion will be crucial in the second part of Theorem 3.4 below. Recall that we write $X \in \mathcal{X}_d$, $d \le n$, if $X \in \mathcal{X}_n$ and $\dim \mathcal{L}(X) = d$.

Definition 3.3 A set $Y = \{y_j\}_{j \in \mathbb{N}} \in \mathcal{X}_d$ is called δ*-regular* if there are constants $c_0 = c_0(d, \delta, Y) > 0$ and $r_0 = r_0(d, Y) \ge 0$, independent of j such that

$$\operatorname{card}(Y_r^{(j)}(\delta)) \ge c_0(d, \delta, Y)\, r^{d-1}, \qquad Y_r^{(j)}(\delta) := \{y_k \in Y : r \le \|y_k - y_j\| < r + \delta\}, \tag{3.4}$$

for $r \ge r_0$ and $j \in \mathbb{N}$.

For instance, the lattice $\mathbb{Z}^n$ and its part $\mathbb{Z}^n_+$ are δ-regular for all $\delta > 0$. On the other hand, if $X = \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$, $\mathcal{L}(X) = \mathbb{R}^n$ but there is a positive integer p such that $X^{(p)} := \{x_k\}_{k \ge p} \subset \mathbb{R}^{n-1}$, then X is certainly irregular.

Note that for any regular set Y the number $N_r^{(j)}$ of points in the set $Y \cap \{y : |y - y_j| \le r\}$ is subject to the bounds

$$c_1 r^d \le N_r^{(j)} \le c_2 r^d \tag{3.5}$$

for all large enough r. Here and in the proof of Theorem 3.4 c_k stand for different positive constants which depend on d, δ, and Y.

Theorem 3.4 (Theorem 1.4) *Let $f \in \mathcal{M}_+$, $X \in \mathcal{X}_d$, $d \le n$.*

(i) *If $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$, then the Schoenberg operator $S_X(f)$ is bounded on ℓ^2 and*

$$\|S_X(f)\| \le 1 + d^2 \left(\frac{5}{d_*(X)}\right)^d \int_0^\infty t^{d-1} f(t)\, dt. \tag{3.6}$$

(ii) *Moreover, $S_X(f)$ has a bounded inverse whenever, in addition,*

$$d_*(X) > 5 d^{2/d} \, \|t^{d-1} f\|_{L^1(\mathbb{R}_+)}^{1/d}. \tag{3.7}$$

(iii) *Conversely, let $S_Y(f)$ be bounded for at least one δ-regular set Y. Then $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$.*

Proof (i). We apply the Schur test to $\mathcal{S}_X(f) = [f(\|x_k - x_j\|)]_{k,j\in\mathbb{N}}$. For a fixed $j \in \mathbb{N}$ and $\varepsilon = d_*(X) > 0$ denote

$$X_m^{(j)} := \{x_k \in X : m\varepsilon \le \|x_k - x_j\| < (m+1)\varepsilon\}, \quad m \in \mathbb{N}, \quad X_0^{(j)} = \{x_j\}. \tag{3.8}$$

By Lemma 3.1 $\mathrm{card}(X_m^{(j)}) < d\, 5^d\, m^{d-1}$. Combining this estimate with the monotonicity of f yields

$$\begin{aligned}\sum_{k=1}^\infty f(\|x_k - x_j\|) &= 1 + \sum_{m=1}^\infty \sum_{x_k \in X_m^{(j)}} f(\|x_k - x_j\|) \le 1 + \sum_{m=1}^\infty \mathrm{card}(X_m^{(j)})\, f(m\varepsilon) \\ &\le 1 + d 5^d \sum_{m=1}^\infty m^{d-1} f(m\varepsilon).\end{aligned} \tag{3.9}$$

The result now follows from the Schur test and Lemma 3.2 with $h(\cdot) = f(\varepsilon \cdot)$.

(ii). Going over to the second statement, one has as above

$$\sum_{k=1}^\infty |f(\|x_k - x_j\|) - \delta_{kj}| = \sum_{\substack{k=1\\k\ne j}}^\infty f(\|x_k - x_j\|) \le d^2 \left(\frac{5}{d_*(X)}\right)^d \int_0^\infty t^{d-1} f(t)\, dt,$$

δ_{kj} is the Kronecker symbol, so $\|S_X(f) - I\| < 1$ as soon as (3.7) holds, I is the unity operator in ℓ^2. Hence $S_X(f)$ is invertible.

(iii). With no loss of generality assume that $\mathcal{L}(X) = \mathbb{R}^d$. At this point we make use of a particular labeling of the set X (generally speaking the way of enumeration of X makes no difference in our setting). Precisely, we label X by increasing of the distance from the origin

$$0 = \|x_1\| < \|x_2\| \le \|x_3\| \le \dots.$$

For a ball $B_r = B_r^d$ of radius $r > 0$ centered at the origin we put $E_r := X \cap B_r$ and $N_r := \mathrm{card}(E_r)$. Given $x_j \in X$, denote by $p(j)$ the number of layers $X_m^{(j)}$ which are contained in B_r. It is clear that for any $x_j \in E_{r/2}$ one has $p(j) \ge [r/2\varepsilon]$.

From the Definition 3.3 and $f \in \mathcal{M}_+$ we see that

$$\sum_{k=1}^{N_r} f(\|x_k - x_j\|) \geq \sum_{m=1}^{p(j)} \sum_{x_k \in X_m^{(j)}} f(\|x_k - x_j\|) \geq c_3 \sum_{m=1}^{p(j)} m^{d-1} f(\varepsilon(m+1))$$
$$\geq c_4 \sum_{m=2}^{p(j)+1} m^{d-1} f(\varepsilon m). \tag{3.10}$$

Since $S_X(f)$ is bounded then on the test vector $h = h_{N_r} = \frac{1}{\sqrt{N_r}}(1, 1, \ldots, 1, 0, 0, \ldots)$, $\|h\| = 1$, we have in view of (3.10) and (3.5) (with $j = 1$, $x_1 = 0$)

$$\|S_X(f)\| \geq |\langle S_X(f)h, h\rangle| = \frac{1}{N_r} \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} f(\|x_k - x_j\|) \geq \frac{1}{N_r} \sum_{|x_j|<r/2} \sum_{k=1}^{N_r} f(|x_k - x_j|)$$
$$\geq \frac{c_5}{N_r} N_{r/2} \sum_{m=2}^{[r/2\varepsilon]} m^{d-1} f(\varepsilon m) \geq c_6 \sum_{m=2}^{[r/2\varepsilon]} m^{d-1} f(\varepsilon m).$$

Since r is arbitrarily large, the result follows from Lemma 3.2. □

Remark 3.5 The statement (iii) of the above theorem is particularly simple for $d = 1$. Let

$$\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}, \quad 0 = \lambda_1 < \lambda_2 < \ldots,$$
$$0 < d_*(\Lambda) := \inf(\lambda_{k+1} - \lambda_k) < d^*(\Lambda) := \sup(\lambda_{k+1} - \lambda_k) < \infty, \tag{3.11}$$

A one-dimensional sequence $X = \{x_k\}_{k\in\mathbb{N}}$, $x_k = \lambda_k e$, $\{\lambda_k\}$ from (3.11), e is a unit vector, is called a Toeplitz-like sequence.

Assume now that the Schoenberg operator $S_X(f)$ is bounded. Take the same test vector $h_N = \frac{1}{\sqrt{N}}(1, 1, \ldots, 1, 0, 0, \ldots)$, $\|h_N\| = 1$ and write

$$\|S_X(f)\| \geq \langle S_X(f)h_N, h_N\rangle = \frac{1}{N} \sum_{i,j=1}^{N} f(\|x_i - x_j\|) = f(0) + \frac{2}{N} \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} f(\lambda_{i+k} - \lambda_i).$$

By (3.11), $kd_*(\Lambda) \leq \lambda_{i+k} - \lambda_i \leq kd^*(\Lambda)$, and in view of monotonicity

$$\|S_X(f)\| \geq 2 \sum_{k=1}^{N-1} \left(1 - \frac{k}{N}\right) f(kd^*(\Lambda)) \geq 2 \sum_{k=1}^{N/2} \left(1 - \frac{k}{N}\right) f(kd^*(\Lambda)) \geq \sum_{k=1}^{N/2} f(kd^*(\Lambda)).$$

Thereby the series $\sum_k f(kd^*(\Lambda))$ converges and Lemma 3.2 gives $f \in L^1(\mathbb{R}_+)$.

For α-stable functions we have a simple condition for the boundedness of $S_X(f)$ in terms of the Schoenberg measure σ_f (2.6).

Corollary 3.6 *Let $f \in \Phi_\infty(\alpha)$, $0 < \alpha \le 2$, $d \in \mathbb{N}$, and let σ_f be the Schoenberg measure in (2.6). Then*

$$\int_0^\infty t^{d-1} f(t)\,dt < \infty \iff \int_0^\infty s^{-\frac{d}{\alpha}}\,\sigma_f(ds) < \infty. \tag{3.12}$$

In particular, the Schoenberg operator $S_X(f)$ is bounded for all $X \in \mathcal{X}_d$, provided that the measure σ_f satisfies (3.12).

Proof It is clear that $f \in \mathcal{M}_+$. Next,

$$\begin{aligned}\int_0^\infty t^{d-1} f(t)\,dt &= \int_0^\infty t^{d-1}\,dt \int_0^\infty e^{-st^\alpha}\,\sigma(ds) = \int_0^\infty \sigma(ds) \int_0^\infty t^{d-1} e^{-st^\alpha}\,dt \\ &= \frac{1}{\alpha}\Gamma\left(\frac{d}{\alpha}\right) \int_0^\infty s^{-\frac{d}{\alpha}}\,\sigma(ds) < \infty. \end{aligned} \tag{3.13}$$

Theorem 3.4 completes the proof. □

Note that the above argument goes through for an arbitrary $\alpha > 0$.

We prove later in Theorem 4.8 that each Schoenberg operator $S_X(f)$ with the symbol as in Corollary 3.6 is actually invertible.

As another direct consequence of Theorem 3.4 we have

Corollary 3.7 *Let $f, g \in \mathcal{M}_+$ and $f(t) = g(t)$ for $t \ge t_0$. If the Schoenberg operator $S_Y(f)$ is bounded for at least one regular set $Y \in \mathcal{X}_d$, then so are $S_X(g)$ for all $X \in \mathcal{X}_d$.*

The monotonicity condition in (1.8) is somewhat restrictive. It is not at all necessary for Schoenberg's operator to be bounded.

Proposition 3.8 *Let f and h be real-valued functions on $\mathbb{R}_+$. Assume that $|f| \le h$ and the operator $S_X(h)$ is bounded. Then so is $S_X(f)$. In particular, let f be a bounded function on $\mathbb{R}_+$, which is monotone decreasing for $t \ge t_0(f)$ and $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$. Then $S_X(f)$ is bounded.*

Proof The Schoenberg matrix $\mathcal{S}_X(h)$ dominates $\mathcal{S}_X(f)$, i.e., $h(\|x_j - x_k\|) \ge |f(\|x_j - x_k\|)|$. Hence if $S_X(h)$ is bounded then so is $S_X(f)$ (see [4, Theorem 29.2]).

Concerning the second statement, it is clear that $f \ge 0$ on $[t_0(f), \infty)$. Put $h(t) := \sup_{t \ge s} |f(s)|$. Then h is a nonnegative function, monotone decreasing on $\mathbb{R}_+$, $h(0) > 0$ (we assume $f \not\equiv 0$), and $h = f$ on $[t_0(f), \infty)$, so (3.2) holds for h. By Theorem 3.4, $S_X(h)$ is bounded and as $h \ge |f|$ on $\mathbb{R}_+$, then by the first part of the proof, so is $S_X(f)$, as needed. □

Corollary 3.9 *Let $g \in \Phi_n$, $\alpha > 0$, and $e_\alpha(t) := e^{-\alpha t}$. Then $f_\alpha := e_\alpha g \in \Phi_n$ and for any $d \in \mathbb{N}$ and any $X \in \mathcal{X}_d$ the Schoenberg operator $S_X(f_\alpha)$ is bounded.*

Proof Since $e_\alpha \in CM_0(\mathbb{R}_+) \subset \Phi_\infty$, then for each finite X the Schoenberg matrix

$$\mathcal{S}_X(f_\alpha) = \mathcal{S}_X(e_\alpha) \circ \mathcal{S}_X(g),$$

being the Schur product of two nonnegative matrices $\mathcal{S}_X(e_\alpha)$ and $\mathcal{S}_X(g)$, is also non-negative. This proves the inclusion $f_\alpha \in \Phi_n$.

Next, since $|f_\alpha(t)| \le Me^{-\alpha t}$ with $M = \|g\|_{C(\mathbb{R}_+)}$, then $t^{d-1} f_\alpha(\cdot) \in L^1(\mathbb{R}_+)$ with an arbitrary $d \in \mathbb{N}$. It remains to apply Proposition 3.8. □

3.2 Fredholm Property

Let $\mathfrak{S}_\infty(\ell^2)$ be the class of compact operators in ℓ^2. We discuss here the situation when $S_X(f)$ is a Fredholm operator, more precisely,

$$S_X(f) = I + T, \qquad T \in \mathfrak{S}_\infty(\ell^2). \tag{3.14}$$

To provide (3.14) one should impose a much stronger condition on X than just $d_*(X) > 0$.

Theorem 3.10 *Let $X = \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^d$ satisfy*

$$\lim_{\substack{i,j\to\infty \\ i\ne j}} \|x_i - x_j\| = +\infty, \tag{3.15}$$

and let $f \in \mathcal{M}_+$ with $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$. Then (3.14) holds. In particular, $S_X(f)$ has bounded inverse whenever $\ker S_X(f) = \{0\}$.

Conversely, let f be a strictly positive, monotone decreasing function on $\mathbb{R}_+$, $f(0) = 1$, and $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$. Then (3.14) implies (3.15).

Proof Note that (3.15) implies $d_*(X) > 0$. To apply Lemma 2.5 we argue as in the proof of the Theorem 3.4. According to Lemma 3.1 for each $p \in \mathbb{N}$ there is $q = q(p) \in \mathbb{N}$ so that for $j \ge p$

$$\begin{aligned}\sum_{k=p}^{\infty} |f(\|x_k - x_j\|) - \delta_{kj}| &= \sum_{\substack{k=p \\ k\ne j}}^{\infty} f(\|x_k - x_j\|) = \sum_{m=q}^{\infty} \sum_{x_k \in X_m^{(j)}} f(\|x_k - x_j\|) \\ &\le d5^d \sum_{m=q}^{\infty} m^{d-1} f(d_*(X)m) \\ &\le d^2 \left(\frac{5}{d_*(X)}\right)^d \int_{(q-1)d_*(X)}^{\infty} t^{d-1} f(t)\, dt\end{aligned}$$

(see 3.3 for the last step). Condition (3.15) implies $q(p) \to \infty$ as $p \to \infty$ and so operator $T = S_X(f) - I$ is compact by Lemma 2.5.

Conversely, suppose that there are two sequences of positive integers $\{i_m\}_m$, $\{j_m\}_m$ so that $i_m \ne j_m$, both tend to infinity as $m \to \infty$ and $\sup_m \|x_{i_m} - x_{j_m}\| \le C < \infty$. Then

$$0 < f(C) \le f(\|x_{i_m} - x_{j_m}\|) = \langle S_X(f) e_{j_m}, e_{i_m}\rangle = \langle T e_{j_m}, e_{i_m}\rangle,$$

which contradicts the compactness of T. The proof is complete. □

Example 3.11 We show that in the converse statement of Theorem 3.10 the condition $f > 0$ cannot be relaxed to $f \geq 0$. Take the truncated power function

$$f(t) = (1-t)_+^l, \qquad l > 0.$$

It is known [17,41] that $f \in \Phi_n$ if and only if $l \geq \frac{n+1}{2}$. As a test sequence $X = \{x_k\}_{k\in\mathbb{N}}$ we put $x_k = a_k\xi$, $\|\xi\| = 1$, with

$$a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_k = k, \quad k = 3, 4, \ldots,$$

so that $f(\|x_2 - x_1\|) = 2^{-l}$, $f(\|x_i - x_j\|) = 0$ for the rest of the pairs $j \neq i$. The Schoenberg operator now takes the form

$$S_X(f) = \begin{bmatrix} A & \\ & I \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2^{-l} \\ 2^{-l} & 1 \end{bmatrix}$$

and I is a unit matrix. It is clear that $S_X(f) = I + T$, rank $T = 2$, but (3.15) is false.

3.3 Unbounded Schoenberg Operators

Conditions on an infinite matrix $\mathcal{A}$ for the corresponding linear operator A on ℓ^2 to be bounded are rather stringent. These conditions fail to hold for a number of Schoenberg's matrices (cf. Example 3.27).

To broaden the area of our study, consider an infinite Hermitian matrix $\mathcal{A} = [a_{kj}]_{k,j\in\mathbb{N}}$, $a_{jk} = \bar{a}_{kj}$, satisfying the following conditions

$$\sum_{k=1}^{\infty} |a_{kj}|^2 < \infty, \qquad \forall j \in \mathbb{N}. \tag{3.16}$$

Such matrix defines in a natural way a linear operator A' on ℓ^2 which act on the standard basis vectors $\{e_k\}_{k\in\mathbb{N}}$, $(e_k)_m = \delta_{km}$, as

$$A'e_j = \sum_{k=1}^{\infty} a_{kj} e_k, \qquad j \in \mathbb{N},$$

extended by linearity to the linear span $\mathcal{L}$ of $\{e_k\}_{k\in\mathbb{N}}$, so A' is densely defined and $\mathrm{dom}(A') \supset \mathcal{L}$. Being symmetric (since $\mathcal{A}$ is a Hermitian matrix), A' is closable, and we denote by $A = \overline{A'}$ its closure. The operator A is called a minimal operator associated with $\mathcal{A}$.

Matrices (3.16) are usually referred to as *unbounded Hermitian matrices* (unless A is a bounded operator).

A maximal operator associated with such matrix $\mathcal{A}$ is defined on the domain

$$\operatorname{dom}(A_{\max}) = \left\{ f = \sum_{k=1}^{\infty} x_k e_k : \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{kj} x_j \right|^2 < \infty \right\}.$$

by

$$A_{\max} f = \sum_{k=1}^{\infty} b_k e_k, \qquad b_k = \sum_{j=1}^{\infty} a_{kj} x_j, \tag{3.17}$$

It is known (see, e.g., [4, Theorem 53.2]) that $A_{\max} = A^*$.

Conversely, given a closed symmetric operator A on a Hilbert space $\mathcal{H}$, an orthonormal basis $\{h_k\}_{k\in\mathbb{N}}$ is called a basis of the matrix representation of A (cf. [4, Sect. IV.53]) if

- $h_k \in \operatorname{dom}(A)$, $k \in \mathbb{N}$;
- A is a minimal closed operator sending h_k to Ah_k, $k \in \mathbb{N}$.

The latter means that if B is a closed symmetric operator such that $B \subseteq A$ and $Bh_k = Ah_k$ then $B = A$.

A curious property of certain Schoenberg's matrices is that the validity of (3.16) for at least one value of j implies relation (3.16) to hold for all $j \in \mathbb{N}$. We begin with a technical lemma.

Let us say that a finite positive Borel measure σ on $\mathbb{R}_+$ possesses a doubling property if there is $\kappa > 0$ so that

$$\sigma\{[2u, 2v]\} \le \kappa\, \sigma\{[u, v]\}, \qquad \forall [u, v] \subset \mathbb{R}_+. \tag{3.18}$$

Lemma 3.12 *Let $f \in CM_0(\mathbb{R}_+)$ and $\xi, \eta \in \mathbb{R}^n$. Then there is a constant $C = C(f, \xi, \eta) > 0$ such that*

$$f(\|x - \xi\|) < Cf(\|x - \eta\|), \qquad \forall x \in \mathbb{R}^n. \tag{3.19}$$

The same conclusion is true for $f \in \Phi_\infty = \Phi_\infty(2)$ as long as its Schoenberg measure σ_f (2.6) possesses the doubling property.

Proof First, let $f \in CM_0(\mathbb{R}_+)$ and τ_f be its Bernstein's measure (2.5). Choose $a = a_f > 0$ so that

$$\int_0^a \tau_f(ds) > \frac{1}{2} \Rightarrow \int_a^\infty \tau_f(ds) < \frac{1}{2}. \tag{3.20}$$

We show that (3.19) actually holds with $C = 2e^{a\|\xi-\eta\|}$. Consider two cases.

1. Let first $\|x-\eta\| \le \|\xi-\eta\|$. Then since $f \le 1$, one has

$$f(\|x-\eta\|) = \int_0^\infty e^{-s\|x-\eta\|}\,\tau_f(ds) \ge \int_0^a e^{-s\|x-\eta\|}\,\tau_f(ds) > \frac{1}{2}\,e^{-a\|x-\eta\|} \ge \frac{1}{2}\,e^{-a\|\xi-\eta\|}\,f(\|x-\xi\|),$$

as needed.
2. Let now $\|x-\eta\| > \|\xi-\eta\|$, so $\|x-\xi\| \ge \|x-\eta\| - \|\xi-\eta\| > 0$. The function f is certainly monotone decreasing, so

$$f(\|x-\xi\|) \le f(\|x-\eta\| - \|\xi-\eta\|) = \int_0^\infty \exp(-s\|x-\eta\| + s\|\xi-\eta\|)\tau_f(ds) = \left\{\int_0^a + \int_a^\infty\right\} \exp(-s\|x-\eta\| + s\|\xi-\eta\|)\tau_f(ds) = I_1 + I_2.$$

Obviously, for every nonnegative and monotone decreasing function u on $\mathbb{R}_+$, condition (3.20) implies

$$\int_0^a u(s)\tau_f(ds) \ge \frac{u(a)}{2} > u(a)\int_a^\infty \tau_f(ds) \ge \int_a^\infty u(s)\tau_f(ds).$$

Hence $I_2 \le I_1$. To bound I_1 note that

$$I_1 \le e^{a\|\xi-\eta\|}\int_0^\infty e^{-s\|x-\eta\|}\,\tau_f(ds) = e^{a\|\xi-\eta\|}\,f(\|x-\eta\|),$$

and (3.19) follows.

Concerning functions $f \in \Phi_\infty$, the reasoning is identical (with the obvious replacement of τ with σ) up to the bound of I_1, where the doubling property comes into play. We now have

$$I_1 = \int_0^a \exp(-s(\|x-\eta\| - \|\xi-\eta\|)^2)\,\sigma_f(ds) \le e^{a\|\xi-\eta\|^2}\int_0^a e^{-\frac{s}{2}\|x-\eta\|^2}\,\sigma_f(ds) \le e^{a\|\xi-\eta\|^2} f\left(\frac{\|x-\eta\|}{\sqrt{2}}\right).$$

It remains only to note that

$$f\left(\frac{r}{\sqrt{2}}\right) = \int_0^\infty e^{-\frac{s}{2}r^2}\,\sigma_f(ds) \le \kappa\int_0^\infty e^{-sr^2}\,\sigma_f(ds) = \kappa f(r), \quad r > 0$$

because of the doubling property (3.18). The proof is complete. □

Proposition 3.13 *Let $f \in CM_0(\mathbb{R}_+)$, $X = \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, and let $\mathcal{S}_X(f)$ be the corresponding Schoenberg matrix. If at least one column of $\mathcal{S}_X(f)$ belongs to ℓ^2, then (3.16) holds and $\{e_k\}_{k\in\mathbb{N}}$ is a basis of the matrix representation for the minimal* *operator A associated with $\mathcal{S}_X(f)$. The same conclusion is true for $f \in \Phi_\infty$ as long as its Schoenberg measure σ possesses the doubling property.*

Proof Let the first column of $\mathcal{S}_X(f)$ belong to ℓ^2. By Lemma 3.12 one has

$$\sum_{j=1}^{\infty} f^2(\|x_j - x_k\|) \le C^2 \sum_{j=1}^{\infty} f^2(\|x_j - x_1\|) < \infty \tag{3.21}$$

for each $k = 2, 3, \ldots$. The statement about the basis of the matrix representation is obvious. □

Remark 3.14 It is easy to see that for a general function in Φ_∞ the doubling property (3.18) for σ_f cannot be dropped for Lemma 3.12 to hold.

Put

$$a_n := \sqrt{\log n + 2 \log\log n}, \qquad n \ge 2.$$

Then clearly

$$\sum_{n=2}^{\infty} e^{-a_n^2} = \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < \infty, \quad \sum_{n=2}^{\infty} e^{-(a_n-1)^2} = \frac{1}{e} \sum_{n=2}^{\infty} \frac{e^{2a_n}}{n \log^2 n} = \infty.$$

Consider now the Schoenberg matrix $\mathcal{S}_X(f)$ with

$$f(t) = e^{-t^2} \in \Phi_\infty \backslash CM_0(\mathbb{R}_+), \qquad X = \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^1: \ x_1 = 0, \ x_2 = 1,$$
$$x_n = a_n, \quad n \ge 3.$$

Then

$$\sum_{n=1}^{\infty} f^2(\|x_n - x_1\|) < \infty, \qquad \sum_{n=1}^{\infty} f^2(\|x_n - x_2\|) = \infty.$$

Certainly, now $\sigma_f = \delta\{1\}$, the Dirac measure at the point 1, has no doubling property. Note that in this instance the conclusion of Lemma 3.12 is false either.

In the above example the set X is not separated, that is, $d_*(X) = 0$. As we will see later in Theorem 4.8, the Schoenberg operator $S_X(e^{-t^2})$ is bounded and invertible whenever $d_*(X) > 0$, so all columns belong to ℓ^2.

There is an intermediate condition on the Schoenberg matrix $\mathcal{S}_X(f)$ between (3.16) and the boundedness. Precisely, let

$$C(f, X) := \sup_j \sum_{k=1}^{\infty} f^2(\|x_k - x_j\|) < \infty. \tag{3.22}$$

In other words, $\sup_j \|S_X(f)e_j\| < \infty$.

Recall that δ-regular sets are defined in Definition 3.3 above.

Proposition 3.15 *Let $f \in \mathcal{M}_+$ (1.8), and for a positive integer d, $1 \le d \le n$,*

$$\int_0^\infty t^{d-1} f^2(t)\,dt < \infty. \tag{3.23}$$

Then (3.22) holds for each separated set $X \in \mathcal{X}_d$. Conversely, assume that

$$\sum_{k=1}^\infty f^2(\|y_k - y_j\|) < \infty \tag{3.24}$$

for some $j \in \mathbb{N}$ and at least one δ-regular set $Y = \{y_k\}_{k\in\mathbb{N}}$. Then (3.23) holds with $d = \dim \mathcal{L}(Y)$.

Proof Let (3.23) hold. We apply Lemma 3.2 with $h = f^2$ and obtain as above

$$\begin{aligned}\sum_{k=1}^\infty f^2(\|x_k - x_j\|) &\le 1 + \sum_{m=1}^\infty \operatorname{card}(X_m^{(j)})\, f^2(d_*(X)m)\\ &\le 1 + d^2\left(\frac{5}{d_*(X)}\right)^d \int_0^\infty s^{d-1} f^2(s)ds,\end{aligned}$$

so (3.22) follows.

Conversely, let f satisfy (3.24) for some $j \in \mathbb{N}$ and some δ-regular set Y. In view of the lower bound (3.3) one has by Lemma 3.2,

$$\begin{aligned}\sum_{k=1}^\infty f^2(\|y_k - y_j\|) &= 1 + \sum_{m=1}^\infty \sum_{y_k \in Y_m^{(j)}} f^2(\|y_k - y_j\|)\\ &\ge 1 + c_2(d)\sum_{m=1}^\infty m^{d-1} f^2(d_*(Y)(m+1))\\ &\ge 1 + c_3(d)\sum_{m=2}^\infty m^{d-1} f^2(d_*(Y)m) \ge 1 + c_4(d)\int_{2d_*(Y)}^\infty s^{d-1} f^2(s)\,ds.\end{aligned}$$

The proof is complete. □

Corollary 3.16 *If $e_j \in \operatorname{dom} S_X(f)$ for some $j \in \mathbb{N}$ and all $X \in \mathcal{X}_n$ then (3.22) holds.*

Remark 3.17 It is easy to manufacture a function $f \in \mathcal{M}_+$ and a separated set X so that (3.22) holds but (3.23) is violated. Indeed, let f tend to zero slow enough as $x \to \infty$, so that $f \notin L^2(\mathbb{R}_+)$. Choose a sequence of positive numbers $\{t_k\}$, $t_1 = 0$ so that $f(t_k) \le e^{-k}$. Now take a set $X = \{x_k\}_{k\in\mathbb{N}}$ with $x_k = t_k\xi$, $k \in \mathbb{N}$, ξ a unit vector. Then

$$\sum_{i=1}^\infty f^2(\|x_k - x_1\|) \le \sum_k e^{-2k} < \infty$$

regardless of whether condition (3.23) holds or not. Due to the second statement of Proposition 3.15 such set is supposed to be irregular.

The example below illustrates Proposition 3.15 and Theorem 3.4.

Example 3.18 Let $h(r) = (1+r)^{-1} \in CM_0(\mathbb{R}_+)$. Take $X = \mathbb{Z}_+^2 = \{(p,q) : p, q \in \mathbb{Z}_+\}$ labeled in the following way

$$X = \bigcup_{m=0}^{\infty} X_m, \qquad X_m = \{x_k^{(m)}\}_{k=0}^m, \quad x_k^{(m)} = (m-k, k), \quad X_0 = \{(0,0)\}.$$

As $|x_k^{(m)}|^2 = (m-k)^2 + k^2 = m^2 + 2k(k-m) \le m^2$, we can easily compute the sum in (3.22)

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m} h^2(\|x_k^{(m)}\|) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{1}{(1+\|x_k^{(m)}\|)^2} \ge \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{1}{(1+m)^2} = \sum_{m=0}^{\infty} \frac{1}{1+m} = +\infty,$$

which is consistent with Proposition 3.15, since $d = \dim \mathcal{L}(X) = 2$, and condition (3.23) is violated.

On the other hand, for $X = \mathbb{Z}_+$ we come to a version of the well-known Hilbert–Toeplitz matrix

$$\mathcal{S}_X(h) = [(1+|i-j|)^{-1}]_{i,j\in\mathbb{N}}, \qquad h(r) = \frac{1}{1+r} = \int_0^{\infty} e^{-sr}\, e^{-s}\, ds. \tag{3.25}$$

Now $d = 1$, so by Proposition 3.15, (3.24) holds. Yet the operator $S_X(h)$ is unbounded in view of Theorem 3.4 ($\mathbb{Z}_+$ is a 1-regular set). We show later in Proposition 3.26 that $S_X(h)$ is a positive definite and self-adjoint operator.

An important property of the minimal Schoenberg operator $S_X(f)$ constitutes the content of the following theorem.

Theorem 3.19 *Let $f \in \Phi_\infty(\alpha)$, $\alpha \in (0,2]$, $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$, and $X \in \mathcal{X}_n$. Assume that the Schoenberg matrix $\mathcal{S}_X(f)$ satisfies condition (3.16). Then the minimal Schoenberg operator $S_X(f)$ associated with the matrix $\mathcal{S}_X(f)$ is a symmetric positive definite operator, i.e.,*

$$\langle S_X(f)\xi, \xi\rangle \ge \varepsilon\|\xi\|^2, \qquad \xi \in \operatorname{dom} S_X(f), \quad \varepsilon > 0. \tag{3.26}$$

In particular, $S_X(f)$ is self-adjoint if and only if $\ker S_X^*(f) = \{0\}$.

Proof According to Theorem 4.8, the function $f \in \Phi_\infty(\alpha)$ is strongly X-positive definite, i.e., there exists $\varepsilon > 0$ such that

$$\sum_{j,k=1}^{N} f(\|x_j - x_k\|)\xi_j\overline{\xi}_k \ge \varepsilon \sum_{j=1}^{N} |\xi_j|^2, \qquad \xi = (\xi_j)_1^N \in \mathbb{C}^N, \quad \forall N \in \mathbb{N}. \tag{3.27}$$

Due to assumption (3.16) the basis $\{e_j\}_{j\in\mathbb{N}}$ is a basis of the matrix representation of the minimal operator $S_X(f)$ associated with the Schoenberg matrix $\mathcal{S}_X(f)$. Therefore inequality (3.27) means that for any finite vector $\xi = (\xi_1, \xi_2, \ldots, \xi_N, 0, 0, \ldots)$

$$\langle S_X(f)\xi, \xi\rangle \geq \varepsilon \sum_{k=1}^{N} |\xi_k|^2 = \varepsilon\, \|\xi\|^2.$$

Taking the closure we get the statement. □

Note that the proof of our main result about Φ_∞-functions (Theorem 4.8) in the next section is completely independent of the above Theorem 3.19.

The converse to Theorem 3.19 is true in a more general setting.

Proposition 3.20 *Assume that the Schoenberg matrix $\mathcal{S}_X(f)$, $f \in \Phi_n$, satisfies condition* (3.16)*, and the minimal Schoenberg operator $S_X(f)$ associated with the matrix $\mathcal{S}_X(f)$ satisfies* (3.26)*, i.e., it is positive definite. Then X is separated, i.e., $d_*(X) > 0$.*

Proof In the above assumptions one has

$$\langle S_X(f)h, h\rangle \geq c\|h\|^2, \qquad 0 < c < \infty \tag{3.28}$$

for each $h \in \mathrm{dom}(S_X(f))$. Hence putting $h = e_k - e_j \in \mathrm{dom}(S_X(f))$ we see that

$$\langle S_X(f)h, h\rangle = 2f(0) - 2f(\|x_k - x_j\|) \geq 2c,$$

so $f(\|x_k - x_j\|) \leq f(0) - c$, $c > 0$, which immediately implies $d_*(X) > 0$. □

Proposition 3.20 says that if $d_*(X) = 0$ and $S_X(f)$ is bounded for $f \in \Phi_n$, then $0 \in \sigma(S_X(f))$, $\sigma(A)$ being the spectrum of operator A. It is easy to manufacture such X for $f(t) = e^{-t}$ (cf. [21, Lemma 3.7]).

There is a simple function theoretic analogue of Proposition 3.20.

Proposition 3.21 *If $f \in \Phi_n$ is strongly X-positive definite, then X is separated.*

Proof By the definition we have for all k, j

$$f(0)(|\xi_1|^2 + |\xi_2|^2) - f(\|x_j - x_k\|)(\xi_1\bar{\xi}_2 + \bar{\xi}_1\xi_2) \geq c(|\xi_1|^2 + |\xi_2|^2), \quad \xi_1, \xi_2 \in \mathbb{C}.$$

By putting $\xi_1 = \xi_2 \neq 0$ we see that

$$f(0) - f(\|x_j - x_k\|) \geq c > 0,$$

so $X \in \mathcal{X}_n$, as needed. □

3.4 Schoenberg–Toeplitz Operators

Although we have no sufficient conditions for general Schoenberg operators $S_X(f)$ to be self-adjoint, Theorem 3.19 gives an essential step toward proving self-adjointness, since it reduces this problem to the study of ker $S_X^*(f)$.

Definition 3.22 (i) Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Recall that a matrix $\mathcal{A} := [a_{jk}]_{j,k\in\mathbb{N}_0}$ is called a *Toeplitz matrix* if there is a sequence $\{a_m\}_{m\in\mathbb{Z}}$ of complex numbers such that $a_{jk} = a_{j-k}$ for every $j, k \in \mathbb{N}_0$.

(ii) An operator A on a space of analytic functions in the unit disk $\mathbb{D}$ such that its domain contains the set of analytic polynomials Pol_+ is called a *Toeplitz operator* if its matrix with respect to the basis $\{z^k\}_{k\in\mathbb{N}_0} = \{e^{ik\varphi}\}_{k\in\mathbb{N}_0}$ is a Toeplitz matrix.

It is known that a Toeplitz operator is characterized by the identity

$$U^* A U = A, \tag{3.29}$$

where U is a unilateral shift in ℓ^2. According to the basic assumption (3.16) the Toeplitz matrix $\mathcal{A}$ defines an operator in ℓ^2 if $\{a_k\} \in \ell^2(\mathbb{Z})$, i.e.,

$$\sum_{j\in\mathbb{Z}} |a_j|^2 < \infty. \tag{3.30}$$

In this case the *Toeplitz symbol* is a function given by

$$a(A, e^{i\varphi}) := \sum_{k\in\mathbb{Z}} a_k e^{ik\varphi} \in L^2[-\pi, \pi]. \tag{3.31}$$

Lemma 3.23 *Let $a_{-j} = \bar{a}_j$, $j \in \mathbb{N}$, i.e., the Toeplitz matrix $\mathcal{A} = [a_{j-k}]_{j,k\in\mathbb{N}_0}$ is a Hermitian matrix. Assume also that $\mathcal{A}$ satisfies (3.30) and the minimal symmetric Toeplitz operator A associated with $\mathcal{A}$ in $\ell^2(\mathbb{N})$ is semibounded from below. Then it is self-adjoint, $A = A^*$.*

Proof Without loss of generality we can assume that A is positive definite. In this case it suffices to make sure that the conjugate (maximal) operator A^* has the trivial kernel. Since $A^* = A_{max}$ acts by means of the same matrix $\mathcal{A}$ (but defined on the maximal domain), the latter property is equivalent to

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots \\ a_{-1} & a_0 & a_1 & \dots \\ a_{-2} & a_{-1} & a_0 & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \Rightarrow p_j \equiv 0, \quad p = \{p_j\} \in \ell^2. \tag{3.32}$$

To prove implication (3.32) it is instructive to rephrase the problem in the function theoretic terms.

Let M denote the multiplication (shift) operator on $L^2(\mathbb{T})$, $\mathbb{T}$ is the unit circle. The equality in (3.32) means that the function

$$p(t) := \sum_{j\ge 0} p_j t^j \in H^2$$

is orthogonal to the system $\{M^k a\}_{k\ge 0}$, where $a \in L^2(\mathbb{T})$ is the Toeplitz symbol (3.31). In other words, the product $p_- := p\,a \in L^1(\mathbb{T})$ is orthogonal to all powers $\{t^k\}_{k\ge 0}$, i.e., $p_- \in H^1_-$ (for the Hardy spaces H^p, H^p_- see [11, Chapter II]).

Positive definiteness of the minimal operator A reads as follows

$$\langle Aq, q\rangle = \sum_{k,j=0}^{N} a_{k-j} q_j \bar{q}_k = \int_{\mathbb{T}} a(t)|q(t)|^2\, m(dt) \ge \varepsilon \|q\|^2_{L^2(\mathbb{T})},$$
$$q(t) := \sum_{j=0}^{N} q_j t^j, \quad \varepsilon > 0, \tag{3.33}$$

for an arbitrary $q \in Pol_+$, m is the normalized Lebesgue measure on $\mathbb{T}$. It is clear from (3.33) that $a(t) \ge \varepsilon$ for a.e. $t = e^{i\varphi} \in \mathbb{T}$. Therefore (see [11, Theorem II.4.6]) there is an outer function D such that

$$a(t) = |D(t)|^2, \qquad D \in H^2, \quad D^{-1} \in H^\infty.$$

We have $p_-(t) = p(t)\,a(t) = p(t)\,|D(t)|^2 \in H^1_-$ and hence

$$p(t)\,D(t) = \frac{p_-(t)}{\overline{D(t)}}\,.$$

But the left-hand side of the latter equality belongs to H^1, whereas the right-hand side lies in H^1_- which yields $p \equiv 0$, as claimed. The proof is complete. □

A sequence $X = \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ is called a *Toeplitz sequence*, if $\|x_i - x_j\| = |i-j|$ for $i, j \in \mathbb{N}$. The latter is equivalent (recall that by our convention $x_1 = 0$) to $x_k = (k-1)\xi$, $\xi \in \mathbb{R}^n$, and $\|\xi\| = 1$, so $d = \dim \mathcal{L}(X) = 1$. In this case $S_X(f)$ is a Toeplitz operator, which will be called a *Schoenberg–Toeplitz operator*. The Toeplitz symbol a (3.31) takes now the form

$$a(f, e^{i\varphi}) := \sum_{k\in\mathbb{Z}} f(|k|)e^{ik\varphi}. \tag{3.34}$$

Remark 3.24 (i) Self-adjointness of not necessarily positive Toeplitz operators with the Toeplitz symbol from $BMO(\mathbb{T})$ (see [11, Chapter VI]) was established by V. Peller [26]. In particular, this is the case for the Hilbert–Toeplitz operator (3.25) with the Toeplitz symbol

$$a(h, t) = 1 - 2\mathrm{Re}\,\frac{\log(1-t)}{t} \in BMO(\mathbb{T}),$$

but not for general Schoenberg–Toeplitz operators with Toeplitz symbols (3.42) below.

(ii) Semibounded Toeplitz operators have been studied in several papers (see [27] and references therein). For instance, it is proved in [28] that the Friedrichs extension A_F of A has absolutely continuous spectrum. However, according to Lemma 3.23, $A_F = A$.

In the rest of the section we will focus on the Schoenberg–Toeplitz operators $S_X(f)$ with symbols $f \in \Phi_\infty = \Phi_\infty(2)$. We clarify and complete Corollary 3.6 for such operators and describe their spectra in terms of the Schoenberg measures σ_f.

Proposition 3.25 *Let $f \in \Phi_\infty$ and let σ_f be its Schoenberg measure (2.6). The Schoenberg–Toeplitz matrix $\mathcal{S}_X(f)$ defines a minimal operator $S_X(f)$ in ℓ^2 if and only if $f \in L^2(\mathbb{R}_+)$. In this case $S_X(f)$ is self-adjoint, its spectrum is purely absolutely continuous and fills in the interval*

$$\sigma(S_X(f)) = \sigma_{ac}(S_X(f)) = [c_-, c_+],$$
$$0 < c_- := \int_0^\infty \vartheta_3\big(\pi, e^{-s}\big)\,\sigma_f(ds) < c_+ := \int_0^\infty \vartheta_3\big(0, e^{-s}\big)\,\sigma_f(ds) \le +\infty, \quad (3.35)$$

where ϑ_3 is the Jacobi theta-function.

Moreover, the operator $S_X(f)$ is bounded if and only if $f \in L^1(\mathbb{R}_+)$, or, equivalently,

$$\int_0^\infty \frac{\sigma_f(ds)}{\sqrt{s}} < \infty. \quad (3.36)$$

Proof As the Schoenberg symbol f is a nonnegative and monotone decreasing function, conditions $f \in L^2(\mathbb{R}_+)$ and $\{f(k)\}_{k\ge 0} \in \ell^2$ are equivalent, so (3.30) is met. Next, for $f \in \Phi_\infty$ the corresponding minimal operator is symmetric and strongly positive definite by Theorem 3.19. Hence $S_X(f)$ is self-adjoint in view of Lemma 3.23.

Consider the kernel function $e_s(u) := e^{-su^2}$, $s > 0$, so $\mathcal{S}_X(e_s) = [e^{-s|i-j|^2}]_{i,j\in\mathbb{N}}$. Since $e_s \in L^1(\mathbb{R}_+)$, the operator $S_X(e_s)$ is bounded by Theorem 3.4. The corresponding Toeplitz symbol is given by (3.34). It can now be expressed by means of the Jacobi theta-function

$$a(e_s, e^{i\varphi}) = \sum_{k\in\mathbb{Z}} e^{-s|k|^2} e^{ik\varphi} = \vartheta_3\Big(\frac{\varphi}{2}, e^{-s}\Big).$$

It is well known (see [38, Chapter 21]) that ϑ_3 is positive on the real line and

$$\frac{\vartheta_3'(\varphi)}{\vartheta_3(\varphi)} = -4\sin 2\varphi \sum_{k=1}^\infty \frac{q^{2k-1}}{1 + 2q^{2k-1}\cos 2\varphi + q^{4k-2}}, \qquad q = e^{-s},$$

so $a(e_s)$ is monotone decreasing on $[0, \pi]$ ($a(e_s)$ is "bell-shaped" on $[-\pi, \pi]$). By the Hartman–Wintner theorem (see, e.g., [24, Theorem 4.2.7]) its spectrum agrees with the range of $a(f)$, so it is the interval

$$\sigma(S_X(e_s)) = a(e_s, \mathbb{T}) = [a(e_s, -1), a(e_s, 1)] = \Big[\vartheta_3\Big(\frac{\pi}{2}, q\Big), \vartheta_3(0, q)\Big].$$

For a general function $f \in \Phi_\infty$ the Toeplitz symbol $a(f)$ of $S_X(f) = [f(|i-j|)]_{i,j\in\mathbb{N}}$ can be computed as

$$\begin{aligned} a(f, e^{i\varphi}) &= \sum_{k\in\mathbb{Z}} f(|k|)e^{ik\varphi} = \sum_{k\in\mathbb{Z}} e^{ik\varphi} \int_0^\infty e^{-s|k|^2}\,\sigma_f(ds) \\ &= \int_0^\infty \vartheta_3\Big(\frac{\varphi}{2}, e^{-s}\Big)\,\sigma_f(ds), \qquad \varphi \neq 0. \end{aligned} \tag{3.37}$$

It is easily seen that

$$\begin{aligned} \vartheta_3\Big(\frac{\pi}{2}, e^{-s}\Big) &\le \vartheta_3\Big(\frac{\varphi}{2}, e^{-s}\Big) \le \vartheta_3(0, e^{-s}) \\ &= \sum_{k\in\mathbb{Z}} e^{-sk^2} \sim \frac{1}{\sqrt{s}}, \quad s \to +0, \qquad 0 \le \varphi \le \pi. \end{aligned} \tag{3.38}$$

Again by the Hartman–Wintner theorem, the spectrum of $S_X(f)$ agrees with the range of $a(f)$, which is exactly the interval given by (3.35). Its absolute continuity is a standard fact in the theory of Toeplitz operators, (see, e.g., [27, p. 64]).

By Theorem 3.4 the boundedness of $S_X(f)$ is equivalent to $f \in L^1(\mathbb{R}_+)$. In turn, the latter is equivalent to (3.36) by Corollary 3.6, applied with $\alpha = 2$ and $d = 1$. The proof is complete. □

It is easy to express the inclusion $f \in \Phi_\infty \cap L^2(\mathbb{R}_+)$ in terms of σ_f (cf. 3.36)

$$\int_{\mathbb{R}_+^2} \frac{\sigma_f(ds_1)\,\sigma_f(ds_2)}{\sqrt{s_1+s_2}} < \infty. \tag{3.39}$$

Next, we provide a similar result for $f \in CM_0(\mathbb{R}_+)$.

Proposition 3.26 *Let $f \in CM_0(\mathbb{R}_+)$, τ_f be its Bernstein measure (2.5). The Schoenberg–Toeplitz matrix $S_X(f)$ defines a minimal operator $S_X(f)$ in ℓ^2 if and only if $f \in L^2(\mathbb{R}_+)$. In this case $S_X(f)$ is self-adjoint, its spectrum is purely absolutely continuous and fills in the interval*

$$\sigma(S_X(f)) = \sigma_{ac}(S_X(f)) = [c_-, c_+], \qquad 0 < c_\pm = \int_0^\infty \frac{1 \pm e^{-s}}{1 \mp e^{-s}}\,\tau_f(ds). \tag{3.40}$$

Moreover, the operator $S_X(f)$ is bounded if and only if $f \in L^1(\mathbb{R}_+)$, or, equivalently,

$$\int_0^\infty \frac{\tau_f(ds)}{s} < \infty. \tag{3.41}$$

Proof As in the proof of the preceding result, we start with the kernel function $e_s(u) := e^{-su}$, $s > 0$, and relate the Schoenberg and Toeplitz symbols:

$$a(e_s, e^{i\varphi}) = \sum_{k\in\mathbb{Z}} e^{-s|k|}e^{ik\varphi} = 1 + \frac{e^{-s+i\varphi}}{1-e^{-s+i\varphi}} + \frac{e^{-s-i\varphi}}{1-e^{-s-i\varphi}}$$
$$= \frac{1-e^{-2s}}{|1-te^{-s+i\varphi}|^2} = P(e^{-s}, e^{i\varphi}),$$

where $P(e^{-s}, e^{i\varphi})$ denotes the Poisson kernel for the unit disk. Hence $S_X(e_s) = [e^{-s|i-j|}]_{i,j\in\mathbb{N}}$ is bounded and its spectrum is the interval

$$\sigma(S_X(e_s)) = a(e_s, \mathbb{T}) = \left[\frac{1-e^{-s}}{1+e^{-s}}, \frac{1+e^{-s}}{1-e^{-s}}\right].$$

The Toeplitz symbol $a(f)$ of the operator $S_X(f) = [f(|i-j|)]_{i,j\in\mathbb{N}}$ can be computed as above

$$a(f, e^{i\varphi}) = \sum_{k\in\mathbb{Z}} f(|k|)e^{ik\varphi} = \sum_{k\in\mathbb{Z}} e^{ik\varphi} \int_0^\infty e^{-s|k|}\,\tau(ds)$$
$$= \int_0^\infty P(e^{-s}, e^{i\varphi})\,\tau_f(ds), \qquad \varphi \neq 0. \tag{3.42}$$

One completes the proof in just the same fashion as in Proposition 3.25. □

Similarly, the condition $f \in CM_0(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is equivalent to (cf. 3.41)

$$\int_{\mathbb{R}_+^2} \frac{\tau_f(ds_1)\,\tau_f(ds_2)}{s_1+s_2} < \infty. \tag{3.43}$$

Example 3.27 It is not hard to manufacture a Schoenberg–Toeplitz matrices with the Schoenberg symbol $f \in CM_0(\mathbb{R}_+)\backslash L^2(\mathbb{R}_+)$. Indeed, one can take

$$\mathcal{S}_X(f_\beta) = [(1+|i-j|)^{-\beta}]_{i,j\in\mathbb{N}}, \qquad f_\beta(r) = \frac{1}{(1+r)^\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-sr}\,s^{\beta-1}e^{-s}\,ds \tag{3.44}$$

with $0 < \beta \le 1/2$. In this example neither column vector belongs to ℓ^2.

Remark 3.28 (i) According to a result of Brown and Halmos (see, e.g., [24, Theorem 4.1.4]) the operator $S_X(f)$ is bounded if and only if $a(f) \in L^\infty(\mathbb{T})$. Due to the asymptotic relation (3.38) for $f \in \Phi_\infty$ the latter is equivalent to (3.36). This observation provides another proof of the last statement of both preceding propositions.

(ii) The relation between the Schoenberg symbol $f \in \Phi_\infty(\alpha)$ for $\alpha = 1, 2$ and the Toeplitz symbol $a(f)$ is implemented by the Poisson kernel and the Jacobi theta-function, respectively. We are unaware of a similar relation for $1 < \alpha < 2$.

(iii) A Schoenberg–Toeplitz operator $S_X(f)$ with $f \in \mathcal{M}_+$ is bounded if and only if the Fourier coefficients of its Toeplitz symbol $a(f)$ (3.34) are positive and monotone decreasing and $a(f) \in W$, the Wiener algebra of absolutely convergent Fourier series. This result stems directly from Theorem 3.4.

Example 3.29 We construct a bounded Schoenberg–Toeplitz operator $S_X(\varphi)$ with $0 \in \sigma(S_X(\varphi))$. Take any Toeplitz sequence $X \subset \mathbb{R}^1$ so that $|x_i - x_j| = |i - j|$ and put

$$\varphi(t) = \left(1 - \frac{t}{2}\right)_+ \in \Phi_1, \qquad (a)_+ := \max(a, 0).$$

Then $S_X(\varphi) = J(\{1\}, \{1/2\})$ is the Jacobi operator with 1's on the main diagonal and 1/2's off the main diagonal. It is well known that $\sigma(S_X(\varphi)) = [0, 2]$, as claimed. Certainly, $\varphi \notin \Phi_\infty$.

4 Schoenberg Matrices and Harmonic Analysis on $\mathbb{R}^n$

4.1 Radial Strongly X-Positive Definite Functions

It turns out that the notions of Riesz–Fischer, Bessel, and Riesz sequences (see Definition 1.8) applied to sequences of exponential functions in L^2-spaces are tightly related to the strong X-positive definiteness.

Given an arbitrary sequence $X = \{x_k\}_{k\in\mathbb{N}}$ of distinct points in $\mathbb{R}^n$, we introduce a system

$$\mathcal{E}_X = \{e(\cdot, x_k)\}_{k\in\mathbb{N}}, \qquad e(x, x_k) = e^{i\langle x, x_k\rangle}, \quad x \in \mathbb{R}^n, \tag{4.1}$$

of exponential functions.

Proposition 4.1 *Let h be a positive definite function (2.2) with the Bochner measure μ_h. For an arbitrary sequence $X = \{x_k\}_{k\in\mathbb{N}}$ of distinct points in $\mathbb{R}^n$ and for the system of exponential functions $\mathcal{E}_X$ (4.1) the following holds.*

- (i) *$\mathcal{E}_X$ is a Riesz–Fischer sequence in $L^2(\mathbb{R}^n, \mu_h)$ if and only if h is strongly X-positive definite.*
- (ii) *$\mathcal{E}_X$ is a Bessel sequence if and only if the Gram matrix*

$$Gr(\mathcal{E}_X, L^2(\mathbb{R}^n, \mu_h)) = [\langle e(\cdot, x_k), e(\cdot, x_j)\rangle_{L^2(\mathbb{R}^n,\mu_h)}]_{k,j\in\mathbb{N}} = [h(x_k - x_j)]_{k,j\in\mathbb{N}} \tag{4.2}$$

defines a bounded, self-adjoint and nonnegative operator on ℓ^2.
- (iii) *$\mathcal{E}_X$ is a Riesz sequence if and only if $Gr(\mathcal{E}_X, L^2(\mathbb{R}^n, \mu_h))$ defines a bounded and invertible, nonnegative operator.*

Proof It is clear that

$$\sum_{k,j=1}^m h(x_k - x_j)\xi_j\overline{\xi}_k = \int_{\mathbb{R}^n} \left|\sum_{k=1}^m \xi_k e(u, x_k)\right|^2 \mu_h(du) = \left\|\sum_{k=1}^m \xi_k e(\cdot, x_k)\right\|^2_{L^2(\mathbb{R}^n,\mu_h)} \tag{4.3}$$

for $\xi = \{\xi_1, \ldots, \xi_m\} \subset \mathbb{C}$ and arbitrary $m \in \mathbb{N}$. All statements are immediate from (4.3). □

The same system $\mathcal{E}$ can be viewed as a system of vectors in another Hilbert space, namely $L^2(S_r^{n-1})$, S_r^{n-1} being a sphere in $\mathbb{R}^n$ of radius r, centered at the origin, with the normalized Lebesgue measure s_n. We denote this system by $\mathcal{E}_X(S_r^{n-1})$. Such approach leads to RPDF's. Note that the assumption $n \geq 2$ looks quite natural now since otherwise the sphere makes no sense.

Definition 4.2 Let $X = \{x_k\}_{k\in\mathbb{N}}$ be a sequence of distinct points in $\mathbb{R}^n$, $\mathcal{K} \subset \mathbb{R}_+$ be a Borel set. $\mathcal{K}$ is called *X-massive* if $\mathcal{E}_X(S_r^{n-1})$ is the Riesz–Fischer sequence for each $r \in \mathcal{K}$.

The following result is borrowed from [21, Proposition 2.14]. We present it with the proof because of its importance in the sequel. Recall that Ω_n is the Schoenberg kernel (1.4).

Proposition 4.3 *Let $f \in \Phi_n$, $n \geq 2$, with the measure ν_f in (1.2). Given an arbitrary sequence $X = \{x_k\}_{k\in\mathbb{N}}$ of distinct points in $\mathbb{R}^n$, the function f is strongly X-positive definite if and only if there exists an X-massive set $\mathcal{K}$ of positive ν_f-measure, $\nu_f(\mathcal{K}) > 0$. In particular, the function $f_\rho(\cdot) = \Omega_n(\rho\cdot)$, $\rho > 0$, is strongly X-positive definite if and only if the system $\mathcal{E}_X(S_\rho^{n-1})$ is the Riesz–Fischer sequence.*

Proof It follows from (1.2) and (1.4) that for $\{\xi_1, \ldots, \xi_m\} \subset \mathbb{C}$ and $m \in \mathbb{N}$

$$\sum_{j,k=1}^{m} f(\|x_k - x_j\|)\xi_j\overline{\xi}_k = \int_0^{+\infty} \left(\int_{S_r^{n-1}} \left|\sum_{k=1}^{m} \xi_k e(u, x_k)\right|^2 s_n(du) \right) \nu_f(dr). \quad (4.4)$$

Suppose that there exists a set $\mathcal{K}$ as stated above. Then for every $r \in \mathcal{K}$ there is a constant $c(r) > 0$ so that

$$\int_{S_r^{n-1}} \left|\sum_{k=1}^{m} \xi_k e(u, x_k)\right|^2 s_n(du) = \left\| \sum_{k=1}^{m} \xi_k e(\cdot, x_k)\right\|^2_{L^2(S_r^{n-1})} \geq c(r) \sum_{k=1}^{m} |\xi_k|^2. \quad (4.5)$$

Choosing $c(r)$ bounded and measurable and combining the latter inequality with (4.4), we obtain

$$\sum_{j,k=1}^{m} f(\|x_j - x_k\|)\xi_j\overline{\xi}_k \geq \int_{\mathcal{K}} \left(\left\| \sum_{k=1}^{m} \xi_k e(\cdot, x_k)\right\|^2_{L_r^2(S^{n-1})} \right) \nu_f(dr) \geq c \sum_{k=1}^{m} |\xi_k|^2,$$
$$c := \int_{\mathcal{K}} c(r)\nu_f(dr). \quad (4.6)$$

Since $\nu_f(\mathcal{K}) > 0$ and $c(r) > 0$, we have $c > 0$, so f is strongly X-positive definite.

Conversely, if

$$\int_0^{\infty} h(r)\, \nu_f(dr) \geq c_1 > 0, \qquad h(r) = \left\| \sum_{k=1}^{m} \xi_k e(\cdot, x_k)\right\|^2_{L_r^2(S^{n-1})},$$

then there is a Borel set $\mathcal{K} \subset (0, +\infty)$ of positive ν_f-measure such that $h \geq c_1$ on $\mathcal{K}$, as claimed. □

We want to lay stress on the fact that the measure ν_f enters this result only via existence of a certain Borel set $\mathcal{K}$ of positive ν_f-measure.

A combination of the latter result with Proposition 3.21 leads to the following

Corollary 4.4 *If $X \notin \mathcal{X}_n$, i.e., $d_*(X) = 0$, then $\mathcal{E}_X(S_r^{n-1})$ is the Riesz–Fischer sequence for neither $r > 0$.*

Corollary 4.5 *Let $f_j \in \Phi_n$, $n \ge 2$, $j = 1, 2$, with the measures ν_1 and ν_2 in (1.2), respectively. Assume that ν_1 is absolutely continuous with respect to ν_2. Given a set $X = \{x_k\}_{k\in\mathbb{N}}$ of distinct points in $\mathbb{R}^n$, if f_1 is strongly X-positive definite then so is f_2. In particular, if ν_1 and ν_2 are mutually absolutely continuous (equivalent), then f_1 and f_2 are strongly X-positive definite simultaneously.*

Proof By Proposition 4.3 there is an X-massive set $\mathcal{K}$, $\nu_1(\mathcal{K}) > 0$. Since ν_1 is absolutely continuous with respect to ν_2, then $\nu_2(\mathcal{K}) > 0$ as well. Now Proposition 4.3 applies in the backward direction and yields strong X-positive definiteness of f_2, as claimed. □

We are in a position now to present the main result of the section.

Theorem 4.6 (Theorem 1.6) *Let* (const $\neq$)$f \in \Phi_n$, $n \ge 2$, *with the representing measure ν_f (1.2). If ν_f is equivalent to the Lebesgue measure on $\mathbb{R}_+$, then f is strongly X-positive definite for each $X \in \mathcal{X}_n$.*

Proof We begin with the function $e_s(r) := e^{-sr} \in \Phi_n$ and show that for each $X \in \mathcal{X}_n$ e_s is strongly X-positive definite for all large enough $s > 0$. Indeed, take s so that

$$\|t^{n-1}e_s\|_{L^1(\mathbb{R}_+)} = \int_0^\infty t^{n-1}e^{-st}\,dt = \frac{\Gamma(n)}{s^n} < \frac{d_*^n(X)}{5^n n^2}.$$

By Theorem 3.4 (see 3.7) the Schoenberg operator $S_X(e_s)$ is bounded and invertible, so (1.11) holds, as needed.

To make use of Corollary 4.5 we compute the measure ν_{e_s}. To this end recall a well-known result from the Fourier transforms theory, which plays a key role in the sequel.

Let $h \in L^1(\mathbb{R}^n)$ and $\widehat{h}$ be its Fourier transform (1.17). If $h(\cdot) = h_0(\|\cdot\|)$ is a radial function, then so is $\widehat{h}(\cdot) = H_0(\|\cdot\|)$. Moreover, H_0 and h_0 are related by (see, e.g., [32, Theorem IV.3.3])

$$\begin{aligned} H_0(r) &= \frac{1}{r^q}\int_0^\infty J_q(ru)u^{q+1}h_0(u)\,du \\ &= \frac{1}{2^q\Gamma(q+1)}\int_0^\infty \Omega_n(ru)u^{n-1}h_0(u)\,du, \quad q := \frac{n}{2} - 1. \end{aligned} \tag{4.7}$$

The latter is usually referred to as the Fourier–Bessel transform.

We apply (4.7) to the pair of functions (see, e.g., [32, Theorem 1.13])

$$h(x) = \frac{2^{n/2}\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}\,\frac{s}{(s^2+\|x\|^2)^{\frac{n+1}{2}}}, \qquad \widehat{h}(t) = e^{-s\|t\|},$$

(this is a particular case of (4.23) below) and come to

$$e_s(r) = e^{-sr} = \frac{2}{B\left(\frac{n}{2},\frac{1}{2}\right)} \int_0^\infty \Omega_n(ru) \frac{su^{n-1}}{(s^2+u^2)^{\frac{n+1}{2}}}\,du, \quad s,t>0,$$
$$B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{4.8}$$

is the Euler beta-function. This is exactly representation (1.2) of e_s with the measure

$$\nu_{e_s}(du) = \frac{2}{B\left(\frac{n}{2},\frac{1}{2}\right)} \frac{su^{n-1}}{(s^2+u^2)^{\frac{n+1}{2}}}\,du,$$

equivalent to the Lebesgue measure. By the assumption of the theorem the measures ν_f and ν_{e_s} are equivalent. Since e_s is strongly X-positive definite for large enough s and each separated set $X \in \mathcal{X}_n$, then by Corollary 4.5, so is f, as claimed. □

Remark 4.7 In fact, Theorem 4.6 remains valid whenever the Lebesgue measure on $\mathbb{R}_+$ is absolutely continuous with respect to the measure ν, that is,

$$\nu_f(ds) = \nu_{f,ac} + \nu_{f,sing} = \nu_f'(s)\,ds + \nu_{f,sing}, \qquad \nu_f'(s) > 0 \text{ a.e.}, \tag{4.9}$$

$\nu_{f,sing}$ is a singular measure. This statement is immediate from the obvious identity $S_X(f) = S_X(f_{ac}) + S_X(f_{sing})$, where f_{ac} and f_{sing} are the Φ_n-functions defined by (1.2) with the measures $\nu_{f,ac}$ and $\nu_{f,sing}$, respectively. It is also a consequence of Corollary 4.5, applied in its full extent.

Theorem 4.8 (Theorem 1.7) *Let $f \in \Phi_\infty(\alpha)$, $0 < \alpha \le 2$. Then*

(i) *f is strongly X-positive definite for each $X \in \mathcal{X}_d$, $d \in \mathbb{N}$. In particular, if $\mathcal{S}_X(f)$ generates an operator $S_X(f)$ on ℓ^2, then it is positive definite and so invertible.*

(ii) *If the Schoenberg measure σ_f in (2.6) satisfies*

$$\int_0^\infty s^{-\frac{d}{\alpha}}\,\sigma_f(ds) < \infty, \qquad d \in \mathbb{N}, \tag{4.10}$$

then the Schoenberg operator $S_X(f)$ is bounded and invertible for each $X \in \mathcal{X}_d$.

(iii) *Conversely, if $S_Y(f)$ is bounded for at least one δ-regular set $Y \in \mathcal{X}_d$, then (4.10) holds.*

Proof (i). We apply again (4.7), now to the pair of functions

$$h(x) = (2s)^{-n/2} \exp\left(-\frac{\|x\|^2}{4s}\right), \qquad \widehat{h}(t) = e^{-s\|t\|^2},$$

to obtain representation (1.2) for the function

$$g_s(r) := e^{-sr^2} = \frac{1}{2^q\Gamma(q+1)} \int_0^\infty \Omega_n(ru) \frac{u^{n-1}}{(2s)^{n/2}} \exp\left(-\frac{u^2}{4s}\right) du, \qquad r,s>0, \tag{4.11}$$

(cf. [3, Sect. V.4.3]). Hence for any $f \in \Phi_\infty$ we can relate integral representations (1.2) and (2.6). Namely, combining (4.11) with (2.6) we arrive at representation (1.2) for $f \in \Phi_\infty$

$$f(r) = \int_0^\infty \Omega_n(ru)\phi_n(u,f)\,du, \quad \phi_n(u,f) := \frac{u^{n-1}}{2^q\Gamma(q+1)} \int_0^\infty (2s)^{-n/2} \exp\left(-\frac{u^2}{4s}\right) \sigma_f(ds). \tag{4.12}$$

Clearly, $\nu_f = \phi_n(\cdot, f)\,du$ is equivalent to the Lebesgue measure, and the density $\phi_{n,\sigma}$ is bounded, strictly positive and continuous on $\mathbb{R}_+$. The rest is Theorem 4.6.

(ii). By Corollary 3.6, the Schoenberg operator $S_X(f)$ is bounded. It is invertible in view of the strong X-positive definiteness of f.

(iii) is a combination of Theorem 3.4, (iii), and Corollary 3.6.

The proof is complete. □

Remark 4.9 As a special case of Theorem 4.8 we get that the function g_s (see 4.11) is strongly X-positive definite for all $s > 0$ and each $X \in \mathcal{X}_n$. The corresponding Schoenberg operator $S_X(g_s)$ is bounded and invertible by Theorem 4.8.

Example 4.10 According to representation (2.6) each $f \in \Phi_\infty(\alpha)$ is monotone decreasing. The following example demonstrates that the monotonicity is not necessary for f to be strongly X-positive definite for each separated set $X \in \mathcal{X}_n$. In particular, it gives an example of strongly X-positive definite function from $\Phi_n \backslash \Phi_\infty$.

Let K_μ be the modified Bessel function of the second kind and order μ (the definition and properties of K_μ are given in the next section). By [35, p. 435, (5)] the following integral representation holds for $n \geq 3$

$$h_s(r) := \Omega_n(rs)M_q(rs) = \frac{2(2s)^{n-2}}{B\left(q, \frac{1}{2}\right)} \int_0^\infty \Omega_n(ru) \frac{u^{n-1}}{(u^4+4s^4)^{\frac{n-1}{2}}}\,du,$$
$$M_q(t) := \frac{t^q K_q(t)}{2^{q-1}\Gamma(q)} \tag{4.13}$$

is the Whittle–Matérn function, well-established in spatial statistics, $q = n/2 - 1$, $s > 0$ is a parameter. We show in the next section that $M_q \in \Phi_\infty$, so the function $h_s \in \Phi_n$. Its representing measure in (1.2) is equivalent to the Lebesgue measure and given explicitly by

$$\nu_{h_s}(du) = \frac{2(2s)^{n-2}}{B\left(q, \frac{1}{2}\right)} \frac{u^{n-1}}{(u^4+4s^4)^{\frac{n-1}{2}}}\,du$$

so by Theorem 4.6 h_s is strongly X-positive definite function for each $X \in \mathcal{X}_n$.

On the other hand, h_s has infinitely many real zeros, so it is not monotone decreasing and hence $f \notin \Phi_\infty$. Thus, by (4.13), $f \in \Phi_n \backslash \Phi_\infty$.

Remark 4.11 If a real-valued function f obeys $|f(r)| \le ce^{-ar}$, $a > 0$, (as in the above example), then by Proposition 3.8, the operator $S_X(f)$ is bounded for each $X \in \mathcal{X}_n$ *and each* $n \in \mathbb{N}$.

4.2 "Grammization" of Schoenberg Matrices

Our goal here is to implement the "grammization" procedure (see Sect. 1) for two positive definite Schoenberg's matrices

$$\mathcal{S}_X(f) = [\exp(-a\|x_i - x_j\|^2)]_{i,j\in\mathbb{N}}, \quad \text{and}$$
$$\mathcal{S}_X(f) = [\exp(-a\|x_i - x_j\|)]_{i,j\in\mathbb{N}}, \qquad a > 0, \tag{4.14}$$

as well as for a certain family of Schoenberg's matrices which contains the second one in (4.14).

A key observation is stated as the following simple result.

Lemma 4.12 *Let* $g \in L^2(\mathbb{R}^n)$ *and* $g_\xi(\cdot) := g(\cdot - \xi)$, $g_\eta(\cdot) := g(\cdot - \eta)$ *be its translates on* $\xi, \eta \in \mathbb{R}^n$. *Then*

$$\langle g_\xi, g_\eta\rangle_{L^2(\mathbb{R}^n)} = f(\eta - \xi) = \widehat{F}(\xi - \eta),$$
$$f(x) := \int_{\mathbb{R}^n} g(x+y)\overline{g(y)}\,dy, \quad F(t) := (2\pi)^{n/2}|\widehat{g}(t)|^2. \tag{4.15}$$

If g is a radial function, then $f \in \Phi_n$ and its Schoenberg's measure ν_f is absolutely continuous with respect to the Lebesgue measure.

Proof The first equality in (4.15) is merely definition of the inner product. By Parseval's equality

$$\langle g_\xi, g_\eta\rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{g_\xi}, \widehat{g_\eta}\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{g}(t)|^2 e^{-i\langle t,\xi-\eta\rangle}\,dt = (2\pi)^{n/2}\widehat{F}(\xi-\eta), \ \ \xi,\eta\in\mathbb{R}^n.$$

The rest is standard (see, e.g., [29, Lemma 3.6.5]). □

Proposition 4.13 *Let* $\xi, \eta \in \mathbb{R}^n$, $a > 0$. *Then*

$$e^{-\frac{a}{2}\|\xi-\eta\|^2} = \left(\frac{2a}{\pi}\right)^{n/2} \langle g_{a,\xi}, g_{a,\eta}\rangle_{L^2(\mathbb{R}^n)}, \qquad g_{a,\xi}(x) = e^{-a\|x-\xi\|^2}. \tag{4.16}$$

The grammization of the first Schoenberg's matrix in (4.14) *reads as follows*

$$\left[\exp\left(-\frac{a}{2}\|x_i - x_j\|^2\right)\right]_{i,j\in\mathbb{N}} = \left(\frac{2a}{\pi}\right)^{n/2} Gr(\{g_j\}, L^2(\mathbb{R}^n)), \quad g_j(x) = e^{-a\|x-x_j\|^2}. \tag{4.17}$$

Proof A combination of Lemma 4.12 (see 4.15) and the well-known formula

$$\widehat{e^{-b\|\cdot\|^2}}(t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-b\|x\|^2 - i\langle x,t\rangle}\, dx = \frac{1}{(2b)^{n/2}}\, e^{-\frac{\|t\|^2}{4b}}, \qquad b > 0, \quad (4.18)$$

yields the result. □

The grammization of the second Schoenberg matrix in (4.14) is similar but technically more involved.

We begin with a brief reminder of the modified Bessel functions K_μ of the second kind of order μ, which solve the differential equations

$$t^2 u''(t) + t u'(t) - (t^2 + \mu^2) u(t) = 0, \qquad t > 0, \quad \mu \in \mathbb{R}.$$

The asymptotics for K_μ is well known (see [1, (9.6.8–9.6.9)], [35, p. 202, (1)])

$$K_\mu(t) = \begin{cases} \frac{\Gamma(\mu)}{2} \left(\frac{t}{2}\right)^{-\mu} + O(t^{-\mu+2}), & \mu > 0; \\ \log \frac{2}{t} + O(1), & \mu = 0; \end{cases} \qquad t \to 0,$$
$$K_\mu(t) = \sqrt{\frac{\pi}{2t}} e^{-t}(1 + O(t^{-1})), \quad t \to \infty. \quad (4.19)$$

The functions K_μ are known to satisfy $K_{-\mu} = K_\mu$ and to admit the following integral representations (see [35, p. 183, (15)], [35, p. 172, (4)])

$$K_\mu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\mu \int_0^\infty \exp\left(-r - \frac{z^2}{4r}\right) r^{-\mu-1}\, dr$$
$$= \frac{\sqrt{\pi}}{\Gamma\left(\mu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^\mu \int_1^\infty e^{-zr}(r^2 - 1)^{\mu - \frac{1}{2}}\, dr, \qquad \mu > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2}. \quad (4.20)$$

Furthermore, K_μ is positive and monotone decreasing function on $\mathbb{R}_+$.

Proposition 4.14 *Let* $n \in \mathbb{N}$ *and* K_μ *be the modified Bessel function of the second kind of order* μ, $0 \le \mu < n/4$. *For* $a > 0$ *and* $\xi \in \mathbb{R}^n$ *put*

$$g_{a,\mu}(x) := \left(\frac{a}{\|x\|}\right)^\mu K_\mu(a\|x\|), \quad g_{a,\mu,\xi}(x) := g_{a,\mu}(x - \xi), \qquad x \in \mathbb{R}^n. \quad (4.21)$$

Then with $p := \frac{n}{2} - 2\mu > 0$ *the following equality holds for all* $\xi, \eta \in \mathbb{R}^n$

$$\left(\frac{\|\xi - \eta\|}{a}\right)^p K_p\left(a\|\xi - \eta\|\right) = \frac{2^{2\tau - \frac{n}{2}}}{\pi^{\frac{n}{2}} B\left(\tau, \frac{1}{2}\right)} \langle g_{a,\mu,\xi}, g_{a,\mu,\eta}\rangle_{L^2(\mathbb{R}^n)}, \qquad \tau := \frac{n}{2} - \mu. \quad (4.22)$$

Proof It follows from (4.19) that $g_{a,\mu} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for $0 \le \mu < n/4$. We begin with the formula for the Fourier transform (cf. [29, Theorem 3.7.5])

$$\widehat{g_{a,\mu}}(t) = \frac{2^{q-\mu}\,\Gamma(q-\mu+1)}{(a^2+\|t\|^2)^{q-\mu+1}} = \frac{2^{\tau-1}\,\Gamma(\tau)}{(a^2+\|t\|^2)^{\tau}} =: h_{a,\mu}(t)\,, \qquad q = \frac{n}{2}-1. \quad (4.23)$$

Due to its importance we outline the proof (cf. [18, p. 7]).

Since $h_{a,\mu} \in L^2(\mathbb{R}^n)$ for $0 \le \mu < n/4$ we can compute its Fourier transform $\widehat{h_{a,\mu}}$. The starting point is the gamma function identity

$$A^{-\tau} = \frac{1}{\Gamma(\tau)} \int_0^\infty e^{-sA} s^{\tau-1}\, ds, \qquad A, \tau > 0,$$

which we use to obtain

$$\frac{1}{(a^2+\|x\|^2)^{\tau}} = \frac{1}{\Gamma(\tau)} \int_0^\infty e^{-s(a^2+\|x\|^2)} s^{\tau-1}\, ds. \quad (4.24)$$

Next, we take the Fourier transform of both sides and apply (4.18) to find

$$\begin{aligned}
\widehat{h_{a,\mu}}(t) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{2^{\tau-1}\,\Gamma(\tau)}{(a^2+\|x\|^2)^{\tau}}\, e^{-i\langle t,x\rangle}\, dx \\
&= \frac{2^{\tau-1}}{(2\pi)^{n/2}} \int_0^\infty e^{-sa^2} s^{\tau-1}\, ds \int_{\mathbb{R}^n} e^{-s\|x\|^2 - i\langle t,x\rangle}\, dx \\
&= 2^{-\mu-1} \int_0^\infty \exp\Big(-sa^2 - \frac{\|t\|^2}{4s}\Big)\, s^{-\mu-1}\, ds \\
&= \frac{a^{2\mu}}{2^{\mu+1}} \int_0^\infty \exp\Big(-r - \frac{a^2\|t\|^2}{4r}\Big)\, r^{-\mu-1}\, dr.
\end{aligned}$$

Equality (4.23) now follows from the first integral representation (4.20).

In view of (4.15) it remains to compute

$$\widehat{F}(t) = \int_{\mathbb{R}^n} |\widehat{g_{a,\mu}}(u)|^2 e^{-i\langle t,u\rangle}\, du = 2^{2(\tau-1)}\Gamma^2(\tau) \int_{\mathbb{R}^n} \frac{e^{-i\langle t,u\rangle}}{(a^2+\|u\|^2)^{2\tau}}\, du$$

by using exactly the same method as above. Precisely, since $\Gamma(2r) = 2^{2r-1}\pi^{-1/2}\,\Gamma(r)\,\Gamma(r+1/2)$, we have

$$\begin{aligned}
2^{2(\tau-1)}\Gamma^2(\tau) \int_{\mathbb{R}^n} \frac{e^{-i\langle t,u\rangle}}{(a^2+\|u\|^2)^{2\tau}}\, du &= 2^{2(\tau-1)}\Gamma^2(\tau)\, \frac{(2\pi)^{n/2}}{2^{2\tau-1}\Gamma(2\tau)} \Big(\frac{a}{\|t\|}\Big)^{\frac{n}{2}-2\tau} \\
K_{\frac{n}{2}-2\tau}(a\|t\|) &= \frac{(2\pi)^{n/2}}{2^{2\tau}}\, B\Big(\tau, \frac{1}{2}\Big) \Big(\frac{\|t\|}{a}\Big)^{p} K_p(a\|t\|),
\end{aligned}$$

as claimed. □

Corollary 4.15 *The grammization for the second Schoenberg matrix in* (4.14) *is*

$$[\exp(-a\|x_j - x_k\|)]_{j,k\in\mathbb{N}} = Gr(\{g_j\}, L^2(\mathbb{R}^n)),$$
$$g_j(x) = \sqrt{\frac{2\Gamma\left(\frac{n+3}{4}\right)a}{\pi^{\frac{n+2}{2}}\,\Gamma\left(\frac{n+1}{4}\right)}}\left(\frac{a}{\|x-x_j\|}\right)^{\frac{n-1}{4}} K_{\frac{n-1}{4}}(a\|x-x_j\|). \tag{4.25}$$

In particular, for $n = 3$

$$e^{-a\|\xi-\eta\|} = \frac{a}{2\pi}\int_{\mathbb{R}^3} \frac{e^{-a\|x-\xi\|}}{\|x-\xi\|}\,\frac{e^{-a\|x-\eta\|}}{\|x-\eta\|}\,dx, \qquad \xi,\eta\in\mathbb{R}^3, \quad a>0. \tag{4.26}$$

Proof Take $\mu = \frac{n-1}{4}$, so $p = 1/2$, and the function in the left side of (4.22) is just the exponential function (cf. [35, p. 80, (13)])

$$\sqrt{\frac{\|\xi-\eta\|}{a}}\,K_{1/2}(a\|\xi-\eta\|) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a\|\xi-\eta\|}}{a}, \tag{4.27}$$

which is (4.25).

If $n = 3$, $\mu = 1/2$, then

$$f_{a,1/2,\xi}(x) = \left(\frac{a}{\|x-\xi\|}\right)^{1/2} K_{1/2}(a\|x-\xi\|) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a\|x-\xi\|}}{\|x-\xi\|}, \tag{4.28}$$

and (4.26) follows. □

Note that (4.26) is one of the cornerstones of [21] (see formula (3.26) therein).

The case $n = 2$, $\mu = 0$ leads to the following

Corollary 4.16 *For all* $\xi,\eta\in\mathbb{R}^2$ *and* $a > 0$

$$\frac{\|\xi-\eta\|}{a}\,K_1(a\|\xi-\eta\|) = \frac{1}{\pi}\,\langle K_0(a\|\cdot-\xi\|),\ K_0(a\|\cdot-\eta\|)\rangle_{L^2(\mathbb{R}^2)}.$$

There is another natural way to view (4.22). For arbitrary $p > 0$ and $a > 0$ consider the Whittle–Matérn function (cf. 4.13)

$$M_{p,a}(r) := \left(\frac{r}{a}\right)^p K_p(ar), \quad r > 0. \tag{4.29}$$

Since $K_{-p} = K_p$, the notation makes sense for negative indices, and another family of the Whittle–Matérn functions comes in

$$\widetilde{M}_{p,a}(r) = M_{-p,a}(r) = \left(\frac{a}{r}\right)^p K_p(ar), \qquad p>0, \qquad \widetilde{M}_{0,a}(r) = K_0(r).$$

Then equality (4.22) with $0 < 2p \le n$ reads

$$M_{p,a}(\|\xi - \eta\|) = \langle c_{n,p}\widetilde{M}_{d,a}(\|\cdot - \xi\|),\ c_{n,p}\widetilde{M}_{d,a}(\|\cdot - \eta\|)\rangle_{L^2(\mathbb{R}^n)},$$
$$0 \le d := \frac{1}{2}\left(\frac{n}{2} - p\right) < \frac{n}{4}, \quad c_{n,p}^2 = \frac{2^p}{\pi^{\frac{n}{2}} B\left(d, \frac{1}{2}\right)} \tag{4.30}$$

for all $\xi, \eta \in \mathbb{R}^n$.

To have a proper normalization at the origin we put (see 4.19 and 4.13)

$$M_p(r) = \frac{M_{p,1}(r)}{2^{p-1}\Gamma(p)} = \frac{r^p K_p(r)}{2^{p-1}\Gamma(p)} = 1 + O(r^2), \quad r \to 0.$$

As a byproduct of Proposition 4.14 we have (cf. [19], [14, Table 2]).

Corollary 4.17 *$M_p \in \Phi_\infty$ for all $p > 0$.*

Proof Take $n > 2p$. By Proposition 4.14, for each finite set $X \subset \mathbb{R}^n$ the Schoenberg matrix $\mathcal{S}_X(M_p)$ is the Gram matrix, so $\mathcal{S}_X(M_p) \ge 0$. Hence $M_p \in \Phi_n$ for all such n, as claimed. □

With regard to Corollary 4.17 one might ask whether the functions M_p belong to certain subclasses of Φ_∞, for instance, to the class $CM_0(\mathbb{R}_+)$ of completely monotone functions. The result below seems interesting on its own.

Proposition 4.18 *For the Whittle–Matérn function M_p the following statements hold.*

(i) *$M_p \in CM(\mathbb{R}_+)$ if and only if $-\infty < p \le 1/2$.*
(ii) *$M_p \in CM_0(\mathbb{R}_+)$ if and only if $0 < p \le 1/2$.*

Proof The assertion for $-\infty < p < 1/2$ follows directly from the second integral representation (4.20) and the Bernstein theorem, if one puts $\mu = -p$. Note that the Bernstein measure is finite if and only if $0 < p < 1/2$. For $p = 1/2$ we have

$$M_{1/2}(r) = e^{-r} \in CM_0(\mathbb{R}_+).$$

Let now $p > 1/2$. We wish to show that inequalities (2.4) are violated for some $k \ge 1$. The argument relies on the differentiation formulae for the Bessel functions, which in our notation look as (see [35, p. 74])

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m M_{p,1}(z) = (-1)^m M_{p-m,1}(z). \tag{4.31}$$

For $m = 1$ it displays the fact that $M_{p,1}$ is monotone decreasing function on $\mathbb{R}_+$. For $m = 2$ we have

$$M''_{p,1}(r) = -M_{p-1,1}(r) + r^2 M_{p-2,1}(r).$$

For $p \geq 2$ obviously $r^2 M_{p-2,1} \to 0$ as $r \to +0$, so $M''_{p,1}(+0) = -2^{p-2}\Gamma(p-1) < 0$, which is inconsistent with (2.4) for $k = 2$. If $1 < p < 2$, then again

$$r^2 M_{p-2,1}(r) = r^p K_{2-p,1}(r) = r^{2p-2} M_{2-p,1}(r) \to 0, \qquad r \to +0,$$

with the same conclusion.

Finally, let $1/2 < p < 1$. From (4.31) with $m = 1$ one has

$$M'_{p,1}(r) = -r M_{p-1,1}(r) = -r^p K_{1-p}(r) = -r^{2p-1} M_{1-p,1}(r) \to 0, \qquad r \to +0$$

so $M'_{p,1}(+0) = 0$ that is impossible for a nonconstant completely monotone function. The proof is complete. □

Remark 4.19 For $0 \leq p \leq 1/2$ a stronger result is proved in [22], namely, $e^r M_{p,1}(r) \in CM(\mathbb{R}_+)$. Our results for the other values of p seem to be new.

4.3 Minimality Conditions and Riesz Sequences in $L^2(\mathbb{R}^n)$

The classical result of Bari (see Sect. 1) states that a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of vectors in a Hilbert space is a Riesz sequence if and only if the corresponding Gram matrix $Gr\{\varphi_k\}_{k\in\mathbb{N}}$ generates a bounded and invertible linear operator on ℓ^2. We examine here certain systems of translates from this viewpoint.

The definitions below are standard (cf. [15, Chap. VI]).

Definition 4.20 A sequence of vectors $\{h_j\}_{j\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called *minimal*, if neither of h_k belongs to the closed linear span $\mathcal{L}(\{h_j\}_{j\neq k})$ of the others. In other words,

$$\delta_k := \operatorname{dist}(h_k/\|h_k\|, \mathcal{L}(\{h_j\}_{j\neq k})) > 0, \qquad k \in \mathbb{N}.$$

$\{h_j\}_{j\in\mathbb{N}}$ is *uniformly minimal*, if $\inf_k \delta_k > 0$.

Recall that Riesz–Fischer systems are defined in (1.15).

Lemma 4.21 *Any bounded Riesz–Fischer sequence $\{h_j\}_{j\in\mathbb{N}}$ is uniformly minimal.*

Proof By the assumption, $\|h_k\| \leq c_1$ with some $c_1 > 0$. Therefore, by Definition 1.8 (i), (see 1.15), for any fixed k and any finite sequence $\{\xi_j\} \subset \mathbb{C}$

$$\left\| \sum_{j\neq k} \xi_j h_j - \frac{h_k}{\|h_k\|} \right\|^2 \geq c^2 \left(\sum_{j\neq k} |\xi_j|^2 + \frac{1}{\|h_k\|^2} \right) \geq \frac{c^2}{\|h_k\|^2} \geq \frac{c^2}{c_1^2}. \tag{4.32}$$

Now the result follows directly from Definition 4.20. □

Given a function $g \in L^2(\mathbb{R}^n)$ and a set $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$, consider a sequence $\mathcal{F}_X(g) = \{g(\cdot - x_j)\}_{j\in\mathbb{N}}$ of translates of g. Denote $g_j(\cdot) = g(\cdot - x_j)$.

Theorem 4.22 (Theorem 1.9) *Let $g \in L^2(\mathbb{R}^n)$, $n \geq 2$, be a real-valued and radial function such that its Fourier transform $\widehat{g}(t) \neq 0$ a.e., and let $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$. Then the following statements are equivalent.*

(i) $\mathcal{F}_X(g) = \{g_j\}_{j\in\mathbb{N}}$ *forms a Riesz–Fischer sequence in* $L^2(\mathbb{R}^n)$;
(ii) $\mathcal{F}_X(g)$ *is uniformly minimal in* $L^2(\mathbb{R}^n)$;
(iii) X *is a separated set, i.e.,* $d_*(X) > 0$.

Proof Implication (i)⇒(ii) is immediate from Lemma 4.21 since now

$$\|g_j\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)}, \qquad j \in \mathbb{N}.$$

(ii)⇒(iii). With no loss of generality we can assume that

$$\|g_j\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} = 1, \qquad j \in \mathbb{N}.$$

The normalization in (4.15) shows that $\widehat{F}(0) = \|g\|^2_{L^2(\mathbb{R}^n)} = 1$.

Let $\mathcal{F}_X(f)$ be uniformly minimal. Then there exists $\varepsilon > 0$ such that $\|g_j - g_k\|^2 \geq 2\varepsilon$ for all $j \neq k \in \mathbb{N}$. A combination of the latter inequality with identity (4.15) yields

$$1 - \widehat{F}(\|x_j - x_k\|) = 1 - \langle g_j, g_k\rangle_{L^2(\mathbb{R}^n)} = \frac{\|g_j - g_k\|^2}{2} \geq \varepsilon, \quad j, k \in \mathbb{N}, \tag{4.33}$$

and so $d_*(X) > 0$.

(iii)⇒(i). Let $d_*(X) > 0$. As all functions in question are radial, we put

$$F_0(\|t\|) := F(t) = (2\pi)^{n/2}|\widehat{g}(t)|^2, \qquad \widetilde{F}_0(\|t\|) := \widehat{F}(t). \tag{4.34}$$

By Lemma 4.12, $\widehat{F}$ is a radial positive definite function on $\mathbb{R}^n$, i.e., $\widetilde{F}_0 \in \Phi_n$, and $F \in L^1(\mathbb{R}^n)$. The equality

$$\widehat{F}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle t,\xi\rangle} F(t)\, dt$$

shows that the measure $\mu_{\widehat{F}}$ from Bochner's representation (2.2) of $\widehat{F}$ is absolutely continuous, $\mu_{\widehat{F}}(dt) = (2\pi)^{-n/2} F\, dt$. Moreover, the condition $\widehat{g} \neq 0$ a.e. implies $F > 0$ a.e. on $\mathbb{R}^n$, that is, $\mu_{\widehat{F}}$ is equivalent to the Lebesgue measure on $\mathbb{R}^n$. Hence, the representing Schoenberg's measure $\nu_{\widetilde{F}_0}$ from (1.2) is equivalent to the Lebesgue measure on $\mathbb{R}_+$ due to the relation $\nu_{\widetilde{F}_0}\{[0, r]\} = \mu_{\widehat{F}}\{\|x\| \leq r\}$. Thereby the conditions of Theorem 4.6 are met, and the function $\widetilde{F}_0$ is strongly X-positive definite. By Lemma 4.12 (see identity (4.15)) and Definition 1.5 of strongly X-positive definite functions, the latter amounts to saying that $\mathcal{F}_X(g)$ is the Riesz–Fischer system. The proof is complete. □

Under certain additional assumptions on g we come to Riesz sequences of translates.

Theorem 4.23 (Theorem 1.10) *Let* $g \in L^2(\mathbb{R}^n)$, $n \geq 2$, *be a real-valued and radial function such that its Fourier transform* $\widehat{g} \neq 0$ *a.e., and let* $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$. *Assume that*

$$v(\xi) := \sup_{\|t\|\ge\|\xi\|} |f(t)| \in L^1(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n, \quad f(t) = \int_{\mathbb{R}^n} g(t+y)g(y)\,dy. \tag{4.35}$$

Then the following statements are equivalent.

(i) $\mathcal{F}_X(g)$ *forms a Riesz sequence in* $L^2(\mathbb{R}^n)$;
(ii) $\mathcal{F}_X(g)$ *forms a basis in its linear span*;
(iii) $\mathcal{F}_X(g)$ *is uniformly minimal in* $L^2(\mathbb{R}^n)$;
(iv) X *is a separated set, i.e.,* $d_*(X) > 0$.

Proof The implications (i)⇒(ii)⇒(iii) are obvious. The implication (iii)⇒(iv) is proved in Proposition 4.22.

It remains to prove that (iv) implies (i). Lemma 4.12 is a key ingredient of the proof. Condition (4.15) now reads

$$Gr(\{g_j, L^2(\mathbb{R}^n)\}) = \mathcal{S}_X(\widetilde{F}_0), \tag{4.36}$$

$\widetilde{F}_0$ is defined in (4.34). In view of the aforementioned theorem of Bari we wish to show that under the hypothesis of Theorem 4.23 the Schoenberg operator $S_X(\widetilde{F}_0)$ is bounded and invertible.

First, v is the radial function, $v(\cdot) = u(\|\cdot\|)$. It is clear that $u^{-1}(0)u(\|\cdot\|) \in \mathcal{M}_+$ and

$$v \in L^1(\mathbb{R}^n) \Longleftrightarrow s^{n-1}u(s) \in L^1(\mathbb{R}_+).$$

Therefore in view of Proposition 3.8, assumption (4.35) implies the boundedness of the Schoenberg matrix $\mathcal{S}_X(\widetilde{F}_0)$, so $\mathcal{F}_X(g)$ is the Bessel sequence.

Secondly, according to Proposition 4.22, the condition $\widehat{g} \neq 0$ a.e. ensures $\mathcal{F}_X(g)$ to be the Riesz–Fischer sequence, i.e., the Schoenberg operator $S_X(\widetilde{F}_0)$ is invertible, as claimed. Thus, by (4.36) the Gram matrix $Gr(\{g_j, L^2(\mathbb{R}^n)\})$ is bounded and invertible, and the Bari theorem completes the proof. □

Corollary 4.24 (Corollary 1.11) *Let* $g \in L^2(\mathbb{R}^n)$, $n \ge 2$, *be a real-valued and radial function with compact support,* $g \not\equiv 0$, *and let* $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$. $\mathcal{F}_X(g)$ *forms a Riesz sequence in* $L^2(\mathbb{R}^n)$ *if and only if* X *is a separated set.*

Proof To verify the conditions of Theorem 4.23 note that the function f in (4.35) is now bounded, continuous and has a compact support. So $v \in L^1(\mathbb{R}^n)$. Next, induction on n and Fubini's theorem show that $\widehat{g} \neq 0$ a.e. □

It might be interesting to point out that the latter result is in general false for $n = 1$. Indeed, let $g = \chi_{[-1,1]}$ equal 1 on $[-1, 1]$ and zero otherwise. Let $X = \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^1$, $x_k = k - 1$. For the system of translates $\mathcal{F}_X(g)$ it is easy to compute the Gram matrix $Gr(\{g_j, L^2(\mathbb{R}^1)\}) = J(\{2\}, \{1\})$ the Jacobi matrix with 2's on the main diagonal and 1's off the main diagonal (cf. Example 3.29). Since its spectrum $\sigma(Gr(\{g_j, L^2(\mathbb{R}^1)\})) = [0, 4]$, the corresponding operator on ℓ^2 is not invertible, so $\mathcal{F}_X(g)$ is not a Riesz sequence, as needed.

Remark 4.25 Condition (4.35) appears in various problems of analysis. For instance, it provides convergence of integral means of an integrable functions in all its Lebesgue points (see [32, Theorem 1.25], [6], and [34, Theorem 8.1.3]).

Example 4.26 The conditions of Theorem 4.23 can be verified for the systems we already encountered in the previous section. For instance, as we have seen in Proposition 4.13,

$$g(x) = e^{-a\|x\|^2} \Longrightarrow \widetilde{F}_0(r) = \left(\frac{1}{4a}\right)^{n/2} e^{-\frac{a}{2}r^2}.$$

Similarly, it is shown in Proposition 4.14 that

$$\begin{aligned} g(x) = \left(\frac{a}{\|x\|}\right)^{\mu} K_\mu(a\|x\|), \;\; 0 \le \mu < \frac{n}{4} &\Longrightarrow \widetilde{F}_0(r) \\ = \frac{B\left(\frac{n}{2}-\mu, \frac{1}{2}\right)}{2^{n-2\mu}} \left(\frac{r}{a}\right)^{p} K_p(ar), \;\; p = \frac{n}{2} - 2\mu. \end{aligned}$$

Since in both cases $\widetilde{F}_0 \in \Phi_\infty \subset \mathcal{M}_+$ (cf. Corollary 4.17) and $\widetilde{F}_0$ decays exponentially fast (see 4.19), Theorem 4.23 applies, so $\mathcal{F}_X(f)$ is the Riesz sequence for each $X \in \mathcal{X}_n$.

In view of applications in the spectral theory let us single out two particular cases of the above example.

Corollary 4.27 *Let*

$$\mathcal{F}_2 = \{K_0(a\|\cdot - x_j\|)\}_{j\in\mathbb{N}}, \qquad \mathcal{F}_3 = \left\{\frac{e^{-a\|\cdot - x_j\|}}{\|\cdot - x_j\|}\right\}_{j\in\mathbb{N}}. \tag{4.37}$$

Then each of the sequences $\mathcal{F}_2$ and $\mathcal{F}_3$ forms a Riesz sequence in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^3)$, respectively, for each $X \in \mathcal{X}_n$.

Remark 4.28 Corollary 4.27 is crucial in the study of certain spectral properties of the Schrödinger operator with point interactions [21]. The statement on the system $\mathcal{F}_3$ was proved in [21, Theorem 3.8] in a different manner. The appearance of such functions takes its origin in the following classical formulae for the resolvent of the Laplace operator $H_0 := -\Delta$ in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively,

$$\begin{aligned} (H_0 - zI)^{-1} f &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}\|x-t\|}}{\|x-t\|} f(t)\, dt, \\ (H_0 - zI)^{-1} f &= \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(\sqrt{-z}\|x-t\|)\, f(t)\, dt, \end{aligned} \tag{4.38}$$

(see [5, formulae (1.1.19), (1.5.15)]).

We show now that a sequence $\mathcal{F}_X(g)$ can be *minimal* but *not uniformly minimal*, (so necessarily $d_*(X) = 0$), whenever $\widehat{g} \neq 0$ a.e. is replaced by the stronger assumption (4.39). Note that in the following proposition a function f is not assumed to be radial.

Proposition 4.29 *Given $g \in L^2(\mathbb{R}^n)$, assume that its Fourier transform $\widehat{g}$ admits a lower bound*

$$|\widehat{g}(t)| \geq \frac{C}{(1+\|t\|)^p} \tag{4.39}$$

for some $p > 0$. Then the system $\mathcal{F}_X(g) = \{g(\cdot - x_j)\}_{j\in\mathbb{N}}$ is minimal in $L^2(\mathbb{R}^n)$ for each set $X = \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ with no finite accumulation points.

Proof Put $g_j(\cdot) = g(\cdot - x_j)$. Since the Fourier transform is a unitary operator in $L^2(\mathbb{R}^n)$, the system $\{g_j\}_{j\in\mathbb{N}}$ is minimal in $L^2(\mathbb{R}^n)$ if and only if so is the system of their Fourier images $\{\widehat{g}_j\}_{j\in\mathbb{N}}$. Note that $\widehat{g}_j = \widehat{g}\, e^{-i\langle\cdot, x_j\rangle}$, $f = f_1$ (recall that $x_1 = 0$).

To prove the minimality of $\{\widehat{g}_j\}_{j\in\mathbb{N}}$, it suffices (in fact is equivalent) to construct a biorthogonal sequence $\{h_j\}_{j\in\mathbb{N}}$,

$$\langle h_j, \widehat{g}_k\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} h_j(t)\overline{\widehat{g}(t)}\, e^{i\langle t, x_k\rangle}\, dt = \delta_{kj}, \qquad h_j \in \mathcal{L}(\{\widehat{g}_j\}_{j\in\mathbb{N}}).$$

To this end take a smoothing function w and its shifts w_j

$$w(x) := \begin{cases} \exp\left(\frac{\|x\|^2}{\|x\|^2-1}\right), & \|x\| < 1; \\ 0, & \|x\| \geq 1. \end{cases} \qquad w_j(x) := w\left(\frac{x - x_j}{\rho_j}\right),$$
$$\rho_j := \operatorname{dist}(x_j, X\backslash\{x_j\}) > 0$$

for each j, since X has no finite accumulation points. By the definition $w_j(x_k) = \delta_{kj}$. Since $w \in C_0^\infty$ (infinitely differentiable with compact support), both w_j and $\widehat{w}_j$ belong to the Schwartz class. Define

$$h_{j,1}(t) := (2\pi)^{-\frac{n}{2}}\, \frac{\widehat{w}_j(t)}{\overline{\widehat{f}(t)}}\,.$$

In view of (4.39), $h_{j,1} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, so

$$\langle h_{j,1}, \widehat{g}_k\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} h_{j,1}(t)\overline{\widehat{f}(t)}\, e^{i\langle t, x_k\rangle}\, dt = (2\pi)^{-\frac{n}{2}}$$
$$\int_{\mathbb{R}^n} \widehat{w}_j(t)\, e^{i\langle t, x_k\rangle}\, dt = w_j(x_k) = \delta_{kj}.$$

We are left with putting $h_j := \mathbb{P}h_{j,1}$, where $\mathbb{P}$ is a projection from $L^2(\mathbb{R}^n)$ onto $\mathcal{L}(\{\widehat{g}_j\}_{j\in\mathbb{N}})$. The proof is complete. □

Example 4.30 Let $g = g_{a,\mu}$ be given by (4.21) with $0 \leq \mu < n/4$. Condition (4.39) follows from (4.23), so the system $\mathcal{F}_X(g)$ is minimal for each set X of distinct points with no finite accumulation points.

Remark 4.31 It is easy to construct a set $X = \{x_k\}_{k\in\mathbb{N}}$ with $d_*(X) = 0$, which has no finite accumulation points. For instance, $x_k = \sqrt{k-1}\, e$, $k \in \mathbb{N}$, e is a unit vector in $\mathbb{R}^n$. Note also that a special case of Proposition 4.29 regarding minimality of the system $\mathcal{F}_3$ (4.37) was proved in [21, Lemma 3.5] in a different manner. In the latter case

$$g(x) = \frac{e^{-a\|x\|}}{\|x\|}, \qquad \widehat{g}(t) = \sqrt{\frac{2}{\pi}}\,\frac{1}{a^2 + \|t\|^2},$$

and (4.39) automatically holds.

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