Complex Anal. Oper. Theory (2018) 12:869–876
https://doi.org/10.1007/s11785-017-0672-6

Complex Analysis and Operator Theory

On a Local Darlington Synthesis Problem

L. Golinskii[1]

Received: 9 March 2017 / Accepted: 1 April 2017 / Published online: 10 April 2017

Abstract The Darlington synthesis problem (in the scalar case) is a problem of embedding a given contractive analytic function to an inner 2×2 matrix function as an entry. A fundamental result of Arov–Douglas–Helton relates this algebraic property to a purely analytic one known as a pseudocontinuation of bounded type. We suggest a local version of the Darlington synthesis problem and prove a local analog of the ADH theorem.

Keywords Darlington synthesis · Pseudocontinuation · Inner matrix function · Unitary matrix · Nevanlinna, Schur and Smirnov classes

Mathematics Subject Classification 30H05 · 30H15 · 30C80

1 Introduction

The Darlington synthesis with its origin in electrical engineering has a long history. The synthesis of non-lossless circuits was a hard problem at the time when computers were unavailable. The idea of the Darlington synthesis was to reduce any such problem to a lossless one.

A mathematical setup in the simplest scalar case looks as follows, see [1–3,5], [6, Section 8.6] and [7, Section 6.7].

Communicated by Vladimir Bolotnikov.

✉ L. Golinskii
golinskii@ilt.kharkov.ua

1 B. Verkin Institute for Low Temperature Physics and Engineering, 47 Science Ave., Kharkov 61103, Ukraine

An analytic function s on the unit disk $\mathbb{D}$ is called a *Schur* (*contractive*) function, $s \in \mathcal{S}$, if $|s| \le 1$ in $\mathbb{D}$. Similarly, an analytic on $\mathbb{D}$ 2×2 matrix function S (throughout this note we deal only with matrices of order 2) is a Schur (contractive) matrix function, $S \in \mathcal{S}^{(m)}$, if

$$I - S^*(z)S(z) \ge 0, \quad z \in \mathbb{D},$$

I is a unity matrix. A function $s \in \mathcal{S}$ (a matrix function $S \in \mathcal{S}^{(m)}$) is said to be *inner (matrix) function* if its boundary values, which exist almost everywhere on the unit circle $\mathbb{T}$, are unimodular (unitary). Given $s \in \mathcal{S}$, the Darlington synthesis problem asks whether there exists an inner matrix function $S \in \mathcal{S}^{(m)}$ so that

$$S(z) = \|s_{ij}(z)\|_{i,j=1}^2 : \quad s_{22}(z) = s(z). \tag{1.1}$$

A seminal result of Arov [1] and Douglas–Helton [3] states that a Schur function admits the Darlington synthesis if and only if it possesses a pseudocontinuation of bounded type across $\mathbb{T}$. Recall that a meromorphic function of bounded type on a region Ω is the quotient of two bounded (or contractive) analytic on Ω functions

$$f(z) = \frac{f_1(z)}{f_2(z)}, \quad f_j \in \mathcal{S}(\Omega). \tag{1.2}$$

Such functions constitute the Nevanlinna class $N(\Omega)$.

The goal of this note is to suggest a local version of the Darlington synthesis problem and to prove a local analog of the Arov–Douglas–Helton theorem. As usual, $\mathbb{D}_e$ stands for the exterior of the unit disk with respect to the extended complex plane $\bar{\mathbb{C}}$.

Definition 1.1 Let γ be an arc of the unit circle (the case $\gamma = \mathbb{T}$ is not excluded). A function $f \in N(\mathbb{D})$ admits a *pseudocontinuation of bounded type across* γ if there is a function $\tilde{f} \in N(\mathbb{D}_e)$ so that their boundary values agree

$$f(t) = \tilde{f}(t) \ \text{ a.e. on } \gamma. \tag{1.3}$$

We write $f \in PC_\gamma$ for such functions. The class PC_γ is nontrivial, see Example 2.1 in Sect. 2.

Theorem 1.2 *Let $s \in \mathcal{S}$. The following conditions are equivalent.*

(1) There is a matrix function $S = \|s_{ij}\|_{i,j=1}^2$ so that $s_{ij} \in \mathcal{S}$, $s_{22} = s$, and S is unitary a.e. on the arc γ;
(2) $s \in PC_\gamma$.

In the case $\gamma = \mathbb{T}$, the above matrix function S is inner due to the Maximum Norm Principle, and we come to the Arov–Douglas–Helton theorem.

Given an arc γ, we denote by $\mathcal{S}_\gamma$ (N_γ) the class of the Schur (Nevanlinna) functions, unimodular a.e. on γ. Similarly, $\mathcal{S}_\gamma^{(m)}$ stands for the class of the Schur matrix functions unitary a.e. on γ.

It is clear that a matrix function with contractive entries does not necessarily belong to $\mathcal{S}^{(m)}$. So the question arises naturally whether the matrix S in Theorem 1.2 can be taken from $\mathcal{S}_\gamma^{(m)}$. If $s \in \mathcal{S}_\gamma$, the answer is affirmative: the matrix function

$$S(z) = \begin{bmatrix} s_{11}(z) & 0 \\ 0 & s(z) \end{bmatrix} \tag{1.4}$$

with an arbitrary inner function s_{11} belongs to $\mathcal{S}_\gamma^{(m)}$. But, in general, the answer is negative. The reason is that s being an entry of a contractive, nondiagonal matrix function is supposed to obey a *global* condition

$$\int_{\mathbb{T}} \log(1 - |s(t)|^2)\, m(dt) > -\infty, \tag{1.5}$$

m is the normalized Lebesgue measure on $\mathbb{T}$. As it turns out, this condition is also sufficient.

Theorem 1.3 *Let $s \in \mathcal{S}'_\gamma = \mathcal{S}\backslash\mathcal{S}_\gamma$. The following conditions are equivalent.*

(1) There is a matrix function $V = \|v_{ij}\|_{i,j=1}^2 \in \mathcal{S}_\gamma^{(m)}$ so that $v_{22} = s$;
(2) $s \in PC_\gamma$ and (1.5) holds.

In contrast to the case $\gamma = \mathbb{T}$ of the whole unit circle, we have neither the model spaces theory nor the Douglas–Shapiro–Shields theorem at hand. So the argument is by and large straightforward and relies upon explicit (in a sense) expressions for the matrix entries of the matrices in question.

2 Local Pseudocontinuation and Darlington Synthesis

Let us begin with the classes $\mathcal{S}_\gamma$ and N_γ, which play the same role as the class of inner functions does in the classical setting of the Darlington synthesis problem.

Example 2.1 Let $a \in N_\gamma$. Write

$$\tilde{a}(\zeta) := \frac{1}{\overline{a(1/\bar{\zeta})}}, \quad \zeta \in \mathbb{D}_e. \tag{2.1}$$

Then $\tilde{a} \in N(\mathbb{D}_e)$ and $\tilde{a} = a$ a.e. on γ, so $N_\gamma \subset PC_\gamma$. In particular, $s \in \mathcal{S}_\gamma$ implies $s \in PC_\gamma \backslash PC_{\mathbb{T}}$, as soon as s is not an inner function.

Proof of Theorem 1.2 (1) $\Rightarrow$ (2). The argument here is standard. By the hypothesis, $\det S \not\equiv 0$, so we write

$$\begin{aligned} U(\zeta) &:= (S^{-1})^*(1/\bar{\zeta}) \\ &= \frac{1}{\overline{\det S(1/\bar{\zeta})}} \begin{bmatrix} \overline{s(1/\bar{\zeta})} & -\overline{s_{21}(1/\bar{\zeta})} \\ -\overline{s_{12}(1/\bar{\zeta})} & \overline{s_{11}(1/\bar{\zeta})} \end{bmatrix}, \quad \zeta \in \mathbb{D}_e. \end{aligned} \tag{2.2}$$

It is clear that all entries of U belong to $N(\mathbb{D}_e)$, and $U = S$ a.e. on γ. Hence, s admits the pseudocontinuation of bounded type across γ, $s \in PC_\gamma$, with

$$\tilde{s}(\zeta) = \frac{\overline{s_{11}(1/\bar{\zeta})}}{\overline{\det S(1/\bar{\zeta})}}.$$

Note that in fact each entry of the bounded matrix function S, unitary a.e. on γ, is in the class PC_γ.

(2) $\Rightarrow$ (1). If $s \in \mathcal{S}_\gamma$, the result follows immediately from (1.4). So we assume further that $s \in \mathcal{S}'_\gamma = \mathcal{S}\backslash\mathcal{S}_\gamma$.

Define a pair of functions on $\mathbb{D}$

$$g(z) := \overline{\tilde{s}(1/\bar{z})} \in N(\mathbb{D}), \quad h(z) := 1 - g(z)s(z) \in N(\mathbb{D}), \tag{2.3}$$

where $\tilde{s}$ is the pseudocontinuation of bounded type of s across γ. Now, $s \notin \mathcal{S}_\gamma$ implies $h \not\equiv 0$, so $\log|h| \in L^1(\mathbb{T})$, see [4, Theorem 2.2], and

$$\int_\gamma \log|h(t)|\, m(dt) = \int_\gamma \log(1 - |s(t)|^2)\, m(dt) > -\infty. \tag{2.4}$$

We see that $\log(1-|s|^2) \in L^1(\gamma)$ as long as $s \in PC_\gamma\backslash\mathcal{S}_\gamma$, which is a local counterpart of the relation (1.5).

In view of (2.4), the function

$$\sigma_\gamma(z) := \exp\left\{\frac{1}{2}\int_\gamma \frac{t+z}{t-z}\log(1-|s(t)|^2)\, m(dt)\right\} \tag{2.5}$$

is a well-defined, outer Schur function, $\sigma_\gamma \in \mathcal{S}_{out}$, with the boundary values

$$|\sigma_\gamma(t)|^2 = 1 - |s(t)|^2 \ \text{ a.e. on } \gamma, \quad |\sigma_\gamma(t)| = 1 \ \text{ a.e. on } \gamma' := \mathbb{T}\backslash\gamma. \tag{2.6}$$

We choose $s_{12} := \sigma_\gamma$.

Going back to the Nevanlinna functions g, h in (2.3), we write

$$g(z) = \frac{g_1(z)}{g_2(z)} = \frac{I_{g_1}(z)O_{g_1}(z)}{I_{g_2}(z)O_{g_2}(z)}, \quad g_j \in \mathcal{S},$$
$$h(z) = \frac{h_1(z)}{h_2(z)} = \frac{I_{h_1}(z)O_{h_1}(z)}{I_{h_2}(z)O_{h_2}(z)}, \quad h_j \in \mathcal{S},$$

where $f = I_f O_f$ is the standard inner-outer factorization of a Schur function f. We proceed with the further factorization of the outer factors with respect to γ, precisely,

$$\begin{aligned} O(z) &= O(z,\gamma)O(z,\gamma'), \\ O(z,\Gamma) &:= \exp\left\{\frac{1}{2}\int_\Gamma \frac{t+z}{t-z}\log|O(t)|\, m(dt)\right\} \end{aligned} \tag{2.7}$$

for the arc $\Gamma = \gamma, \gamma'$. We have $O(\cdot, \Gamma) \in \mathcal{S}_{out}$ and

$$|O(t, \gamma)| = 1 \ \text{ a.e. on } \gamma', \quad |O(t, \gamma')| = 1 \ \text{ a.e. on } \gamma. \tag{2.8}$$

Hence,

$$\begin{aligned} g(z) &= \frac{I_{g_1}(z) O_{g_1}(z, \gamma) O_{g_1}(z, \gamma')}{I_{g_2}(z) O_{g_2}(z, \gamma) O_{g_2}(z, \gamma')}, \\ h(z) &= \frac{I_{h_1}(z) O_{h_1}(z, \gamma) O_{h_1}(z, \gamma')}{I_{h_2}(z) O_{h_2}(z, \gamma) O_{h_2}(z, \gamma')}. \end{aligned} \tag{2.9}$$

Put

$$p(z) := I_{g_2}(z) I_{h_2}(z)\, O_{g_2}(z, \gamma') O_{h_2}(z, \gamma'), \tag{2.10}$$

so $|p| = 1$ a.e. on γ. Our choice of s_{11} and s_{21} is

$$\begin{aligned} s_{11}(z) &:= -p(z) g(z) = -I_{g_1}(z) I_{h_2}(z)\, O_{g_1}(z, \gamma') O_{h_2}(z, \gamma')\, \frac{O_{g_1}(z, \gamma)}{O_{g_2}(z, \gamma)}, \\ s_{21}(z) &:= p(z)\, \frac{h(z)}{\sigma_\gamma(z)} = I_{g_2}(z) I_{h_1}(z)\, O_{g_2}(z, \gamma') O_{h_1}(z, \gamma')\, \frac{O_{h_1}(z, \gamma)}{O_{h_2}(z, \gamma)\sigma_\gamma(z)}. \end{aligned} \tag{2.11}$$

It is clear that $s_{22} = s$ and $s_{12} = \sigma_\gamma$ in (2.5) are contractive functions. As for s_{11} and s_{21} (2.11), we note that they belong to an important subclass $N^+(\mathbb{D}) \subset N(\mathbb{D})$ of the Nevanlinna class, which is usually referred to as the Smirnov class, see [4, Section 2.5]. It is characterized by the denominator in (1.2) being an outer Schur function, which is exactly the case in (2.11). The main feature of this class is the Smirnov maximum modulus principle, [4, Theorem 2.11], which states that

$$f \in N^+(\mathbb{D}), \quad |f(t)| \le 1 \ \text{ a.e. on } \mathbb{T} \ \Rightarrow \ f \in \mathcal{S}. \tag{2.12}$$

In view of (2.8), we obtain $|s_{11}| \le 1$, $|s_{21}| \le 1$ a.e. on γ'. Next, as we have already mentioned, $|p| = 1$ a.e. on γ, so

$$|s_{11}(t)| \le |g(t)| = |s(t)| \le 1, \quad |s_{21}(t)| \le \frac{|h(t)|}{|\sigma_\gamma(t)|} = (1 - |s(t)|^2)^{1/2} \le 1$$

a.e. on γ, and the first claim of the Theorem follows from (2.12).

To show that S is unitary a.e. on γ, we put

$$S^*(t)S(t) = \begin{bmatrix} |s_{11}(t)|^2 + |s_{21}(t)|^2 & \overline{s_{11}(t)} s_{12}(t) + \overline{s_{21}(t)} s(t) \\ s_{11}(t)\overline{s_{12}(t)} + s_{21}(t)\overline{s(t)} & |s_{12}(t)|^2 + |s(t)|^2 \end{bmatrix}.$$

By (2.6),

$$|s_{12}(t)|^2 + |s(t)|^2 = |\sigma_\gamma(t)|^2 + |s(t)|^2 = 1 - |s(t)|^2 + |s(t)|^2 = 1$$

a.e. on γ. Next, $|p| = 1$ a.e. on γ implies

$$|s_{11}(t)|^2 + |s_{21}(t)|^2 = |g(t)|^2 + \frac{|h(t)|^2|}{|\sigma_\gamma(t)|^2} = |s(t)|^2 + 1 - |s(t)|^2 = 1.$$

Finally, by (2.6) and the definition of h,

$$\begin{aligned} s_{11}(t)\overline{s_{12}(t)} + s_{21}(t)\overline{s(t)} &= p(t)\Big(-g(t)\overline{\sigma_\gamma(t)} + \frac{h(t)}{\sigma_\gamma(t)}\,\overline{s(t)}\Big) \\ &= p(t)\overline{s(t)}\Big(-\overline{\sigma_\gamma(t)} + \frac{h(t)}{\sigma_\gamma(t)}\Big) = 0 \end{aligned}$$

a.e. on γ. So, $S^*S = I$, as claimed. The proof is complete. □

Proof of Theorem 1.3 (1) $\Rightarrow$ (2). By Theorem 1.2, $s \in PC_\gamma$, so we should verify condition (1.5). Note that at least one of the functions v_{12}, v_{21} is not identically zero (otherwise, $s \in \mathcal{S}_\gamma$). Assume that $v_{12} \not\equiv 0$ and write

$$I - V^*(t)V(t) = \begin{bmatrix} * & * \\ * & 1 - |v_{12}(t)|^2 - |s(t)|^2 \end{bmatrix} \ge 0, \quad t \in \mathbb{T},$$

so $1 - |s(t)|^2 \ge |v_{12}(t)|^2$. Since $\log|v_{12}| \in L^1(\mathbb{T})$, the condition (1.5) follows.

(2) $\Rightarrow$ (1). The matrix V arises as an appropriate modification of the matrix S from Theorem 1.2. By (1.5), the function

$$\sigma_{\gamma'}(z) := \exp\left\{\frac{1}{2}\int_{\gamma'} \frac{t+z}{t-z}\,\log(1 - |s(t)|^2)\,m(dt)\right\}$$

is well-defined and lies in $\mathcal{S}_{out}$. Denote by e the outer Schur function with

$$|e(t)| = 1 \ \text{ a.e. on } \gamma, \quad |e(t)| = \varepsilon \ \text{ a.e. on } \gamma',$$

where $0 < \varepsilon < 1/3$ is a small enough positive constant, and put $r := e\sigma_{\gamma'}$. Take the matrix V in question as

$$V(z) = \begin{bmatrix} r(z) & 0 \\ 0 & 1 \end{bmatrix} S(z) \begin{bmatrix} r(z) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r^2(z)s_{11}(z) & r(z)s_{12}(z) \\ r(z)s_{21}(z) & s(z) \end{bmatrix},$$

As both e and $\sigma_{\gamma'}$ are unimodular on γ, then so is r, and thereby V is unitary a.e. on γ.

It remains to check that $V \in \mathcal{S}^{(m)}$. To this end we put for $t \in \gamma'$

$$\begin{aligned} W(t) = \begin{bmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{bmatrix} &:= I - V^*(t)V(t) \\ &= \begin{bmatrix} 1 - |r^2 s_{11}|^2 - |rs_{21}|^2 & -\bar{r}|r|^2 s_{12}\overline{s_{11}} - \overline{rs_{21}}s \\ -r|r|^2 s_{11}\overline{s_{12}} - rs_{21}\bar{s} & 1 - |rs_{12}|^2 - |s|^2 \end{bmatrix}. \end{aligned}$$

Since

$$r(z)s_{12}(z) = e(z)\sigma_{\gamma'}(z)\sigma_{\gamma}(z) = e(z)\,\exp\left\{\frac{1}{2}\int_{\mathbb{T}}\frac{t+z}{t-z}\,\log(1-|s(t)|^2)\,m(dt)\right\},$$

then $|rs_{12}|^2 = \varepsilon^2(1-|s|^2)$, and so

$$w_{22}(t) = 1 - |r(t)s_{12}(t)|^2 - |s(t)|^2 = (1-\varepsilon^2)(1-|s(t)|^2) \ge 0 \tag{2.13}$$

a.e. on γ'. Next, the functions r, s_{11}, s_{21} are contractive, hence,

$$\begin{aligned} w_{11}(t) &= 1 - |r^2(t)s_{11}(t)|^2 - |r(t)s_{21}(t)|^2 = 1 - |r(t)|^2\big(|r(t)s_{11}(t)|^2 + |s_{21}(t)|^2\big) \\ &= 1 - \varepsilon^2(1-|s(t)|^2)(|r(t)s_{11}(t)|^2 + |s_{21}(t)|^2) \ge 1 - 2\varepsilon^2(1-|s(t)|^2) \end{aligned}$$

and for $0 < \varepsilon < 1/3$

$$w_{11}(t) \ge 1 - 2\varepsilon^2(1-|s(t)|^2) > \frac{7}{9} \tag{2.14}$$

a.e. on γ'. Finally,

$$-w_{21}(t) = r(t)\big(s_{21}(t)\overline{s(t)} + |r(t)|^2 s_{11}(t)\overline{s_{12}(t)}\big) = r(t)v(t), \quad |v(t)| \le 2,$$

and thereby,

$$W(t) \ge \begin{bmatrix} \frac{7}{9} & -\overline{r(t)v(t)} \\ -r(t)v(t) & (1-\varepsilon^2)(1-|s(t)|^2) \end{bmatrix} = \widetilde{W}(t) = \|\tilde{w}_{ij}(t)\|_{i,j=1}^2.$$

To show that $\widetilde{W} \ge 0$ a.e. on γ', given $\tilde{w}_{11} \ge 0$, $\tilde{w}_{22} \ge 0$, we compute the determinant of $\widetilde{W}$, keeping in mind $0 < \varepsilon < 1/3$:

$$\begin{aligned} \tilde{w}_{11}(t)\tilde{w}_{22}(t) - |\tilde{w}_{12}(t)|^2 &= \frac{7}{9}(1-\varepsilon^2)(1-|s(t)|^2) - |\varepsilon v(t)|^2(1-|s(t)|^2) \\ &\ge \Big(\frac{7}{9}(1-\varepsilon^2) - 4\varepsilon^2\Big)(1-|s(t)|^2) \ge \frac{2}{9}(1-|s(t)|^2) \ge 0 \end{aligned}$$

a.e. on γ'. So, $V \in \mathcal{S}_{\gamma}^{(m)}$, and the proof is complete. □

We complete this note with some properties of the pseudocontinuation of bounded type across an arc.

Proposition 2.2 *Let $s_1, s_2 \in \mathcal{S}$ and $|s_1| = |s_2|$ a.e. on an arc γ. Then s_1 and s_2 belong to PC_{γ} simultaneously.*

Proof Let $s_1 \in PC_{\gamma}$. We have the canonical factorization

$$s_k(z) = I_k(z)O_k(z,\gamma)O_k(z,\gamma'), \quad k = 1, 2,$$

and, by the assumption, $O_1(\cdot,\gamma) = O_2(\cdot,\gamma)$. Hence,

$$s_2(z) = a(z)s_1(z), \quad a(z) := \frac{I_2(z)O_2(z,\gamma')}{I_1(z)O_1(z,\gamma')}.$$

The function $a \in N_\gamma$, so, see Example 2.1, $a \in PC_\gamma$. The later class is closed under multiplication, so $s_2 \in PC_\gamma$, which is our claim. □

Recall that σ_γ is defined in (2.5) under the condition (2.4).

Proposition 2.3 *Let* $s \in \mathcal{S}$ *and* $\log(1-|s|^2) \in L^1(\gamma)$. *Then*

$$s \in PC_\gamma \Leftrightarrow \sigma_\gamma \in PC_\gamma.$$

Proof As we mentioned earlier in the proof of Theorem 1.2, each entry of the bounded matrix function S, unitary a.e. on γ, is in the class PC_γ. If $s \in PC_\gamma$, the matrix function S in Theorem 1.2 contains both s and σ_γ as its entries, and we are done.

Conversely, let $\sigma_\gamma \in PC_\gamma$. By Theorem 1.2, there is a matrix function Σ with contractive entries, unitary a.e. on γ, and

$$\Sigma(z) = \begin{bmatrix} \sigma_{11}(z) & \sigma_{12}(z) \\ \sigma_{21}(z) & \sigma_\gamma(z) \end{bmatrix}.$$

In particular, $|\sigma_{12}|^2 + |\sigma_\gamma|^2 = 1$, and so $|\sigma_{12}| = |s|$ a.e. on γ. The function σ_{12}, being the entry of Σ, belongs to the class PC_γ. By Proposition 2.2, so does s, as claimed. □

Remark 2.4 The fact that γ is the arc of the unit circle is obviously immaterial. The argument works for an arbitrary Borel set $\gamma \subset \mathbb{T}$ of positive Lebesgue measure.

Acknowledgements I thank the participants of the Analysis Seminar at Kharkiv National University for valuable discussions.

References

1. Arov, D.Z.: Darlington's method in the study of dissipative systems. Dokl. Akad. Nauk SSSR **201**(3), 559–562 (1971)
2. Arov, D.Z.: Realization of matrix-valued functions according to Darlington. Izv. Akad. Nauk SSSR Ser. Mat. **7**(6), 1295–1326 (1973)
3. Douglas, R.G., Helton, J.W.: Inner dilations of analytic matrix functions and Darlington synthesis. Acta Sci. Math. (Szeged) **34**, 61–67 (1973)
4. Duren, P.: Theory of H^p Spaces, Pure and Applied Mathematics, vol. 38. Academic Press, New York (1970)
5. Garcia, S.R.: Inner matrices and Darlington synthesis. Methods Funct. Anal. Topol. **11**(1), 37–47 (2005)
6. Garcia, S.R., Mashreghi, J., Ross, W.T.: Introduction to Model Spaces and Their Operators, Cambridge Studies in Advanced Mathematics, vol. 148. Cambridge University Press, Cambridge (2016)
7. Ross, W.T., Shapiro, H.S.: Generalized Analytic Continuation, University Lecture Series, vol. 25. AMS, Providence (2002)