Central limit theorems for the free energy of the modified Gardner problem

M.Shcherbina*, B.Tirozzi†

Abstract

Fluctuations of the free energy of the modified Gardner model for any $\alpha < \alpha_c$ are studied. It is proved that they converge in distribution to a Gaussian random variable.

1 Introduction and Main Results

In the paper [S-T2] we introduced some modification of the famous Gardner model:

$$\mathcal{H}(\boldsymbol{J},k,z,\varepsilon) \equiv -\sum_{\mu=1}^{p} \log \mathrm{H}\left(\frac{k-(\boldsymbol{\xi}^{(\mu)},\boldsymbol{J})N^{-1/2}}{\sqrt{\varepsilon}}\right) + \frac{z}{2}(\boldsymbol{J},\boldsymbol{J}), \quad p \sim \alpha N \tag{1.1}$$

where $\boldsymbol{J},\boldsymbol{\xi}^{(\mu)} \in \mathbf{R}^N$, $\{\xi_i^{(\mu)}\}_{i=1,\dots,N,\,\mu=1,\dots p}$ are taken to be independent random variables with zero mean and variance 1 and ε, z, k are some positive parameters. The function $\mathrm{H}(x)$ is defined as

$$\mathrm{H}(x) \equiv \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt, \tag{1.2}$$

The partition function of the Hamiltonian (1.1) is

$$Z_{N,p}(k,z,\varepsilon) = \sigma_N^{-1}\int d\boldsymbol{J}\exp\{-\mathcal{H}(\boldsymbol{J},k,z,\varepsilon)\}, \tag{1.3}$$

where σ_N is the Lebesgue measure of the N-dimensional sphere of radius $N^{1/2}$.

We denote also by $\langle \dots \rangle$ the corresponding Gibbs averaging and

$$f_{N,p}(k,z,\varepsilon) \equiv \frac{1}{N}\log Z_{N,p}(k,z,\varepsilon). \tag{1.4}$$

In [S-T2] we used this model to find the free energy of the initial Gardner model. This model was introduced in [G] to study the typical volume of interactions between each pair of N Ising spins which solve the problem of storing a given set of p random patterns $\{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^p$. In this initial model the partitian function has the form (cf 1.3).

$$\Theta_{N,p}(k) = \sigma_N^{-1}\int_{(\boldsymbol{J},\boldsymbol{J})=N} d\boldsymbol{J}\prod_{\mu=1}^{p}\theta(N^{-1/2}(\boldsymbol{\xi}^{(\mu)},\boldsymbol{J})-k), \tag{1.5}$$

*Institute for Low Temperature Physics,Ukr. Ac. Sci., 47 Lenin ave., Kharkov, Ukraine
†Department of Physics of Rome University "La Sapienza", 5, p-za A.Moro, Rome, Italy

where the function $\theta(x)$ is zero in the negative semi-axis and 1 in the positive. By by using the model (1.1)-(1.4) we prove mathematically the following result obtained by Gardner with a replica trick: there exists a critical value of α

$$\alpha_c(k) \equiv (\frac{1}{\sqrt{2\pi}} \int_{-k}^{\infty} (u+k)^2 e^{-u^2/2} du)^{-1}, \tag{1.6}$$

such that for any $\alpha < \alpha_c(k)$ there exists the limit

$$\begin{aligned} \lim_{N,p\to\infty, p/N\to\alpha} \frac{1}{N} E\{\log \Theta_{N,p}(k)\} &= \min_{0\le q\le 1} \Big[\alpha E\Big\{ \log \mathrm{H}\Big(\frac{u\sqrt{q}+k}{\sqrt{1-q}}\Big)\Big\} \\ &\quad + \frac{1}{2}\frac{q}{1-q} + \frac{1}{2}\log(1-q)\Big]. \end{aligned} \tag{1.7}$$

Here u is a Gaussian random variable with zero mean and variance 1, and here and below we denote by the symbol $E\{...\}$ the averaging with respect to all random parameters of the problem and also with respect to u. And for $\alpha \ge \alpha_c(k)$ $\frac{1}{N}\log \Theta_{N,p}(k)$ tends to minus infinity.

Comparing (1.3) with (1.5), one can see that we replace the functions $\theta(x)$ by $\mathrm{H}(-x/\sqrt{\varepsilon})$, to have possibility to study the model (1.5) by means of statistical mechanics of the systems with random interaction (see [S-T1]-[T5]). When $\varepsilon \to 0$, $\mathrm{H}(-x/\sqrt{\varepsilon}) \to \theta(x)$, so the model (1.5) is the limiting case of (1.1)-(1.4). The other difference from (1.5) is that we introduce an additional parameter $z > 0$ to replace the integration over the sphere $(\boldsymbol{J},\boldsymbol{J}) = N$ in (1.5) by the integration in the whole space $\mathbf{R}^N$ in (1.3).

In [S-T2] we have proved that for any $\alpha < 2$, $k \ge 0$, there exists $\varepsilon^*(\alpha,k)$ such that for any $\varepsilon \le \varepsilon^*(\alpha,k)$ and $z \le \varepsilon^{-1/3}$ there exists

$$\begin{aligned} \lim_{N,p\to\infty, p/N\to\alpha} E\{f_{N,p}(k,z,\varepsilon)\} &= F(\alpha,k,z,\varepsilon), \\ F(\alpha,k,z,\varepsilon) &\equiv \max_{R>0} \min_{0\le q\le R} \Big[\alpha E\Big\{ \log H\Big(\frac{u\sqrt{q}+k}{\sqrt{\varepsilon+R-q}}\Big)\Big\} \\ &\quad + \frac{1}{2}\frac{q}{R-q} + \frac{1}{2}\log(R-q) - \frac{z}{2}R\Big], \end{aligned} \tag{1.8}$$

where u is a Gaussian random variable with zero mean and variance 1.

Similar results for small α were obtained in [T2] for the so-called Gardner-Derrida model. In the paper [T4] the fluctuation of the order parameters for the Gardner-Derrida model were studied, but only for small enough α.

We would like to mention here also the work [Gu-T], where the fluctuations of the overlap parameters and of the free energy for the Sherrington-Kirkpatrick model in the high temperature region were studied by the method of characteristic functions.

In the paper [S-T4] we study the fluctuations of the order parameters of the model (1.1) for all $\alpha < 2$, $k > 0$, $\varepsilon \le \varepsilon^*(\alpha,k)$ and $z \le \varepsilon^{-1/3}$. In particular, we consider the family of the order parameters

$$R_{l,m} = \frac{1}{N}(\boldsymbol{J}^{(l)}, \boldsymbol{J}^{(m)}), \quad (l,m = 1,\dots n), \tag{1.9}$$

where the upper indexes of the variables $\boldsymbol{J}$ mean that we consider n replicas of the Hamiltonian (1.1) with the same random parameters $\{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^p$, but different $\boldsymbol{J}^{(1)},\dots,\boldsymbol{J}^{(n)}$. We have proved that if we fix n and study the limit $N,p\to\infty$, then the family of the random variables $u_{l,m} = N^{1/2}(R_{l,m} - q)$ $(l \ne m)$ converges in distribution to the family of Gaussian random variables. Here and below we denote by (q,R) the unique solution of the system of equations

$$\frac{\partial \mathcal{F}}{\partial q} = 0, \quad \frac{\partial \mathcal{F}}{\partial R} = 0, \tag{1.10}$$

for the function $\mathcal{F}(q,R;k,z,\varepsilon)$ which is defined by the expression in the r.h.s. of (1.8) before taking $\max_R \min_q$. It is proven in [S-T2] that if $\alpha<2$, $\varepsilon\le\varepsilon^*(\alpha,k)$ and $z\le\varepsilon^{1/3}$, the the system (1.10) has a unique solution.

This work is based on the results of [S-T2], and [S-T4]. The main result of the paper is the following theorem:

thm:1 **Theorem 1.** *Consider the model (??)-(1.4) with Gaussian independent $\{\xi_i^{(\mu)}\}_{i=1,\dots,N,\mu=1,\dots,p}$ with zero mean and variance 1. Then for any $\alpha<2$, $k>0$, $\varepsilon\le\varepsilon^*(\alpha,k)$ and $z\le\varepsilon^{-1/3}$. Then the random variable*

$$v_{N,p}=N^{1/2}(f_{N,p}-E\{f_{N,p}\}) \tag{1.11}$$ v

converges in distribution, as $N,p\to\infty, p/N\to\alpha$, to a Gaussian random variable with zero mean and the variance

$$V^2=\alpha E\left\{\left(\log \mathrm{H}\left(\frac{u\sqrt{q}+k}{\sqrt{\varepsilon+R-q}}\right)\right)^2\right\}-\alpha E^2\left\{\log \mathrm{H}\left(\frac{u\sqrt{q}+k}{\sqrt{\varepsilon+R-q}}\right)\right\} \tag{1.12}$$ var

Remark. In fact Theorem 1 can be proved for any independent $\{\xi_i^{(\mu)}\}_{i=1,\dots,N,\mu=1,\dots,p}$ with zero mean and variance 1 under condition $E\{|\xi_i^{(\mu)}|^4\}\le C$. We use the Gaussian random variable to integrate by parts, i.e. to use the formula

$$E\left\{\xi_i^{(\mu)}\phi\left(\xi_i^{(\mu)}\right)\right\}=E\left\{\phi'\left(\xi_i^{(\mu)}\right)\right\} \tag{1.13}$$ ibp

valid for any smooth function $\phi(x)$ which grouth like polynomial at infinity. But in all our considerations we have $\phi(x)=\tilde\phi(xN^{-1/2})$, where $\tilde\phi$ is some smooth independent of N function. So, if $\xi_i^{(\mu)}$ is nongaussian, instead of (1.13) one can use the formula

$$E\left\{\xi_i^{(\mu)}\tilde\phi\left(\xi_i^{(\mu)}N^{-1/2}\right)\right\}=N^{-1/2}E\left\{\tilde\phi'\left(\xi_i^{(\mu)}N^{-1/2}\right)\right\}+\frac{1}{2N}E\left\{\left(\xi_i^{(\mu)}\right)^3\tilde\phi''\left(\zeta(\xi_i^{(\mu)})\xi_i^{(\mu)}N^{-1/2}\right)\right\} \tag{1.14}$$ ibp'

where $|\zeta(\xi_i^{(\mu)})|\le 1$. Since the second term in the r.h.s. of (1.14) is of order $O(N^{-1})$ while the first one is $O(N^{-1/2})$ it is easy to believe that the terms the second type will give a smaller contribution than the terms of the first type. But to make corresponding bounds rigorously we need to do some standard but rather tedious work. That is why in order to avoid additional technical difficulties in Theorem 1 we assume that $\{\xi_i^{(\mu)}\}$ are independent Gaussian random variables.

2 Proof of Theorem 1

For the proof of Theorem 1 we need some apriory estimates for the fluctuation of the free energy. Namely, we need the bound

$$E\{|v_{N,p}|^m\}\le\Gamma(m)C^m. \tag{2.1}$$ b_v

We remark that here and below we use C to denote some constant whose value is not important for us, but which is independent of N,p. This value can be different in different formulas.

The bound (2.1) one can be obtained by using the lemma, proven in [S-T4].

lem:1 **Lemma 1.** *Let $\mathbf{u}\in\mathbf{R^m}$ be a Gaussian random vector with normally distributed independent components $\{u_i\}_{i=1}^m$ and $f(\mathbf{x})$ be some function defined in $\mathbf{R}^m$. If there exist A_0 and $s_0>0$ such that $f(\mathbf{x})$ satisfies the conditions:*

$$P(A)=\mathrm{Prob}\{|\nabla f(\mathbf{u})|^2\ge A\}\le e^{-C(A-A_0)}\ (\forall A>A_0) \tag{2.2}$$ p1.3

$$E\{e^{\pm s_0 f}\}\le e^{s_0 B}, \tag{2.3}$$ p1.3a

Then for $s \le \frac{1}{4}s_0$ *we have*

$$E\{e^{\pm s(f(\mathbf{u})-Ef(\mathbf{u}))}\} \le e^{2A_0 s^2}(1+(A_0+C^{-1})e^{2sB-CA_0/2}). \qquad (2.4)$$ p1.4

Using this lemma for $\mathbf{u} = (\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(p)})$, $f(\mathbf{u}) = Nf_{N,p}$ and $A_0 = Na_0$, $B = Nb$ (a_0, b some N-independent constants) and $s = N^{-1/2}$, one can get easily that

$$E\{\cosh v_{N,p}\} \le C$$

which implies (2.1).

Hence, we need only to check (2.2) and (2.3) for $f(\mathbf{u}) = Nf_{N,p}$.

Let us write

$$|\nabla f(\mathbf{u})|^2 = \frac{1}{\varepsilon N}\sum_{\mu,i}\langle J_i A_\mu\rangle^2 \le \frac{1}{N}\sum_{\mu}\langle A_\mu^2\rangle\langle(\boldsymbol{J},\boldsymbol{J})\rangle,$$

where we denote $A_\mu = \mathrm{A}((k-S_\mu)\varepsilon^{-1/2})$ with

$$\mathrm{A}(x) = -\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\log \mathrm{H}(x).$$

and here and below

$$S_\mu = \frac{1}{N^{1/2}}(\boldsymbol{J}, \boldsymbol{\xi}^{(\mu)}). \qquad (2.5)$$ ti_R

It is easy to see that

$$A_\mu^2 \le C_1 - C_2 \log \mathrm{H}((k-S_\mu)\varepsilon^{-1/2}), \qquad (2.6)$$ in1

Let us use a bound valid for any function $\phi(\boldsymbol{J})$ and any Gibbs averaging $\langle\dots\rangle_0$:

$$-\frac{\langle\phi(\boldsymbol{J})e^{\phi(\boldsymbol{J})}\rangle_0}{\langle e^{\phi(\boldsymbol{J})}\rangle_0} \le -\langle\phi(\boldsymbol{J})\rangle_0. \qquad (2.7)$$ gen_b

If we denote by $\langle\dots\rangle_0$ the Gibbs averaging with respect to the Hamiltonian $H_0 = z(\boldsymbol{J},\boldsymbol{J})$, we get

$$\frac{1}{N}\sum_{\mu}\langle A_\mu^2\rangle \;\le\; C_1 - C_2\langle\log \mathrm{H}((k-S_\mu)\varepsilon^{-1/2})\rangle_0 \;\le\; C_1 + C_2'\langle(S_\mu)^2\rangle_0 \;\le\; C_1 + \frac{C_2'}{N}\sum_{\mu}(\boldsymbol{\xi}^{(\mu)},\boldsymbol{\xi}^{(\mu)})$$

Besides, it was proven in [S-T2] that there exist constants M_0 and m_0 such that for any $M > M_0$

$$\mathrm{Prob}\{\langle(\boldsymbol{J},\boldsymbol{J})\rangle \ge MN\} \le e^{-Nm_0(M-M_0)}. \qquad (2.8)$$ b_J

Using this bound and the above inequality for $\frac{1}{N}\sum_\mu\langle A_\mu^2\rangle$, it is easy to obtain the inequality (2.2).

To obtain (2.3) let us observe, that since $\log \mathrm{H}((k-S_\mu)\varepsilon^{-1}) \le 0$, $Nf_{N,p} \le -N\log z$. So we need only to obtain a bound from below. By the Jensen inequality

$$\Big\langle \exp\Big\{\sum_{\mu=1}^{p}\log \mathrm{H}((k-S_\mu)\varepsilon^{-1/2})\Big\}\Big\rangle_0 \ge \exp\Big\{\sum_{\mu=1}^{p}\langle\log \mathrm{H}((k-S_\mu)\varepsilon^{-1/2})\rangle_0\Big\}$$
$$\ge \exp\Big\{-C\Big\langle\sum_{\mu=1}^{p}(S_\mu)^2\Big\rangle_0\Big\} \ge \exp\Big\{-Cz^{-1}\sum_{\mu=1}^{p}(\boldsymbol{\xi}^{(\mu)},\boldsymbol{\xi}^{(\mu)})\Big\}$$

This inequality implies (2.3) for small enough s.

To prove Theorem 1 we use the method, proposed initially in [P-S] to prove the self-averaging property of the free energy of the Sherrington-Kirkpatrick model. We denote by E_ℓ the averaging with respect to $\boldsymbol{\xi}^{(1)},\dots,\boldsymbol{\xi}^{(\ell)}$ (E_0 means no averaging at all) and write

$$v_{N,p}=N^{-1/2}\sum_{\ell=1}^{p}L_\ell,\quad L_\ell=E_{\ell-1}\{\log Z_{N,p}\}-E_\ell\{\log Z_{N,p}\}. \tag{2.9}$$

So

$$E\{v_{N,p}^m\}=N^{-m/2}\sum_{\ell_1,\dots,\ell_m=1}^{p}E\{L_{\ell_1}\dots L_{\ell_m}\} \tag{2.10}$$

Consider the term $E\{L_{\ell_1}\dots L_{\ell_m}\}$. Let the minimal index is ℓ_j. One can see easily, that if no other index coincides with ℓ_j, then

$$E\{L_{\ell_1}\dots L_{\ell_m}\}=0.$$

Let $\Sigma_{\ell,m}$ be the sum of the terms in which two indexes are equal to ℓ and all the rest indexes $\ell_j\ge\ell$, and $\Sigma_{\ell,m}^{(l)}$ be the sum of the terms which are included in $\Sigma_{\ell,m}$ and have just l indexes equal to ℓ. Than, since for $l\ge 3$

$$\begin{aligned}\Sigma_{\ell,m}^{(l)}&=(m-1)\dots(m-l+1)N^{-m/2}\sum_{\ell_1,\dots,\ell_{m-l}>\ell}^{p}E\Big\{E_\ell\{L_\ell^l\}L_{\ell_1}\dots L_{\ell_{m-l}}\Big\}\\&=(m-1)\dots(m-l+1)N^{-l/2}E\Big\{E_\ell\{L_\ell^l\}\Big(N^{-1/2}\sum_{j>\ell}L_j\Big)^{m-l}\Big\}\\&\le CN^{-l/2}E^{1/2}\{L_\ell^{2l}\}E\Big\{\Big(N^{-1/2}\sum_{p\ge j>\ell}L_j\Big)^{2(m-l)}\Big\}\le C'N^{-l/2}\end{aligned} \tag{2.11}$$

Here we have used the bound (2.1), which implies

$$\begin{aligned}E\Big\{\Big(N^{-1/2}\sum_{p\ge j>\ell}L_j\Big)^{2(m-l)}\Big\}&=E\Big\{\Big(N^{-1/2}(E_\ell\{\log Z_{N,p}\}-E\{\log Z_{N,p}\})\Big)^{2(m-l)}\Big\}\\&\le E\Big\{\Big(N^{-1/2}(\log Z_{N,p}-E\{\log Z_{N,p}\})\Big)^{2(m-l)}\Big\}\le C\end{aligned} \tag{2.12}$$

Besides, we use a simple observation that

$$L_\ell=E_{\ell-1}\Big\{\log\frac{Z_{N,p}}{Z_{N,p-1}^{(\ell)}}\Big\}-E_\ell\Big\{\log\frac{Z_{N,p}}{Z_{N,p-1}}\Big\}, \tag{2.13}$$

where $Z_{N,p-1}^{(\ell)}$ is the partition function of the Hamiltonian (1.1) in which $\log\mathrm{H}((k-N^{-1/2}(\boldsymbol{\xi}^{(\ell)},\boldsymbol{J}))/\sqrt{\varepsilon})$ is replaced by 0 and so it is independent of $\boldsymbol{\xi}^{(\ell)}$. Now one can get easily, that

$$E\{L_\ell^l\}\le C.$$

Thus, we derive from (2.11) that

$$\Sigma_{\ell,m}=\Sigma_{\ell,m}^{(2)}+O(N^{-3/2}). \tag{2.14}$$

Let us find $\Sigma_{\ell,m}^{(2)}$. To this end we need to compute $E_\ell\{L_\ell^2\}$. We use the method developed in [S-T4].

Let us consider a standard Gaussian variable u and introduce the function

$$G^{(t)}(S,u)=\frac{1}{\sqrt{(1-t)(R-q+\varepsilon)}}\int e^{-x^2/2(R-q+\varepsilon)(1-t)}G(\sqrt{t}S+u\sqrt{q(1-t)}+x)dx, \tag{2.15}$$

where

$$G(S)=\mathrm{H}\Big(\frac{k-S}{\sqrt{\varepsilon}}\Big), \tag{2.16}$$

and (q,r) are the unique solution of the system (1.10).

We have $G^{(1)}(S,u)=G(S)$ and

$$G^{(0)}(S,u)=\frac{1}{\sqrt{(R-q+\varepsilon)}}\int e^{-x^2/2(R-q+\varepsilon)}G(u\sqrt{q}+x)dx$$

independent of S.

We remark, that definition (2.15) becomes more natural, if we introduce it through the Fourier transform $\hat{G}(\lambda)$ of $G(S)$:

$$\begin{aligned}
G^{(t)}(S,u)&=\frac{1}{\sqrt{2\pi}}\int \hat{G}(\lambda)\exp\Big\{-i\lambda\Big(S\sqrt{t}+u\sqrt{q(1-t)}\Big)-\frac{\lambda^2}{2}(1-t)(R-q+\varepsilon)\Big\}d\lambda\\
&=\frac{1}{\sqrt{2\pi}}\int \hat{G}(\lambda)F(S,u,t,\lambda)d\lambda,\\
F(S,u,t,\lambda)&\equiv\exp\Big\{-i\lambda\Big(S\sqrt{t}+u\sqrt{q(1-t)}\Big)-\frac{\lambda^2}{2}(1-t)(R-q+\varepsilon)\Big\}.
\end{aligned} \tag{2.17}$$

Denote

$$\mathcal{L}_\ell(t)=\log\langle G^{(t)}(S_\ell,u)\rangle_{-\ell},$$

where symbol $\langle\ldots\rangle_{-\ell}$ means the Gibbs averaging, corresponding to the Hamiltonian (1.1) in which $\log\mathrm{H}((k-N^{-1/2}(\boldsymbol{\xi}^{(\ell)},\boldsymbol{J}))/\sqrt{\varepsilon})$ is replaced by 0. It is evident that $\mathcal{L}_\ell(1)=\log\dfrac{Z_{N,p}}{Z^{(\ell)}_{N,p-1}}$. Now, denoting $E_{\ell*}$ the averaging with respect to $\boldsymbol{\xi}^{(\ell)}$, and u, let us compute $\dfrac{d}{dt}E_{\ell*}\{\mathcal{L}_\ell(t)\}$ and $\dfrac{d}{dt}E_{\ell*}\{\mathcal{L}^2_\ell(t)\}$.

Differentiating by t and then integrating by parts (i.e. using that $E\{v\phi(v)\}=E\{\phi'(v)\}$ for any normal random variable v), we get

$$\begin{aligned}
\frac{d}{dt}E_{\ell*}\{\mathcal{L}_\ell(t)\}&=\frac{1}{2}E_{\ell*}\Big\{\frac{\langle\int d\lambda(-i\lambda S_\ell t^{-1/2}+iu\sqrt{q}(1-t)^{-1/2}+\lambda^2(R-q))\hat{G}(\lambda)F(S_\ell,u,t,\lambda)\rangle_{-\ell}}{\langle\int d\lambda\hat{G}(\lambda)F(S_\ell,u,t,\lambda)\rangle_{-\ell}}\Big\}\\
&=\frac{1}{2}E_{\ell*}\Big\{\frac{\langle\int d\lambda\lambda^2(R-R_{11})F(S,u,t,\lambda)\rangle_{-\ell}}{\langle\int d\lambda\hat{G}(\lambda)F(S,u,t,\lambda)\rangle_{-\ell}}\Big\}\\
&\quad-\frac{1}{2}E_{\ell*}\Big\{\frac{\langle\int d\lambda_1d\lambda_2\lambda_1\lambda_2(q-R_{12})\hat{G}(\lambda_1)\hat{G}(\lambda_2)F(S,u,t,\lambda)F(S,u,t,\lambda)\rangle^{(2)}_{-\ell}}{\langle\int d\lambda_1d\lambda_2\hat{G}(\lambda_1)\hat{G}(\lambda_2)F(S,u,t,\lambda)F(S,u,t,\lambda)\rangle^{(2)}_{-\ell}}\Big\}\\
&=\frac{1}{2}E_{\ell*}\Big\{\frac{\langle(\mathcal{D}^{(t)}_1)^2G^{(t)}(S^{(1)}_\ell,u)(R_{11}-R)\rangle_{-\ell}}{\langle G^{(t)}(S^{(1)}_\ell,u)\rangle_{-\ell}}\Big\}\\
&\quad-\frac{1}{2}E_{\ell*}\Big\{\frac{\langle\mathcal{D}^{(t)}_1\mathcal{D}^{(t)}_2G^{(t)}(S^{(1)}_\ell,u)G^{(t)}(S^{(2)}_\ell,u)(R_{12}-q)\rangle^{(2)}_{-\ell}}{\langle G^{(t)}(S^{(1)}_\ell,u)G^{(t)}(S^{(2)}_\ell,u)\rangle^{(2)}_{-\ell}}\Big\}\\
&=\frac{1}{2}E_{\ell*}\Big\{\langle(R_{11}-R)(\mathcal{D}^{(t)}_1)^2\rangle_{(-k,t)}\Big\}-\frac{1}{2}E_{\ell*}\Big\{\langle(R_{12}-q)\mathcal{D}^{(t)}_1\mathcal{D}^{(t)}_2\rangle_{(-\ell,t)}\Big\},
\end{aligned} \tag{2.18}$$

where the function $F(S,u,t,\lambda)$ was defined in (2.17) and we denote

$$\mathcal{D}^{(t)}_l=\frac{1}{\sqrt{t}}\frac{d}{dS^{(l)}_\ell},$$

with $S_\ell^{(l)}$ being the l-th replica of S_ℓ, and $\langle \dots \rangle_{(-\ell,t)}$ being the Gibbs averaging with respect to the Hamiltonian

$$\begin{aligned} -\mathcal{H}(t) &= \textstyle\sum_{\mu\neq\ell} g(S_\mu) + g(t,S_\ell,u) + \frac{z}{2}(\boldsymbol{J},\boldsymbol{J}), \\ g(x) &= -\log \mathrm{H}(\tfrac{k-x}{\sqrt{\varepsilon}}), \quad g(t,S_\ell,u) = -\log G^t(S,u) \end{aligned} \tag{2.19}$$

Now if take into account that $|t^{-1}\frac{d^2}{dS^2}g(t,S,u)| \le C$, we get from (2.18)

$$\begin{aligned} \left|\frac{d}{dt}E_{\ell*}\{\mathcal{L}_\ell(t)\}\right| \le C E_{\ell*}^{1/2}\Big\{\Big\langle (R_{11}-R)^2 + (R_{12}-q)^2 \Big\rangle_{(-\ell,t)}\Big\} \\ E_{\ell*}^{1/2}\Big\{\Big\langle t^{-2}\Big|\frac{d}{dS_\ell^{(1)}}g(t,S_\ell^{(1)},u)\Big|^4\Big\rangle_{(-\ell,t)}\Big\} \end{aligned} \tag{2.20}$$

Proposition 1. *For any n*

$$E_{\ell*}\Big\{\Big\langle \Big|t^{-1/2}\frac{d}{dS_\ell^{(1)}}g(t,S_\ell^{(1)},u)\Big|^{2n}\Big\rangle_{(-\ell,t)}\Big\} \le C_1(n) + C_2(n)\langle N^{-1}(\boldsymbol{J},\boldsymbol{J})\rangle_{-\ell}^n \tag{2.21}$$

This proposition is proven in [S-T4], but since the proof is not complicated, in order to have self-consistent proof of Theorem 1 we repeat it here.

Proof of Proposition 1

One of the most important feature of our Hamiltonian $\mathcal{H}(t)$, is that $g(S)$ and $g(t,S,u)$ are concave function with respect to S. It allows us to use the Brascamp-Lieb inequalities (see [B-L]), according to which for any smooth function f

$$\langle (f-\langle f\rangle)^2\rangle \le \frac{1}{z}\langle |\nabla f|^2\rangle. \tag{2.22}$$

Thus, using this inequality and the fact that $|t^{-1}\frac{d^2}{dS^2}g(t,S,u)| \le C$, one can get easily that

$$\Big\langle \Big|t^{-1/2}\frac{d}{dS_\ell^{(1)}}g(t,S_\ell^{(1)},u)\Big|^{2n}\Big\rangle_{(-\ell,t)} \le C(n)\Big\langle \Big|t^{-1/2}\frac{d}{dS_\ell^{(1)}}g(t,S_\ell^{(1)},u)\Big|^{2}\Big\rangle_{(-\ell,t)}^n. \tag{2.23}$$

Besides, since $g(t,S,u)$ is a concave function with a bounded second derivative, we have

$$\Big|t^{-1/2}\frac{d}{dS_\ell^{(1)}}g(t,S_\ell^{(1)},u)\Big|^2 \le C_1 + C_2 g(t,S_\ell^{(1)},u).$$

Thus,

$$\begin{aligned} \Big\langle \Big|t^{-1/2}\frac{d}{dS_\ell^{(1)}}g(t,S_\ell^{(1)},u)\Big|^2\Big\rangle_{(-\ell,t)} \le C_1 + C_2\Big\langle g(t,S_\ell^{(1)},u)\Big\rangle_{(-\ell,t)} \\ \le C_1 + C_2\langle g(t,S_\ell^{(1)},u)\rangle_{-\ell} \le C_1' + C_2'\langle |S_\ell^{(1)}|^2\rangle_{-\ell}, \end{aligned} \tag{2.24}$$

where we used (2.7)

Now, taking the nth power of the r.h.s.(2.24) and averaging with respect $\boldsymbol{\xi}_\ell$, we obtain (2.21).

Using (2.20) and (2.21) one can write

$$E_{\ell*}\{\mathcal{L}_\ell(1)\} = E_{\ell*}\{\mathcal{L}_\ell(0)\} + \Delta_\ell^{(1)} \tag{2.25}$$

with

$$|\Delta_\ell^{(1)}| \le \int_0^1 dt E_{\ell *}^{1/2}\Big\{\Big\langle (R_{11}-R)^2+(R_{12}-q)^2\Big\rangle_{(-\ell,t)}\Big\}\Big[C_1(n)+C_2(n)\langle N^{-1}(\boldsymbol{J},\boldsymbol{J})\rangle_{-\ell}^n\Big].$$

Similarly we get

$$E_{\ell *}\{\mathcal{L}_\ell(1)^2\} = E_{\ell *}\{\mathcal{L}_\ell(0)^2\} + \Delta_\ell^{(2)} \tag{2.26}$$

with

$$|\Delta_\ell^{(2)}| \le \int_0^1 E_{\ell *}^{1/2}\Big\{\Big\langle (R_{11}-R)^2+(R_{12}-q)^2\Big\rangle_{(-\ell,t)}\Big\}\Big[C_1(n)+C_2(n)\langle N^{-1}(\boldsymbol{J},\boldsymbol{J})\rangle_{-\ell}^n\Big]$$

Observing, that

$$E_{\ell *}\{\mathcal{L}_\ell(0)^2\} - E_{\ell *}^2\{\mathcal{L}_\ell(0)\} = \alpha^{-1}V^2,$$

we get now from (2.14) and (2.26)

$$\Sigma_{\ell,m} = \frac{(m-1)V}{N}\Sigma_{\ell+1,m-2} + E\{\Delta_\ell \Sigma_{\ell+1,m-2}\} + O(N^{-3/2}) \tag{2.27}$$

with

$$\Delta_\ell \le |\Delta_\ell^{(1)}|^2 + \Delta_\ell^{(2)}$$

Now applying the Schwartz inequality and using the bound (2.12) and 2.8, we have

$$\begin{aligned} E\left\{\Delta_\ell \Sigma_{\ell+1,m-2}\right\} &\le E\{\Delta_\ell^2\}\, E\left\{\Sigma_{\ell+1,m-2}^2\right\} \le E\left\{\Delta_\ell^2\right\} C \\ &\le C \max_t \Big[E^{1/2}\left\{\langle (R_{11}-R)^2\rangle_{(-\ell,t)}\right\} + E^{1/2}\left\{\langle (R_{12}-q)^2\rangle_{(-\ell,t)}\right\}\Big] \end{aligned}$$

But it was proven in [S-T4] (see Proposition 1 and Lemma 1 of [S-T4]) that

$$E\left\{\langle (R_{11}-R)^2\rangle_{(-\ell,t)}\right\} \le CN^{-1}, \quad E\left\{\langle (R_{12}-q)^2\rangle_{(-\ell,t)}\right\} \le CN^{-1}$$

Hence, (2.27) and the above estimates imply

$$\Sigma_{\ell,m} = \frac{(m-1)V^2\alpha^{-1}}{N}\Sigma_{\ell+1,m-2} + O(N^{-3/2}) \tag{2.28}$$

Now, taking the sum with respect to ℓ and observing that

$$|\Sigma_{\ell+1,m-2} - \Sigma_{\ell,m-2}| \le O(N^{-1/2}),$$

we get finally

$$E\{v_{N,p}^m\} = (m-1)VE\{v_{N,p}^{m-2}\} + O(N^{-1/2})$$

Theorem 1 follows.

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