Central limit theorems for order parameters of the Gardner problem

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Abstract

Fluctuations of the order parameters of the Gardner model for any $\alpha < \alpha_c$ are studied. It is proved that they converge in distribution to a family of jointly Gaussian random variables.

1 Introduction and Main Results

The Gardner model was introduced in [G] to study the typical volume of interactions between each pair of N Ising spins which solve the problem of storing a given set of p random patterns $\{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^{p}$. The components $\xi_i^{(\mu)}$ of the patterns are taken usually to be independent random variables with zero mean and variance 1. After a simple transformation this problem is reduced to the analysis of the asymptotic behaviour of the random variable

$$\Theta_{N,p}(k) = \sigma_N^{-1} \int_{(\boldsymbol{J},\boldsymbol{J})=N} d\boldsymbol{J} \prod_{\mu=1}^{p} \theta(N^{-1/2}(\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J}) - k), \tag{1.1}$$

where $\boldsymbol{J} \in \mathbf{R}^N$, the function $\theta(x)$ is zero in the negative semi-axis and 1 in the positive, and σ_N is the Lebesgue measure of the N-dimensional sphere of radius $N^{1/2}$. Then, the question of interest is the behaviour of $\frac{1}{N}\log\Theta_{N,p}(k)$ in the limit $N, p \to \infty$, $\frac{p}{N} \to \alpha$. Gardner [G] had solved this problem by using the so-called replica trick, which is completely non-rigorous from the mathematical point of view but sometimes very useful in the physics of spin glasses (see [M-P-V] and references therein).

She obtained that if $\alpha < \alpha_c(k)$, with

$$\alpha_c(k) \equiv (\frac{1}{\sqrt{2\pi}} \int_{-k}^{\infty} (u+k)^2 e^{-u^2/2} du)^{-1}, \tag{1.2}$$

then the following limit exists

$$\begin{aligned} \lim_{N,p\to\infty, p/N\to\alpha} \frac{1}{N} E\{\log\Theta_{N,p}(k)\} \quad = \quad & \min_{0\le q\le 1} \left[\alpha E\left\{ \log \mathrm{H}\left(\frac{u\sqrt{q}+k}{\sqrt{1-q}} \right) \right\} \right. \\ & \left. + \frac{1}{2}\frac{q}{1-q} + \frac{1}{2}\log(1-q) \right]. \end{aligned} \tag{1.3}$$

Here u is a Gaussian random variable with zero mean and variance 1, $\mathrm{H}(x)$ is defined as

$$\mathrm{H}(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt \tag{1.4}$$

and here and below we denote by the symbol $E\{...\}$ the averaging with respect to all random parameters of the problem and also with respect to u. And $\frac{1}{N}\log\Theta_{N,p}(k)$ tends to minus infinity for $\alpha \ge \alpha_c(k)$.

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In the paper [S-T2] (see also [S-T3]) we have studied the Gardner problem in a regular mathematical way and proved that for any $\alpha < \alpha_c$ formula (1.3) is valid while for $\alpha > \alpha_c$ any $\frac{1}{N}E\{\log \Theta_{N,p}(k)\} \to -\infty$, as $N, p \to \infty, p/N \to \alpha$. We studied the case $\xi_i^{(\mu)} = \pm 1$, but of course the same results are valid for any distribution $\xi_i^{(\mu)}$ with $E\{\xi_i^{(\mu)}\} = 0$, $E\{(\xi_i^{(\mu)})^2\} = 1$, $E\{|\xi_i^{(\mu)}|^4\} < \infty$.

To obtain this results we introduced an intermediate "modified" Hamiltonian depending on the parameters $\varepsilon, z, h > 0$:

$$\mathcal{H}(\boldsymbol{J}, k, z, \varepsilon) \equiv -\sum_{\mu=1}^{p} \log \mathrm{H}\left(\frac{k - (\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J})N^{-1/2}}{\sqrt{\varepsilon}}\right) + \frac{z}{2}(\boldsymbol{J}, \boldsymbol{J}), \tag{1.5}$$

where the function $\mathrm{H}(x)$ is defined in (1.4).

The partition function for this Hamiltonian is

$$Z_{N,p}(k, z, \varepsilon) = \sigma_N^{-1} \int d\boldsymbol{J} \exp\{-\mathcal{H}(\boldsymbol{J}, k, z, \varepsilon)\}. \tag{1.6}$$

We denote also by $\langle \ldots \rangle$ the corresponding Gibbs averaging and

$$f_{N,p}(k, z, \varepsilon) \equiv \frac{1}{N} \log Z_{N,p}(k, z, \varepsilon). \tag{1.7}$$

Comparing with the model (1.1), one can see that we replace the functions $\theta(x)$ by $\mathrm{H}(x/\sqrt{\varepsilon})$, to have possibility to study the model (1.1) by means of statistical mechanics of the systems with random interaction (see [S-T1]-[T5]). When $\varepsilon \to 0$, $\mathrm{H}(x/\sqrt{\varepsilon}) \to \theta(x)$, so the model (1.1) is the limiting case of (1.5)-(1.7). But the corresponding statement (see Theorem 3 of [S-T2]) is rather nontrivial and in this paper we are going to study the model (1.5)-(1.7) itself with some fixed positive ε. The other difference from (1.1) is that we introduce an additional parameter $z > 0$ to replace the integration over the sphere $(\boldsymbol{J}, \boldsymbol{J}) = N$ in (1.1) by the integration in the whole space $\mathbf{R}^N$ in (1.6). It is proven in [S-T2] that if we find the thermodynamic limit

$$\lim_{N,p\to\infty, p/N\to\alpha} E\{f_{N,p}(k, z, \varepsilon)\} = F(\alpha, k, z, \varepsilon)$$

and choose z^* from the condition

$$F(\alpha, k, z^*, \varepsilon) + \frac{z^*}{2} = \min_{z>0}\{F(\alpha, k, z, \varepsilon) + \frac{z}{2}\},$$

then

$$\lim_{N,p\to\infty,p/N\to\alpha} N^{-1}E\Big\{ \log \sigma_N^{-1} \int_{(\boldsymbol{J},\boldsymbol{J})=N} d\boldsymbol{J} \exp\{-\mathcal{H}(\boldsymbol{J}, k, 0, \varepsilon)\Big\} = F(\alpha, k, z^*, \varepsilon) + \frac{z^*}{2}.$$

In [S-T2] we have proved the theorem:

Theorem 1 *For any $\alpha < 2$, $k \geq 0$, there exists $\varepsilon^*(\alpha, k)$ such that for any $\varepsilon \leq \varepsilon^*(\alpha, k)$ and $z \leq \varepsilon^{-1/3}$ there exists*

$$\begin{aligned} \lim_{N,p\to\infty,p/N\to\alpha} E\{f_{N,p}(k, z, \varepsilon)\} &= F(\alpha, k, z, \varepsilon), \\ F(\alpha, k, z, \varepsilon) &\equiv \max_{R>0} \min_{0\leq q\leq R} \Big[\alpha E\left\{\log \mathrm{H}\left(\frac{u\sqrt{q} + k}{\sqrt{\varepsilon + R - q}}\right)\right\} \\ &\quad + \frac{1}{2}\frac{q}{R-q} + \frac{1}{2}\log(R-q) - \frac{z}{2}R\Big], \end{aligned} \tag{1.8}$$

where u is a Gaussian random variable with zero mean and variance 1.

Similar results for small α were obtained in [T2] for the so-called Gardner-Derrida model. In the paper [T4] the fluctuation of the order parameters for the Gardner-Derrida model were studied, but only for small enough α.

An important ingredient of the analysis of the free energy of the model (1.5) in [S-T2] was the proof of the fact that the variance of its order parameters (or the overlap parameters) disappears in the thermodynamic limit. In the present paper we study the behaviour of fluctuations of the overlap parameters, defined as

$$R_{l,m} = \frac{1}{N}(\boldsymbol{J}^{(l)}, \boldsymbol{J}^{(m)}), \quad (l, m = 1, \dots n), \tag{1.9}$$

where the upper indexes of the variables $\boldsymbol{J}$ mean that we consider n replicas of the Hamiltonian (1.5) with the same random parameters $\{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^{p}$, but different $\boldsymbol{J}^{(1)}, \dots, \boldsymbol{J}^{(n)}$.

We introduce also the notations:

$$\begin{aligned} \dot{q} &= N^{1/2}(\langle R_{1,2}\rangle - q), \\ T_{l,m} &= \frac{1}{N^{1/2}}(\dot{\boldsymbol{J}}^{(l)}, \dot{\boldsymbol{J}}^{(m)}), \quad T_l = \frac{1}{N^{1/2}}(\dot{\boldsymbol{J}}^{(l)}, \langle \boldsymbol{J}\rangle). \end{aligned} \tag{1.10}$$

Here and below $\dot{\mathbf{J}} \equiv \mathbf{J} - \langle \mathbf{J}\rangle$ and $\langle \boldsymbol{J}\rangle = (\langle J_1\rangle, \dots \langle J_N\rangle) \in \mathbf{R}^N$, where $\langle \dots \rangle$ is the Gibbs averaging with respect to the Hamiltonian (1.5). (q, R) is the solution of the system of equations:

$$\begin{aligned} q &= (R-q)^2\left[\frac{\alpha}{R-q+\varepsilon} E\left\{ \mathrm{A}^2\left(\frac{\sqrt{q}u+k}{\sqrt{R-q+\varepsilon}}\right)\right\}\right], \\ z &= \frac{\alpha}{(R-q+\varepsilon)^{3/2}} E\left\{(\sqrt{q}u+k)\mathrm{A}\left(\frac{\sqrt{q}u+k}{\sqrt{R-q+\varepsilon}}\right)\right\} \\ &\quad -\frac{q}{(R-q)^2} + \frac{1}{R-q}, \end{aligned} \tag{1.11}$$

with

$$\mathrm{A}(x) = -\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\log \mathrm{H}(x).$$

These equations are equivalent to $\frac{\partial \mathcal{F}}{\partial q} = 0$ $\frac{\partial \mathcal{F}}{\partial R} = 0$, for the function $\mathcal{F}(q, R; k, z, \varepsilon)$ which is defined by the expression in the r.h.s. of (1.8) before taking $\max_R \min_q$. It is proven in [S-T2] that if $\alpha < 2$, $\varepsilon \le \varepsilon^*(\alpha, k)$ and $z \le \varepsilon^{1/3}$, the the system (1.11) has a unique solution.

To avoid additional technical difficulties below we assume that $\{\xi_i^{(\mu)}\}$ are independent Gaussian random variables with zero mean and variance 1.

The main result of the paper is

Theorem 2 *Consider any $\alpha < 2$, $k > 0$, $\varepsilon \le \varepsilon^*(\alpha, k)$ and $z \le \varepsilon^{-1/3}$. Then for any integer n the families of random variables $\{\sqrt{N}(R_{l,m} - E\langle R_{l,m}\rangle)\}_{l<m\le n}$, converges in distribution, as $N, p \to \infty, p/N \to \alpha$, to the Gaussian family of random variables $\{v_{l,m}\}_{l<m\le n}$, with the covariance matrix:*

$$\begin{aligned} E\{v_{l,m}v_{l,m}\} &= A^*, \\ E\{v_{l,m}v_{l,m'}\} &= B^* \ (m \ne m'), \\ E\{v_{l,m}v_{l',m'}\} &= C^* \ (m, m', l, l' \ \mathit{are\ different}). \end{aligned} \tag{1.12}$$

In particular,

$$\begin{aligned} \lim_{N,p\to\infty, p/N\to\alpha} E\left\{\langle T_{1,2}^{2n}\rangle\right\} &= \frac{\Gamma(2n-1)}{\Gamma(n-1)} A_*^n \\ \lim_{N,p\to\infty, p/N\to\alpha} E\left\{\langle T_{1}^{2n}\rangle\right\} &= \frac{\Gamma(2n-1)}{\Gamma(n-1)} B_*^n \\ \lim_{N,p\to\infty, p/N\to\alpha} E\{\dot{q}^{2n}\} &= \frac{\Gamma(2n-1)}{\Gamma(n-1)} C_*^n, \end{aligned} \tag{1.13}$$

where the constants A^*, B^*, C^*, A_*, B_*, C_* depend on $\alpha, k, z, \varepsilon$ and all odd moments for these random variables tend to zero.

Remark 1 *In fact it follows from our proof (see proofs of Lemmas 3,4,5 in Sec.2) that $\{T_{l,m}\}_{l<m\le n}$ and $\{T_l\}_{l\le n}$ in some sense do not depend on the random variables $\{\xi_i^{(\mu)}\}$, i.e. if we consider P- some product of $\{T_{l,m}\}_{l<m\le n}$ and $\{T_l\}_{l\le n}$, then*

$$\lim_{N,p\to\infty,p/N\to\alpha} E\{(\langle P\rangle - E\langle P\rangle)^2\} = 0. \tag{1.14}$$

As it was mentioned above, similar results were obtained in [T4] for the Gardner-Derrida model for small α and for the Sherrington-Kirkpatrick model for the high temperature. We would like to mention also the work [Gu-T], where the fluctuations of the overlap parameters for the Sherrington-Kirkpatrick model in the high temperature region were studied by the method of characteristic functions.

One of the most important feature of our Hamiltonian (1.5), which allows us to prove Theorems 1 and 2 for any $\alpha < \alpha_c(k)$ is that it has the form

$$-\mathcal{H} = \sum_\mu g(S_\mu) - \frac{z}{2}(\boldsymbol{J},\boldsymbol{J}), \quad S_\mu = \frac{1}{N^{1/2}}(\boldsymbol{J},\boldsymbol{\xi}^{(\mu)}), \quad g(x) = -\log \mathrm{H}(\frac{x}{\sqrt{\varepsilon}}), \tag{1.15}$$

where $g(x)$ is a concave function. It allows us to use the Brascamp-Lieb inequalities (see [1]), according to which for any integer n and any $\mathbf{x}\in\mathbf{R}^N$

$$\Big\langle\Big(\frac{(\dot{\boldsymbol{J}},\mathbf{x})}{\sqrt{N}}\Big)^{2n}\Big\rangle \le \frac{\Gamma(2n-1)}{z^n\Gamma(n-1)}\Big(\frac{|\mathbf{x}|^2}{N}\Big)^n. \tag{1.16}$$

Besides, for any smooth function f

$$\langle (f-\langle f\rangle)^2\rangle \le \frac{1}{z}\langle|\nabla f|^2\rangle. \tag{1.17}$$

Below we use the representation (1.15) of $\mathcal{H}$ and the following properties of the functions $g(x)$:

$$g(x)\le 0, \ -C\le g''\le 0, \ |g^{(s)}(x)|\le C, \ (s=3,\dots,6). \tag{1.18}$$

In fact the only place, where we use the real form of $g(x)$, is that the limiting system of equations (1.11) has the unique solution (see [S-T2]).

We would like to mention that here and below we use C to denote some constant whose value is not important for us, but which is independent of N,p. This value can be different in different formulas.

We use also notations:

$$S_\mu^{(l)} = \frac{1}{N^{1/2}}(\boldsymbol{J}^{(l)},\boldsymbol{\xi}^{(\mu)}), \quad \tilde{R}_{l,l'} = \frac{1}{N}\sum_\mu g'(S_\mu^{(l)})g'(S_\mu^{(l')}), \quad \tilde{U}_l = \frac{1}{N}\sum_\mu (g''(S_\mu^{(l)}) + g'^2(S_\mu^{(l)})). \tag{1.19}$$

An important ingredient of our proof is the following proposition:

Proposition 1 *There exists $d_0>0$, $C>0$,$N_0>0$ such that for any $\delta<d_0$, $N>N_0$*

$$\begin{aligned}
&\mathrm{Prob}\{|\langle R_{1,2}\rangle - E\langle R_{1,2}\rangle| > \delta\} \le e^{-N\delta^2/2C},\\
&\mathrm{Prob}\{|\langle R_{1,1}\rangle - E\langle R_{1,1}\rangle| > \delta\} \le e^{-N\delta^2/2C},\\
&\mathrm{Prob}\{|\langle \tilde{R}_{1,2}\rangle - E\langle \tilde{R}_{1,2}\rangle| > \delta\} \le e^{-N\delta^2/2C},\\
&\mathrm{Prob}\{|\langle \tilde{U}_1\rangle - E\langle \tilde{U}_1\rangle| > \delta\} \le e^{-N\delta^2/2C}.
\end{aligned} \tag{1.20}$$

Corollary 1 *There exist $C>O$, $N_0>0$ such that for any $N>N_0$*

$$E\{|\langle R_{1,2}\rangle - E\langle R_{1,2}\rangle|^n\} \le \frac{C^n\Gamma(n)}{N^{n/2}}. \tag{1.21}$$

2 Proof of Main Results

Proof of Proposition 1
For the proof of Proposition 1 we need the following remark:

Remark 2 *It was proven in [S-T2] that there exist constants M_0, m_0 such that for any $M > M_0$, $N \geq 1$*

$$\text{Prob}\{\langle(\boldsymbol{J},\boldsymbol{J})\rangle \geq MN\} \leq e^{-Nm_0(M-M_0)}. \tag{2.1}$$

Besides, it is well known that if we define

$$\mathcal{A}_{*i,j} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^{(\mu)}\xi_j^{(\mu)}, \quad \tilde{\mathcal{A}}_{*\mu,\nu} = \frac{1}{N}\sum_{i=1}^{N} \xi_i^{(\mu)}\xi_i^{(\nu)}, \tag{2.2}$$

then there exist C_0, c_0 such that for any $C > C_0$, $N \geq 1$

$$\text{Prob}\{||\mathcal{A}_*|| \geq C\} \leq e^{-Nc_0(C-C_0)}, \quad \text{Prob}\{||\tilde{\mathcal{A}}_*|| \geq C\} \leq e^{-Nc_0(C-C_0)} \tag{2.3}$$

(see e.g. [S-T1]).

We prove Proposition 1 using a method proposed in [T4] with small modifications which we need to study the case, when the variables $\{J_i\}$ are unbounded.

Lemma 1 *Let $\mathbf{u} \in \mathbf{R}^{\mathbf{m}}$ be a Gaussian random vector with normally distributed independent components $\{u_i\}_{i=1}^m$ and $f(\mathbf{x})$ be some function defined in $\mathbf{R}^m$. If there exist A_0 and $s_0 > 0$ such that $f(\mathbf{x})$ satisfies the conditions:*

$$P(A) = \text{Prob}\{|\nabla f(\mathbf{u}))|^2 \geq A\} \leq e^{-C(A-A_0)} \ (\forall A > A_0) \tag{2.4}$$

$$E\{e^{\pm s_0 f}\} \leq e^{s_0 B}, \tag{2.5}$$

Then for $s \leq \frac{1}{4}s_0$

$$E\{e^{\pm s(f(\mathbf{u})-Ef(\mathbf{u}))}\} \leq e^{2A_0 s^2}(1 + (A_0 + C^{-1})e^{2sB - CA_0/2}). \tag{2.6}$$

Proof.
Consider $\mathbf{v} \in \mathbf{R}^m$ -another Gaussian random vector with normally distributed independent components $\{v_i\}_{i=1}^m$ which are independent of $\{u_i\}_{i=1}^m$. Define

$$G_{s,t}(\mathbf{u},\mathbf{v}) = \exp\{s(f(\mathbf{u}) - f(\sqrt{1-t}\mathbf{u} + \sqrt{t}\mathbf{v}))\}, \quad \varphi_s(t) = E\{G_{s,t}(\mathbf{u},\mathbf{v})\}.$$

Then, integrating by parts, we get

$$\begin{aligned}
\varphi_s'(t) &= \frac{s}{2}E\left\{\frac{\partial}{\partial x_i}f(\sqrt{1-t}\mathbf{u} + \sqrt{t}\mathbf{v})(\frac{u_i}{\sqrt{1-t}} - \frac{v_i}{\sqrt{t}})G_{s,t}(\mathbf{u},\mathbf{v})\right\} \\
&= \frac{s^2}{2\sqrt{1-t}}E\left\{\frac{\partial}{\partial x_i}f(\sqrt{1-t}\mathbf{u} + \sqrt{t}\mathbf{v})\frac{\partial}{\partial x_i}f(\mathbf{u})G_{s,t}(\mathbf{u},\mathbf{v})\right\} \\
&\leq \frac{A_0 s^2}{\sqrt{1-t}}\varphi_s(t) + \frac{s^2 e^{2sB}}{2\sqrt{1-t}}E^{1/2}\left\{|\nabla f(u)|^4\theta(|\nabla f(u)|^2 - 2A_0)\right\} \\
&\leq \frac{A_0 s^2}{\sqrt{1-t}}\varphi_s(t) + \frac{s^2 e^{2sB}}{2\sqrt{1-t}}\left(\int_{A>2A_0} A^2 dP(A)\right)^{1/2}.
\end{aligned}$$

Thus we obtain

$$E\{e^{s(f(\mathbf{u})-f(\mathbf{v}))}\} = \varphi_s(1) \leq e^{2A_0 s^2}(\varphi_s(0) + (A_0 + C^{-1})e^{2sB - CA_0/2}).$$

Since, by definition, $\varphi_s(0) = 1$, averaging first with respect to $\mathbf{u}$ and then with respect to $\mathbf{v}$ and using the Iensen inequality, we get (2.6). Lemma 1 is proven.

To apply this result to $f = N\langle R_{1,2}\rangle$ and $\mathbf{u} = (\xi^{(1)}, \ldots, \xi^{(p)}) \in \mathbf{R}^{Np}$ we have to check the condition (2.4). We write

$$\begin{aligned}
\frac{1}{N}\sum_{i,\mu}\left|\frac{\partial f}{\partial \xi_i^{(\mu)}}\right|^2 &= \frac{4}{N^2}\sum_{i,j,k,\mu}\langle J_i\rangle\Big\langle \dot{J}_i g'(S_\mu) J_j\Big\rangle\Big\langle J_j g'(S_\mu)\dot{J}_k\Big\rangle\langle J_k\rangle \\
&= \frac{4}{N^2}\sum_{i,j,k,\mu}\langle J_i\rangle\Big\langle \dot{J}_i g'(S_\mu)(\dot{J}_j + \langle J_j\rangle)\Big\rangle\Big\langle (\dot{J}_j + \langle J_j\rangle) g'(S_\mu)\dot{J}_k\Big\rangle\langle J_k\rangle \\
&\le \frac{8}{N^2}\sum_{i,j,k,\mu}\langle J_i\rangle\Big\langle \dot{J}_i g'(S_\mu)\dot{J}_j\Big\rangle\Big\langle \dot{J}_j g'(S_\mu)\dot{J}_k\Big\rangle\langle J_k\rangle \\
&\quad + \frac{8}{N^2}\sum_{i,j,k,\mu}\langle J_i\rangle\Big\langle \dot{J}_i (g'(S_\mu) - \langle g'(S_\mu)\rangle)\Big\rangle\Big\langle (g'(S_\mu) - \langle g'(S_\mu)\rangle)\dot{J}_k\Big\rangle\langle J_k\rangle\langle J_j\rangle^2 \\
&= 8I + 8II.
\end{aligned} \tag{2.7}$$

To estimate the r.h.s. we use the following proposition:

Proposition 2 *Consider the matrices $\mathcal{A}^{(f)} : \mathbf{R}^N \to \mathbf{R}^N$, $\mathcal{B}^{(f)} : \mathbf{R}^p \to \mathbf{R}^N$ and $\mathcal{C} : \mathbf{R}^p \to \mathbf{R}^p$ of the form*

$$\begin{aligned}
&\mathcal{A}_{i,j}^{(f)} = \Big\langle \dot{J}_i \dot{J}_j f(\boldsymbol{J})\Big\rangle, \quad \mathcal{B}_{i,\mu}^{(f)} = \Big\langle \dot{J}_i (g'(S_\mu) - \langle g'(S_\mu)\rangle) f(\boldsymbol{J})\Big\rangle, \\
&\mathcal{C}_{\mu,\nu} = \Big\langle (g'(S_\mu) - \langle g'(S_\mu)\rangle)(g'(S_\nu) - \langle g'(S_\nu)\rangle)\Big\rangle.
\end{aligned} \tag{2.8}$$

Then

$$||\mathcal{A}^{(f)}|| \le \frac{\langle 3f^2\rangle^{1/2}}{z}, \quad ||\mathcal{B}^{(f)}|| \le \frac{||\mathcal{A}_*||^{1/2}\langle |g''|^2\rangle^{1/2}\langle 3f^4\rangle^{1/4}}{z}, \quad ||\mathcal{C}|| \le \frac{||\tilde{\mathcal{A}}_*||\langle |g''|^2\rangle}{z}, \tag{2.9}$$

where the matrices $\mathcal{A}_*$, $\tilde{\mathcal{A}}_*$ are defined by (2.2).

We prove this proposition in the next section.

Denoting $\mathcal{A}_{i,j}^{\mu} = \langle \dot{J}_i \dot{J}_j g'(S_\mu)\rangle$ and using (2.9), we obtain

$$I = \frac{1}{N^2}\sum(\mathcal{A}^\mu * \mathcal{A}^\mu\langle \boldsymbol{J}\rangle, \langle \boldsymbol{J}\rangle) \le \frac{3}{z^2N^2}(\langle \boldsymbol{J}\rangle, \langle \boldsymbol{J}\rangle)\Big\langle \sum g'^2(S_\mu)\Big\rangle.$$

Similarly, taking $f = 1$ in the definition (2.9) of $\mathcal{B}$, we get

$$II = \frac{1}{N^2}(\langle \boldsymbol{J}\rangle, \langle \boldsymbol{J}\rangle)(\mathcal{B} * \mathcal{B}^*\langle \boldsymbol{J}\rangle, \langle \boldsymbol{J}\rangle) \le \frac{2||\mathcal{A}_*||\langle |g''|^2\rangle}{z^2N^2}(\langle \boldsymbol{J}\rangle, \langle \boldsymbol{J}\rangle)^2.$$

Proposition 3

$$\frac{1}{N}\sum\langle g'^2(S_\mu)\rangle \le \frac{C}{N^2}\sum(\boldsymbol{\xi}^{(\mu)}, \boldsymbol{\xi}^{(\mu)}).$$

Since g'' is a bounded function, by using this proposition (see the next section for the proof), one can easily check the condition (2.4) for the terms I and II.

Thus we have proved the first line of (2.6). Now by using the standard Chebyshev inequality we get (1.20). The other inequalities in (1.20) can be proven similarly.

To prove Theorem 2 we need to make some preliminary work.
Denote

$$d = q(R - q)^{-1}, \quad U = d + z - (R - q)^{-1}. \tag{2.10}$$

Lemma 2 *For any $0<\epsilon<1$ there exists a constant C_ϵ such that uniformly in N*

$$\begin{array}{ll} |E\{\langle R_{1,2}\rangle\}-q|\le \dfrac{C_\epsilon}{N^{1-\epsilon}}, & |E\{\langle R_{1,1}\rangle\}-R|\le \dfrac{C_\epsilon}{N^{1-\epsilon}},\\ |E\{\langle g'(S_\mu)\rangle^2\}-d|\le \dfrac{C_\epsilon}{N^{1-\epsilon}}, & |E\{\langle g''(S_\mu)+g'^2(S_\mu)\rangle\}-U|\le \dfrac{C_\epsilon}{N^{1-\epsilon}}. \end{array} \tag{2.11}$$

Remark 3 *Here and below we mean that all our inequalities are valid for $N\ge N_0$ with some N,p-independent N_0.*

For the proof of this lemma see Section 3.

Using this lemma and inequality (1.21), we get

$$E\{|\dot{q}^n|\}\le 2C^n\Gamma(n). \tag{2.12}$$

Besides, using inequalities (1.16) one can get easily:

$$E\{\langle T_{1,2}^n\rangle\}\le C^n\Gamma(n),\quad E\{\langle T_1^n\rangle\}\le C^n\Gamma(n). \tag{2.13}$$

Then, since $R_{1,2}-q=N^{-1/2}(T_{1,2}+T_1+T_2+\dot{q})$, we obtain

$$E\{\langle|R_{1,2}-q|^n\rangle\}\le \frac{C^n\Gamma(n)}{N^{n/2}}. \tag{2.14}$$

Besides, on the basis of (1.17) and Lemma 2, we have

$$E\{\langle(\tilde{R}_{l,l'}-d)^2\rangle\}\le \frac{C}{N},\quad E\{\langle\dot{U}_l^2\rangle\}\le\frac{C}{N},\quad E\{\langle(R_{l,l}-R)^2\rangle\}\le\frac{C}{N}. \tag{2.15}$$

Here and below we denote

$$\dot{U}_l=\tilde{U}_l-U.$$

From this inequality, using the bound

$$|\tilde{R}_{l,l'}|,|\dot{U}^2|\le N^{-1}\langle(\boldsymbol{J},\boldsymbol{J})\rangle||\mathcal{A}_*||$$

and inequalities (2.1) and (2.3), we obtain for any $r>2$

$$E\left\{\langle|\tilde{R}_{l,l'}-d|^r\rangle\right\}\le\frac{C_r}{N},\quad E\left\{\langle|\dot{U}|^r\rangle\right\}\le\frac{C_r}{N},\quad E\left\{\langle|R_{l,l}-R|^r\rangle\right\}\le\frac{C_r}{N}. \tag{2.16}$$

Following the method of [T5], we introduce the Hamiltonian

$$\begin{aligned} -H_t &= \sum g(S_\mu^-+J_1\sqrt{t}\xi_1^{(\mu)}N^{-1/2})+\sqrt{d(1-t)}uJ_1\\ &\quad+\frac{1-t}{2}(U-d)J_1^2-\frac{z}{2}J_1^2-\frac{z}{2}(\mathbf{J}^-,\mathbf{J}^-), \end{aligned} \tag{2.17}$$

where u is a normally distributed random variable, independent of $\boldsymbol{\xi}^{(\mu)}$ and $\mathbf{h}$ and $S_\mu^-=N^{-1/2}(\boldsymbol{\xi}^{(\mu)},\mathbf{J}^-)$ (with $\mathbf{J}^-=(0,J_2,\dots,J_N)$ do not depend on $\xi_1^{(\mu)}$.

Denote $\langle\ldots\rangle_t$ the Gibbs averaging corresponding to H_t (or n replicas of H_t), and for any $\xi_1^{(\mu)}$-independent function defined of $\mathbf{R}^{N\times n}$ define

$$\nu_t(f)=E\langle f\rangle_t,\quad \nu_t'(f)=\frac{d}{dt}\nu_t(f). \tag{2.18}$$

Besides, to simplify notation we denote

$$s_i=J_1^{(i)}. \tag{2.19}$$

Proposition 4 *There exists a constant C such that for any integer n*

$$|\nu_t(s_1^{2n})| \le C^n\Gamma(n). \tag{2.20}$$

For the proof of this proposition see the next section.

Let us compute $\nu_t'(f)$. Differentiating and then integrating by parts with respect to $\xi_1^{(\mu)}$ and u, similarly to [T4] we get

$$\begin{aligned}\nu_t'(f) &= \frac{1}{2}\sum_{l=1}^{n}\nu_t(fs_l^2\dot{U}_l^-) - \frac{n}{2}\nu_t(fs_{n+1}^2\dot{U}_{n+1}^-)\\ &+\sum_{l<l'}^{n}\nu_t(fs_ls_{l'}(\tilde{R}_{l,l'}^- - d)) - n\sum_{l=1}^{n}\nu_t(fs_ls_{n+1}(\tilde{R}_{l,n+1}^- - d))\\ &+\frac{n(n+1)}{2}\nu_t(fs_{n+1}s_{n+2}(\tilde{R}_{n+1,n+2}^- - d)).\end{aligned} \tag{2.21}$$

Here

$$\tilde{R}_{l,l'}^- = \frac{1}{N}\sum_\mu g'(S_\mu^{-(l)})g'(S_\mu^{-(l')}),\quad \tilde{U}_l^- = \frac{1}{N}\sum_\mu (g''(S_\mu^{-(l)}) + g'^2(S_\mu^{-(l)})),\quad \dot{U}_l^- = \tilde{U}_l^- - U$$

with U, d defined by (2.10) and $S_\mu^{-(l)}$ denoting the lth replica of S_μ^- (the definition of S_μ^- was given after formula (2.17)).

Since the Hamiltonian (2.17) has the form (1.15), the inequalities (1.16) and (1.17) for this Hamiltonian are also valid. Therefore the estimate (2.14)- (2.16) are fulfilled and so, using the Schwartz inequality and (2.15), we get

$$|\nu_1(f) - \nu_0(f)| \le C\max_t \nu_t^{1/2}(|f|^2)N^{-1/2}. \tag{2.22}$$

Using the same formula to compute the second derivative of $\nu_t(f)$ with respect to t, we obtain the expression in each term of which we have as a multipliers $(\tilde{R}_{l,l'} - d)(\tilde{R}_{l_1,l_1'} - d)$ or $(\tilde{R}_{l,l'} - d)\dot{U}_{l_1}$ or $\dot{U}_l\dot{U}_{l_1}$. Using the Hölder inequality with $p = (1-\epsilon)^{-1}$, $q = \epsilon^{-1}$, and then (2.16) with $r = 2(1-\epsilon)^{-1}$ we obtain for any $0 < \epsilon < 1$:

$$\begin{aligned}|\nu_1(f) - \nu_0(f) - \nu_0'(f)| &\le C_\varepsilon \max\nolimits_t \nu_t^{\epsilon/2}(|f|^{1/\epsilon})|s_1|^{4/\epsilon})N^{-1+\epsilon}\\ &\le C_\varepsilon \max\nolimits_t \nu_t^{\epsilon}(|f|^{2/\epsilon})\nu_t^{\epsilon/2}(|s_1|^{8/\epsilon})N^{-1+\epsilon}.\end{aligned} \tag{2.23}$$

To compute the averages of the type $\langle\tilde{R}_{l,l'}\rangle$ we use another tool. Denote $\mathcal{D}_l^{(t)} = \frac{1}{\sqrt{t}}\frac{d}{dS_1^{(l)}}$. One can see easily that, e.g., $E\{\langle\tilde{R}_{1,2}\rangle\}$ can be represented in the form

$$E\{\langle\tilde{R}_{1,2}\rangle\} = E\Big\{\frac{\langle\mathcal{D}_1^{(1)}\mathcal{D}_2^{(1)}G(S_1^{(1)})G(S_1^{(2)})\rangle_-}{\langle G(S_1^{(1)})G(S_1^{(2)})\rangle_-}\Big\},$$

where

$$G(S) = e^{g(S)} \tag{2.24}$$

and the symbol $\langle\ldots\rangle_-$ means the Gibbs averaging, corresponding to the Hamiltonian (1.15) in which $g(S_1)$ is replaced by 0.

Let us again consider a standard Gaussian variable u and introduce a function

$$G^{(t)}(S,u) = \frac{1}{\sqrt{(1-t)(R-q+\varepsilon)}}\int e^{-x^2/2(R-q+\varepsilon)(1-t)}G(\sqrt{t}S + u\sqrt{q(1-t)} + x)dx. \tag{2.25}$$

We have $G^{(1)}(S,u)=G(S)$ and

$$G^{(0)}(S,u)=\frac{1}{\sqrt{(R-q+\varepsilon)}}\int e^{-x^2/2(R-q+\varepsilon)}G(u\sqrt{q}+x)dx$$

independent of S. Below we denote

$$G_0(u)=G^{(0)}(S,u)=\frac{1}{\sqrt{(R-q+\varepsilon)}}\int e^{-x^2/2(R-q+\varepsilon)}G(u\sqrt{q}+x)dx. \tag{2.26}$$

We remark, that the definition (2.25) becomes more natural, if we introduce it through the Fourier transform $\hat{G}(\lambda)$ of $G(S)$:

$$G^{(t)}(S,u)=\frac{1}{\sqrt{2\pi}}\int \hat{G}(\lambda)\exp\Big\{-i\lambda\Big(S\sqrt{t}+u\sqrt{q(1-t)}\Big)-\frac{1-t}{2}(R-q+\varepsilon)\lambda^2\Big\}. \tag{2.27}$$

Now for any $\xi_i^{(1)}$-independent function $f:\mathbf{R}^{N\times n}\to\mathbf{R}$ and some polynomial $P(x_1,\dots,x_n)$ consider the operator $\mathcal{P}^{(t)}=P(\mathcal{D}_1^{(t)},...,\mathcal{D}_n^{(t)})$

$$\begin{aligned}\varphi_t^{(n)}(f\mathcal{P}_t) &= E\Big\{\frac{\langle f\,\mathcal{P}^{(t)}G^{(t)}(S_1^{(1)},u)...G^{(t)}(S_1^{(n)},u)\rangle_-}{\langle G^{(t)}(S_1^{(1)},u)...G^{(t)}(S_1^{(n)},u)\rangle_-}\Big\}\\ &= E\Big\{\Big\langle f\,\frac{\mathcal{P}^{(t)}G^{(t)}(S_1^{(1)},u)...G^{(t)}(S_1^{(n)},u)}{G^{(t)}(S_1^{(1)},u)...G^{(t)}(S_1^{(n)},u)}\Big\rangle_{(t)}\Big\},\end{aligned} \tag{2.28}$$

where $\langle...\rangle_{(t)}$ means the Gibbs averaging corresponding to the n replicas of the Hamiltonian (1.15) in which $g(S_1^{(l)})$ is substituted by $-\log G^{(t)}(S_1^{(l)},u)$. According to the result of [1], this function is also concave with respect to S_1 and so inequalities (2.14) and (2.15) for it are also valid.

We remark here that due to the definition $G^{(t)}$ (2.27) the operator $\mathcal{P}^{(t)}$ has a natural form:

$$\begin{aligned}\mathcal{P}^{(t)}G^{(t)}(S_1^{(1)},u)...G^{(t)}(S_1^{(n)},u) &= \frac{1}{(\sqrt{2\pi})^n}\int \hat{G}(\lambda_1)\dots\hat{G}(\lambda_n)P(-i\lambda_1,\dots,-i\lambda_n)\\ &\quad \exp\{-i\sum\lambda_l S_1^{(l)}\sqrt{t}-iu\sum\lambda_l\sqrt{q(1-t)}-\frac{1-t}{2}(R-q)\lambda^2\}.\end{aligned} \tag{2.29}$$

So for $t=0$ it is well defined:

$$\mathcal{P}^{(0)}G^{(0)}(S_1^{(1)},u)...G^{(0)}(S_1^{(n)},u)\Big|_{t=0}=P(q^{-1/2}\frac{d}{dx_1},\dots,q^{-1/2}\frac{d}{dx_1})G_0(x_1)...G_0(x_n)\Big|_{x_1=...=x_n=u}.$$

Let us compute the derivative with respect to t of $\varphi_t^{(n)}(f\mathcal{P}_t)$.

$$\begin{aligned}\frac{d}{dt}\varphi_t^{(n)}(f\mathcal{P}^{(t)}) &= \frac{1}{2}\sum_{l=1}^n\varphi_t^{(n)}(f(R_{l,l}-R)(\mathcal{D}_l^{(t)})^2\mathcal{P}^{(t)})-\frac{n}{2}\varphi_t^{(n+1)}(f(R_{n+1,n+1}-R)(\mathcal{D}_{n+1}^{(t)})^2\mathcal{P}^{(t)})\\ &+\sum_{l<l'}^n\varphi_t^{(n)}(f(R_{l,l'}-q)\mathcal{D}_l^{(t)}\mathcal{D}_{l'}^{(t)}\mathcal{P}^{(t)})-n\sum_{l=1}^n\varphi_t^{(n+1)}(f(R_{l,n+1}-q)\mathcal{D}_l^{(t)}\mathcal{D}_{n+1}^{(t)}\mathcal{P}^{(t)})\\ &+\frac{n(n+1)}{2}\varphi_t^{(n+2)}(f(R_{n+1,n+2}-q)\mathcal{D}_{n+1}^{(t)}\mathcal{D}_{n+2}^{(t)}\mathcal{P}^{(t)}).\end{aligned} \tag{2.30}$$

This formula is obtained by differentiation with respect to t and then integration by parts with respect to $\xi_i^{(1)}$ and u in the expressions (2.27) and (2.29).

Proposition 5 *For any polynomial $P(\lambda_1,\dots\lambda_n)$*

$$|\varphi_t^{(n)}(\mathcal{P}^{(t)})|\le C, \tag{2.31}$$

where the constant C depends on n and on the polynomial $P(x_1,...,x_n)$.

As it was mentioned above the inequalities (1.16) and (1.17) for $\langle ... \rangle_{(t)}$ are also valid. Therefore the estimate (2.14), and (2.16) are fulfilled and so, using the Schwartz inequality, Proposition 5 and (2.15), we obtain:

$$|\varphi_1^{(n)}(f\mathcal{P}^{(1)}) - \varphi_0^{(n)}(f\mathcal{P}^{(0)})| \le C \max_t \varphi_t^{1/4}(|f|^4)N^{-1/2}. \tag{2.32}$$

Using the same formula to compute the second derivative of $\varphi_t^{(n)}(f\mathcal{P}_t)$ with respect to t, we obtain the expression in each term of which we have $(R_{l,l'}-q)(R_{l_1,l_1'}-q)$ or $(R_{l,l'}-q)(R_{l_1,l_1}-R)$ or $(R_{l_1,l_1}-R)^2$. Using the Hölder inequality with $p=(1-\epsilon)^{-1}$, $q=\epsilon^{-1}$, and then (2.16) with $r=2(1-\epsilon)^{-1}$, we obtain for any $0\le\epsilon\le 1$

$$\begin{aligned}
|\varphi_1^{(n)}(f\mathcal{P}^{(1)}) - \varphi_0^{(n)}(f\mathcal{P}^{(0)}) - \tfrac{d}{dt}\varphi_0^{(n)}(f\mathcal{P}^{(t)})| &\le C_\epsilon \max\nolimits_t \varphi_t^{\epsilon}(|f|^{1/\epsilon}(\mathcal{P}^{(t)})^{1/\epsilon})N^{-1+\epsilon} \\
&\le C_\epsilon \max\nolimits_t \varphi_t^{\epsilon/2}(|f|^{2/\epsilon})\varphi_t^{\epsilon/2}(\mathcal{P}_t^{2/\epsilon})N^{-1+\epsilon} \\
&\le C'_\epsilon \max\nolimits_t \varphi_t^{\epsilon/2}(|f|^{2/\epsilon})N^{-1+\epsilon}.
\end{aligned} \tag{2.33}$$

The last inequality here follows from Proposition5.

Proof of Theorem 2
We prove Theorem 2 in 3 steps which are Lemma 3, 4 and 5.

Lemma 3 *Consider an expression of the form $T_{1,2}^k P$ where P is some product of the terms $T_{i,j}$ (with $T_{i,j}\ne T_{1,2}$ $i,j\le m$), T_i and $\dot q$. Then for any $0<\epsilon<\frac12$*

$$E\langle T_{1,2}^k P\rangle = (k-1)A_* E\langle T_{1,2}^{k-2}P\rangle + O(N^{-1/2(1+\epsilon)}) + O(\frac{p}{N}-\alpha), \tag{2.34}$$

where

$$A_* = \frac{b_0}{1-\alpha b_0 c_0}, \tag{2.35}$$

with

$$b_0=(R-q)^2, \quad c_0=q^{-2}E\{(g_0''(u))^2\}, \quad g_0(u)=\log G_0(u), \tag{2.36}$$

where $G_0(u)$ was defined in (2.26).

Lemma 4 *Consider an expression of the form $T_1^k P$ where P is some product of the terms T_i with $1<i\le m$ and $\dot q^r$. Then for any $0<\epsilon<\frac12$*

$$E\langle T_1^k P\rangle = (k-1)B_* E\langle T_1^{k-2}P\rangle + O(N^{-1/2(1+\epsilon)}) + O(\frac{p}{N}-\alpha), \tag{2.37}$$

where B_* is some absolute constant which does not depend on P, k, m and N and which is an algebraic function of the coefficients b_0, c_0 and

$$\begin{aligned}
b_1 &= E\{\langle \dot s^2\rangle_0 \langle s\rangle_0^2\} = q(R-q)\\
c_1 &= E\{\langle(\mathcal{D}_1^{(0)}-\mathcal{D}_2^{(0)})\mathcal{D}_1^{(0)}\mathcal{D}_3^{(0)}\mathcal{D}_4^{(0)}\rangle_{(0)}\} = q^{-2}E\{g_0''(g_0')^3\}\\
c_2 &= E\{\langle(\mathcal{D}_1^{(0)}-\mathcal{D}_2^{(0)})(\mathcal{D}_1^{(0)})^2\mathcal{D}_3^{(0)}\rangle_{(0)}\} = q^{-2}E\{(g_0'''+2g_0''g_0')g_0'\}\\
c_3 &= E\{\langle((\mathcal{D}_1^{(0)})^2-(\mathcal{D}_2^{(0)})^2)(\mathcal{D}_1^{(0)})^2\rangle_{(0)}\} = q^{-2}E\{(g_0^{(4)}+4g_0'''g_0'+2(g_0'')^2+4g_0''(g_0')^2)\}\\
c_4 &= E\{\langle\mathcal{D}_1^{(0)}\mathcal{D}_2^{(0)}\mathcal{D}_3^{(0)}\mathcal{D}_4^{(0)}\rangle_{(0)}\} = q^{-2}E\{(g_0')^4\}.
\end{aligned} \tag{2.38}$$

Lemma 5 *For any $0<\epsilon<\frac12$*

$$E\langle \dot q^k\rangle = (k-1)C_* E\langle \dot q^{k-2}\rangle + O(N^{-1/2(1+\epsilon)}) + O(\frac{p}{N}-\alpha), \tag{2.39}$$

where C_* is some absolute constant which does not depend on N, k and which is an algebraic function of the coefficients b_0, b_1, c_0,..., c_4.

One can see easily that the statement of Theorem 2 follows from these lemmas by induction. So our goal is to prove the lemmas.

Proof of Lemma 3.

Let us remark that $\langle T_{1,2}P\rangle$ does not change, if we substitute all the multipliers here by the following expressions:

$$\begin{aligned}
T_{l,l'} &\to N^{-1/2}((\boldsymbol{J}^{(l)}-\boldsymbol{J}^{(k)}),(\boldsymbol{J}^{(l)}-\boldsymbol{J}^{(k')}))\\
&= N^{-1/2}((\boldsymbol{J}^{-(l)}-\boldsymbol{J}^{-(k)}),(\boldsymbol{J}^{-(l)}-\boldsymbol{J}^{-(k')}))+N^{-1/2}(s_l-s_k)(s_l-s_{k'})\\
T_l &\to N^{-1/2}((\boldsymbol{J}^{(l)}-\boldsymbol{J}^{(k_1')}),\boldsymbol{J}^{(k_1)})\\
&= N^{-1/2}((\boldsymbol{J}^{-(l)}-\boldsymbol{J}^{-(k_1')}),\boldsymbol{J}^{-(k_1)})+N^{-1/2}(s_l-s_{k_1})s_{k_1'}\\
\dot{q} &\to N^{-1/2}((\boldsymbol{J}^{(k_2)},\boldsymbol{J}^{(k_2')})-Nq)\\
&= N^{-1/2}((\boldsymbol{J}^{-(k_2)},\boldsymbol{J}^{-(k_2')})-(N-1)q)+N^{-1/2}(s_{k_2}s_{k_2'}-q),
\end{aligned}\tag{2.40}$$

where indexes $k,k',k_1,k_1',k_2,k_2'>m$ are different for each term in the product and we have used notations of formulas (2.17) and (2.19). We denote the last term in i-th expression by $N^{-1/2}f_i(s)$. Using the symmetry of the Hamiltonian and the above representation, we can write

$$\begin{aligned}
E\{\langle T_{1,2}^k P\rangle\} &= N^{-1/2}E\Big\{\Big\langle\sum_{i=1}^N(J_i^{(1)}-\langle J_i\rangle)(J_i^{(2)}-\langle J_i\rangle)T_{1,2}^{k-1}P\Big\rangle\Big\}\\
&= \sqrt{N}\nu_1((s_1-\langle s_1\rangle)(s_2-\langle s_2\rangle)\tilde{P}^-)\\
&= \sqrt{N}\nu_1((s_1-s_k)(s_2-s_{k'})\tilde{P}^-)\\
&\quad+\sum_i\nu_1((s_1-s_k)(s_2-s_{k'})f_i(s)\tilde{P}_i^-)+O(N^{-1/2})\\
&= I+II+O(N^{-1/2}),
\end{aligned}\tag{2.41}$$

where $\tilde{P}^-$ denotes the product only of such terms of (2.40) which does not contain s_l and $\tilde{P}_i^-$ means the product of the same terms except the i-th one. The term $O(N^{-1/2})$ appears because of the products which contain more than 1 term $f_i(s)$. Applying twice formula (2.22) to the term II, we get

$$II = \sum_i\nu_0((s_1-s_k)(s_2-s_{k'})f_i(s))\nu_0(\tilde{P}_i^-)+O(N^{-1/2}),$$

where we used that the Hamiltonian which corresponds n replicas of H_0 (see (2.17)) gives us the Gibbs averages factorised with respect to s_i (see (2.17)). One can get easily that, if $f_i(s)$ does not contain both s_1 and s_2, then

$$\nu_0((s_1-s_k)(s_2-s_{k'})f_i(s))=0.$$

So, since both s_1 and s_2 are contained only in $f_2,\dots f_k$, which correspond to $T_{1,2}$, we obtain

$$\begin{aligned}
II &= \sum_{i=2}^k\nu_0((s_1-s_k)(s_2-s_{k'})(s_1-s_{k_i})(s_2-s_{k_i'}))\nu_0(\tilde{P}_i^-)+O(N^{-1/2})\\
&= (k-1)b_0\nu_1(\tilde{P}_i^-)+O(N^{-1/2}).
\end{aligned}\tag{2.42}$$

with b_0 defined by (2.36). Here we used (2.22) to replace $\nu_0(\tilde{P}_i^-)$ by $\nu_1(\tilde{P}_i^-)$. Then, using (2.40) and the bounds (2.12), (2.13) we can substitute in (2.42) $\tilde{P}_i$ by $T_{1,2}^{k-2}P$.

Now let us analyze the term I of (2.41) with formula (2.23). It is evident that

$$\nu_0((s_1-s_k)(s_2-s_{k'}))=0$$

and so the term ν_0 in the l.h.s. of (2.23) disappears. Calculating ν_0', by formula (2.21), we get

$$I=\sqrt{N}\sum_{l<l'}^{n}\nu_0((s_1-s_k)(s_2-s_{k'})s_ls_{l'})\nu_0((\tilde{R}^-_{l,l'}-d)\tilde{P}^-)+O(N^{-1/2+\epsilon}),$$

where we choose $\epsilon<\frac{1}{2}$. All the rest terms in (2.21) disappear because

$$\nu_0((s_1-s_k)(s_2-s_{k'})s_ls_{l'})=b_0(\delta_{l,1}\delta_{l',2}+\delta_{l,k}\delta_{l',k'}-\delta_{l,k}\delta_{l',2}-\delta_{l,k'}\delta_{l',1}).$$

So, we get

$$I\quad=\quad b_0\sqrt{N}\nu_0((\tilde{R}^-_{1,2}-\tilde{R}^-_{1,k'}-\tilde{R}^-_{2,k}+\tilde{R}^-_{k,k'})\tilde{P}^-)+O(N^{-1/2+\epsilon}). \tag{2.43}$$

Now our goal is to substitute $\nu_0(\ldots)$ in the r.h.s. of (2.43) by $\nu_1(\ldots)$. To this end we apply formula (2.21) to the function $f=f_1f_2$ with

$$f_1=\sqrt{N}(\tilde{R}^-_{1,2}-\tilde{R}^-_{1,k'}-\tilde{R}^-_{2,k}+\tilde{R}^-_{k,k'}),\quad f_2=\tilde{P}^-$$

To estimate the the r.h.s. of (2.21) we use the Hölder inequality of the form

$$\begin{gathered}|\nu_t(f_1f_2f_3f_4)|\le\nu_t^{1/p_1}(|f_1|^{p_1})\cdot\nu_t^{1/p_2}(|f_2|^{p_2})\cdot\nu_t^{1/p_3}(|f_3|^{p_3})\cdot\nu_t^{1/p_4}(|f_4|^{p_4}),\\ \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}=1\end{gathered} \tag{2.44}$$

for f_1 and f_2 defined above and $f_3=s_ls_{l'}$, $f_4=R_{l,l'}-d$ or $f_4=\dot{U}_l^-$ and $p_1=2$, $p_2=p_3=4(1+\epsilon)/\epsilon$, $p_4=2(1+\epsilon)$. Then, according to (2.20)

$$\nu_t^{\epsilon/4(1+\epsilon)}(f_3^{4(1+\epsilon)/\epsilon})\le C_\epsilon.$$

According to (2.12) and (2.13),

$$\nu_t^{\epsilon/4(1+\epsilon)}(f_2^{4(1+\epsilon)/\epsilon})\le C_\epsilon,$$

and according to (2.16),

$$\nu_t^{1/2(1+\epsilon)}(f_4^{2(1+\epsilon)})\le C_\epsilon N^{-1/2(1+\epsilon)}.$$

Thus, we derive from (2.44)

$$|\nu_t'(f)|\le C_\epsilon N^{-1/2(1+\epsilon)}\nu_t^{1/2}(f_1^2). \tag{2.45}$$

But, using the Schwartz inequality and (2.9),(2.3), we obtain

$$\begin{aligned}\nu_t^{1/2}(f_1^2)\quad&=\quad E\Big\{N^{-1}\sum_{\mu,\nu}\Big\langle(g'(S^-_\mu)-\langle g'(S^-_\mu)\rangle_t)(g'(S^-_\nu)-\langle g'(S^-_\nu)\rangle_t)\Big\rangle_t^2\Big\}\\ &=\quad E\Big\{N^{-1}\sum_{\mu,\nu}\mathcal{C}^2_{\mu,\nu}\Big\}\le E\{||\mathcal{C}||^2\}\le C,\end{aligned} \tag{2.46}$$

where the matrix $\mathcal{C}$ was defined by (2.8)). Thus we derive from (2.43)

$$I\quad=\quad b_0\sqrt{N}E\{\langle(\tilde{R}^-_{1,2}-\tilde{R}^-_{1,k'}-\tilde{R}^-_{2,k}+\tilde{R}^-_{k,k'})\tilde{P}^-\rangle_1\}+O(N^{-1/2+\epsilon}). \tag{2.47}$$

To substitute here $\tilde{R}^-_{l,l'}$ by $\tilde{R}_{l,l'}$ we write

$$\begin{aligned}&E\{\langle(\tilde{R}_{1,2}-\tilde{R}_{1,k'}-\tilde{R}_{2,k}+\tilde{R}_{k,k'})\tilde{P}^-\rangle_1\}\\ &=E\Big\{N^{-1/2}\Big\langle\sum_\mu(g'(S_\mu^{(1)})-\langle g'(S_\mu)\rangle)(g'(S_\mu^{(2)})-\langle g'(S_\mu)\rangle)\tilde{P}^-\Big\rangle_1\Big\}\\ &=E\Big\{N^{-1/2}\Big\langle\sum_\mu(g'(S_\mu^{-(1)})-\langle g'(S^-_\mu)\rangle)(g'(S_\mu^{-(2)})-\langle g'(S^-_\mu)\rangle)\tilde{P}^-\Big\rangle_1\Big\}\\ &+E\Big\{N^{-1}\Big\langle\sum_\mu\xi_1^{(\mu)}(s_1g''(S_\mu^{-(1)})-\langle sg''(S^-_\mu)\rangle_1)(g'(S_\mu^{-(2)})-\langle g'(S_\mu-)\rangle)\tilde{P}^-\Big\rangle_1\Big\}\\ &+E\Big\{N^{-1}\Big\langle\sum_\mu\xi_1^{(\mu)}(s_2g''(S_\mu^{-(2)})-\langle sg''(S^-_\mu)\rangle_1)(g'(S_\mu^{-(1)})-\langle g'(S^-_\mu)\rangle)\tilde{P}^-\Big\rangle_1\Big\}+O(N^{-1/2})\end{aligned} \tag{2.48}$$

Here the terms of the second and higher orders with respect to $N^{-1/2}\xi_1^{(\mu)}$ give us $O(N^{-1/2})$. But due to the Schwartz inequality we have

$$
\begin{aligned}
&E\Big\{N^{-1}\Big\langle\sum_\mu \xi_1^{(\mu)}(s_1 g''(S_\mu^{-(1)})-\langle sg''(S_\mu^-)\rangle_1)(g'(S_\mu^{-(2)})-\langle g'(S_\mu^-)\rangle)\tilde{P}^-\Big\rangle_1\Big\}\\
&\le E^{1/2}\Big\{N^{-2}\sum_{\mu,\nu}\Big\langle(sg''(S_\mu^-)-\langle sg''(S_\mu^-)\rangle_1)(sg''(S_\nu^-)-\langle sg''(S_\nu^-)\rangle_1)\Big\rangle_1\\
&\cdot\Big\langle(g'(S_\mu^-)-\langle g'(S_\mu^-)\rangle)(g'(S_\nu^{-(1)})-\langle g'(S_\nu^-)\rangle)\Big\rangle_1\Big\}E^{1/2}\Big\{\Big\langle(\tilde{P}^-)^2\Big\rangle_1\Big\}\\
&=E^{1/2}\Big\{N^{-2}\sum_{\mu,\nu=1}^p \xi_1^{(\mu)}\xi_1^{(\nu)}\mathcal{C}_{\mu,\nu}\mathcal{C}'_{\mu,\nu}\Big\}\le N^{-1/2}E^{1/4}\{||||^2\}E^{1/4}\{||\mathcal{C}'||^2\},
\end{aligned}
\tag{2.49}
$$

where the matrix $\mathcal{C}$ is defined by (2.8) and

$$
\mathcal{C}'_{\mu,\nu}=\Big\langle(sg''(S_\mu^-)-\langle sg''(S_\mu^-)\rangle_1)(sg''(S_\nu^-)-\langle sg''(S_\nu^-)\rangle_1)\Big\rangle.
$$

By using (1.17) one can prove that $||\mathcal{C}'||\le C||\mathcal{A}_*||$ (see the proof of Proposition 2 in the next section). Then (2.48), (2.49) combined with (2.9), (2.3) and (2.13) imply that we can substitute $\tilde{R}^-_{l,l'}$ by $\tilde{R}_{l,l'}$ in (2.47).

Hence, we get

$$
\begin{aligned}
&\sqrt{N}\nu_0((\tilde{R}_{1,2}-\tilde{R}_{1,k'}-\tilde{R}_{2,k}+\tilde{R}_{k,k'})\tilde{P}^-)\\
&=\sqrt{N}E\{\langle(\tilde{R}_{1,2}-\tilde{R}_{1,k'}-\tilde{R}_{2,k}+\tilde{R}_{k,k'})\tilde{P}^-\rangle_1\}+O(N^{-1/2(1+\epsilon)})\\
&=III+O(N^{-1/2(1+\epsilon)}).
\end{aligned}
\tag{2.50}
$$

To analyze III we use again the symmetry of (1.15) and notations of (2.28) to write

$$
\begin{aligned}
III &= \frac{1}{\sqrt{N}}\sum_{\mu=1}^p E\{\langle(g'(S_\mu^{(1)})-g'(S_\mu^{(k)}))(g'(S_\mu^{(2)})-g'(S_\mu^{(k')}))\tilde{P}^-\rangle\}\\
&= \tfrac{p}{N}\sqrt{N}E\{\langle(g'(S_1^{(1)})-g'(S_1^{(k)}))(g'(S_1^{(2)})-g'(S_1^{(k')}))\tilde{P}^-\rangle\}\\
&= \tfrac{p}{N}\sqrt{N}\varphi_1((\mathcal{D}_1^{(1)}-\mathcal{D}_k^{(1)})(\mathcal{D}_2^{(1)}-\mathcal{D}_{k'}^{(1)})\tilde{P}^-).
\end{aligned}
$$

Now, applying formula (2.33), we can write

$$
\frac{N}{p}III=\sqrt{N}\sum_{l<l'}\varphi_0((\mathcal{D}_1^{(0)}-\mathcal{D}_k^{(0)})(\mathcal{D}_2^{(0)}-\mathcal{D}_{k'}^{(0)})\mathcal{D}_l^{(0)}\mathcal{D}_{l'}^{(0)})\varphi_0((R_{l,l'}-q)\tilde{P}^-)+O(N^{-1/2+\epsilon}).
\tag{2.51}
$$

All the rest terms in (2.30) disappear because

$$
\varphi_0((\mathcal{D}_1^{(0)}-\mathcal{D}_k^{(0)})(\mathcal{D}_2^{(0)}-\mathcal{D}_{k'}^{(0)}))=0,\quad \varphi_0((\mathcal{D}_1^{(0)}-\mathcal{D}_k^{(0)})(\mathcal{D}_2^{(0)}-\mathcal{D}_{k'}^{(0)})(\mathcal{D}_l^{(0)})^2)=0.
$$

Let us note that

$$
\varphi_0((\mathcal{D}_1^{(0)}-\mathcal{D}_k^{(0)})(\mathcal{D}_2^{(0)}-\mathcal{D}_{k'}^{(0)})\mathcal{D}_l^{(0)}\mathcal{D}_{l'}^{(0)})=c_0(\delta_{l,1}\delta_{l',2}+\delta_{l,k}\delta_{l',k'}-\delta_{l,k}\delta_{l',2}-\delta_{l,k'}\delta_{l',1}),
$$

with c_0 defined in (2.36). Besides, by using (2.32) one can get easily that

$$
\varphi_0((R_{1,2}-R_{1,k'}-R_{2,k}+R_{k,k'})\tilde{P}^-)=\varphi_1((R_{1,2}-R_{1,k'}-R_{2,k}+R_{k,k'})\tilde{P}^-)+O(N^{-1/2}).
$$

By using the representation (2.40), (2.46) and (2.13) one can also substitute $\tilde{P}^-$ by $T_{1,2}^{k-1}P$ in (2.47). Hence, we get from (2.51) that

$$\begin{aligned}
&\tfrac{N}{p}\sqrt{N}E\{\langle(\tilde{R}_{1,2}-\tilde{R}_{1,k'}-\tilde{R}_{2,k}+\tilde{R}_{k,k'})\tilde{P}^-\rangle\}\\
&=c_0\sqrt{N}E\{\langle(R_{1,2}-R_{1,k'}-R_{2,k}+R_{k,k'})\tilde{P}^-\rangle\}+O(N^{-1/2+\epsilon})\\
&=c_0E\{\langle T_{1,2}^kP\rangle\}+O(N^{-1/2+\epsilon}).
\end{aligned} \tag{2.52}$$

Now, on the basis of (2.40), (2.41), (2.42), (2.43), (2.50) and (2.52), we get (2.34). Lemma 3 is proven.

Proof of Lemma 4

Like in the proof of Lemma 2 we use representation (2.40) for all terms and change the numbers of replicas to have for the first T_1 the numbers $1,2,3$ and for the i-th T_1 – $(1,k_i,k_i')$. Using the symmetry of the problem, we write (similarly to (2.41)):

$$\begin{aligned}
E\{T_1^kP\} &= \sqrt{N}\nu_1((s_1-s_2)s_3\tilde{P}^-)\\
&\quad+\sum_i\nu_1((s_1-s_2)s_3(s_1-s_{k_i})s_{k_i'}f_i(s)\tilde{P}_i^-)+O(N^{-1/2})\\
&= I+II+O(N^{-1/2}),
\end{aligned} \tag{2.53}$$

where $\tilde{P}^-$ means the product only of such terms of (2.40) which does not contain s_1 and $\tilde{P}_i^-$ means the product of the same terms except the i-th one. By the same way as in Lemma 3, we get

$$II=(k-1)b_1E\{\langle T_1^{k-2}P\rangle\}+O(N^{-1/2}), \tag{2.54}$$

where b_1 is defined by (2.38). Using formula (2.23) for $0<\epsilon<\frac{1}{2}$ we obtain

$$\begin{aligned}
I &= \sqrt{N}\Big[2b_1\nu_0((\tilde{U}_1^- -\tilde{U}_2^-)\tilde{P}^-)+b_0\nu_0((\tilde{R}_{1,3}^- -\tilde{R}_{2,3}^-)\tilde{P}^-)\\
&\quad+b_1\sum_{l=3}^{n}\nu_0((\tilde{R}_{1,l}^- -\tilde{R}_{2,l}^-)\tilde{P}^-)-nb_1\nu_0((\tilde{R}_{1,n+1}^- -\tilde{R}_{2,n+1}^-)\tilde{P}^-)\Big]+O(N^{-1/2+\epsilon})\\
&= 2b_1\sqrt{N}\nu_0((\tilde{U}_1^- -\tilde{U}_2^-)\tilde{P}^-)+(b_0-2b_1)\sqrt{N}\nu_0((\tilde{R}_{1,3}^- -\tilde{R}_{2,3}^-)\tilde{P}^-)\\
&\quad+b_1\sum_{l=3}^{n}\sqrt{N}\nu_0((\tilde{R}_{1,l}^- -\tilde{R}_{2,l}^- -\tilde{R}_{1,n+1}^- +\tilde{R}_{2,n+1}^-)\tilde{P}^-)+O(N^{-1/2+\varepsilon}),
\end{aligned} \tag{2.55}$$

where we used that

$$\nu_0((\tilde{R}_{1,3}^- -\tilde{R}_{2,3}^-)\tilde{P}^-)=\nu_0((\tilde{R}_{1,n+1}^- -\tilde{R}_{2,n+1}^-)\tilde{P}^-),$$

because $\tilde{P}^-$ does not contains both the third and the $(n+1)$th replica. Besides, on the basis of (2.52), applied to the Hamiltonian H_0 (see (2.17)), one can conclude that

$$b_0\sqrt{N}\nu_0((\tilde{R}_{1,l}^- -\tilde{R}_{2,l}^- -\tilde{R}_{1,n+1}^- +\tilde{R}_{2,n+1}^-)\tilde{P}^-)=\sqrt{N}\nu_0((R_{1,l}^- -R_{2,l}^- -R_{1,n+1}^- +R_{2,n+1}^-)\tilde{P}^-)+O(N^{-1/2+\epsilon})$$

Let us rewrite all the terms in $\tilde{P}^-$ as $N^{-1/2}(J^{-(i)},J^{-(j)})=T_{i,j}^- +T_i^- +T_j^- +N^{-1,2}(\langle J^-\rangle_0,\langle J^-\rangle_0)$ $(i,j\le n)$, where T_{ij}^-, T_i^- are defined by (1.10) if we replace there $\boldsymbol{J}^{(i)}$ by $\boldsymbol{J}^{-(i)}$. Then we obtain

$$\begin{aligned}
&\sqrt{N}\nu_0((R_{1,l}^- -R_{2,l}^- -R_{1,n+1}^- +R_{2,n+1}^-)\tilde{P}^-)\\
&=\nu_0((T_{1,l}^- -T_{2,l}^- -T_{1,n+1}^- +T_{2,n+1}^-)\tilde{P}^-)=\delta_{l,k_i}A_*\nu_0((T_1^-)^{k-2}P^-)+O(N^{-1/2+\epsilon})\\
&\delta_{l,k_i}A_*E\{\langle T_1^{k-2}P\rangle\}+O(N^{-1/2+\epsilon}).
\end{aligned} \tag{2.56}$$

We recall that k_i is the number of replica which appears because of the substitution (2.40) of ith T_1 in our product by $N^{-1/2}((\boldsymbol{J}^{(1)} - \boldsymbol{J}^{(k_i')}), \boldsymbol{J}^{(k_i)})$. Here we have used Lemma 3, which imply that for any product P (defined in this lemma) which doe not contain $T_{i,j}$ $E\{\langle T_{i,j}P\rangle\} = O(N^{-1/2+\epsilon})$ and $E\{\langle T_{i,j}^2 P\rangle\} = A_* E\{\langle P\rangle\} + O(N^{-1/2+\epsilon})$. To write the last equality in (2.56) we similarly to Lemma 3 use the representation (2.40), formula (2.22) and the bounds (2.13) and (2.14).

Thus, we obtain from (2.55)-(2.56)

$$\begin{aligned} I &= 2b_1\sqrt{N}E\{\langle(\tilde{U}_1^- - \tilde{U}_2^-)\tilde{P}^-\rangle_0\} + (b_0 - 2b_1)\sqrt{N}E\{\langle(\tilde{R}_{1,3}^- - \tilde{R}_{2,3}^-)\rangle_0\} \\ &\quad +(k-1)b_1 b_0^{-1} A_* E\{\langle T_1^{k-2}P\rangle\} + O(N^{-1/2+\varepsilon}). \end{aligned} \tag{2.57}$$

Then, repeating corresponding conclusions of Lemma 3, one can get the representation

$$\begin{aligned} \tfrac{N}{p} I &= 2b_1\sqrt{N}\varphi_1(((\mathcal{D}_1^{(1)})^2 - (\mathcal{D}_2^{(1)})^2)\tilde{P}^-) \\ &\quad +(b_0 - 2b_1)\sqrt{N}\varphi_1((\mathcal{D}_1^{(1)} - \mathcal{D}_2^{(1)})\mathcal{D}_3^{(1)}\tilde{P}^-) \\ &\quad +(k-1)b_1 b_0^{-1} A_* E\{\langle T_1^{k-2}P\rangle\} + O(N^{-1/2+\epsilon}). \end{aligned} \tag{2.58}$$

Applying formula (2.30) to the first term here, we get

$$\begin{aligned} &\sqrt{N}\varphi_1(((\mathcal{D}_1^{(1)})^2 - (\mathcal{D}_2^{(1)})^2)\tilde{P}^-) = \\ &\sqrt{N}\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)(\mathcal{D}_1^{(0)})^2)\varphi_0((R_{1,1} - R)\tilde{P}^-) \\ &+\sqrt{N}\sum_{l=3}^{n}\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)\mathcal{D}_1^{(0)}\mathcal{D}_l^{(0)})\varphi_0((R_{1,l} - R_{2,l})\tilde{P}^-) \\ &-n\sqrt{N}\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)\mathcal{D}_1^{(0)}\mathcal{D}_{n+1}^{(0)})\varphi_0((R_{1,n+1} - R_{2,n+1})\tilde{P}^-) \\ &= c_3\sqrt{N}\varphi_0((R_{1,1} - R)\tilde{P}^-) + c_2\sqrt{N}\sum_{l=3}^{n}\varphi_0((R_{1,l} - R_{2,l} - R_{1,n+1} + R_{1,n+1})\tilde{P}^-) \\ &-2c_2\sqrt{N}\varphi_0((R_{1,1} - R)\tilde{P}^-). \end{aligned} \tag{2.59}$$

Here we have used that

$$\begin{aligned} &\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)\mathcal{D}_l^{(0)}\mathcal{D}_{l'}^{(0)}) = 0, \qquad \{l, l'\} \cap \{1, 2\} = \emptyset \\ &\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)\mathcal{D}_1^{(0)}\mathcal{D}_l^{(0)}) \\ &= -\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)\mathcal{D}_2^{(0)}\mathcal{D}_l^{(0)}) = c_2, \quad (l \geq 3) \\ &\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)\mathcal{D}_1^{(0)}\mathcal{D}_2^{(0)}) = 0 \\ &\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)(\mathcal{D}_1^{(0)})^2) \\ &= -\varphi_0(((\mathcal{D}_1^{(0)})^2 - (\mathcal{D}_2^{(0)})^2)(\mathcal{D}_2^{(0)})^2) = c_3 \end{aligned}$$

Similarly to (2.56) $\sqrt{N}\varphi_0((R_{1,l} - R_{2,l} - R_{1,n+1} + R_{1,n+1})\tilde{P}^-) = \delta_{l,k_i} A_* E\{\langle T_1^{k-2}P\rangle\} + O(N^{-1/2+\varepsilon})$. Thus we get from (2.59)

$$\begin{aligned} &\sqrt{N}\varphi_1(((\mathcal{D}_1^{(1)})^2 - (\mathcal{D}_2^{(1)})^2)\tilde{P}^-) = (k-1)A_* E\{\langle T_1^{k-2}P\rangle\} \\ &+c_3\sqrt{N}\varphi_0((R_{1,1} - R)\tilde{P}^-) - 2c_2\sqrt{N}\varphi_0((R_{1,3} - R_{2,3})\tilde{P}^-). \end{aligned} \tag{2.60}$$

By the same way, applying formula (2.30) to $\varphi_1((\mathcal{D}_1^{(1)}-\mathcal{D}_2^{(1)})\mathcal{D}_3^{(1)}\tilde{P}^-)$, we get

$$\begin{aligned}
&\sqrt{N}\varphi_1((\mathcal{D}_1^{(1)}-\mathcal{D}_2^{(1)})\mathcal{D}_3^{(1)}\tilde{P}^-)\\
&= c_2\sqrt{N}\varphi_0((R_{1,1}-R)\tilde{P}^-)+c_1\sum_{l\geq 3}\sqrt{N}\varphi_0((R_{1,l}-R_{2,l})\tilde{P}^-)\\
&\quad -nc_1\sqrt{N}\varphi_0((R_{1,n+1}-R_{2,n+1})\tilde{P}^-)+O(N^{-1/2+\epsilon})\\
&= c_2\sqrt{N}\varphi_0((R_{1,1}-R)\tilde{P}^-)-2c_1\sqrt{N}\varphi_0((R_{1,l}-R_{2,l})\tilde{P}^-)\\
&\quad +c_3\sum_{l\geq 3}\sqrt{N}\varphi_0((R_{1,l}-R_{2,l}-R_{1,n+1}+R_{2,n+1})\tilde{P}^-)+O(N^{-1/2+\epsilon})
\end{aligned} \tag{2.61}$$

But similarly to (2.56) the last sum here is $(k-1)A_*E\{\langle T_1^{k-2}P\rangle\}+O(N^{-1/2+\epsilon})$. Thus, using (2.58), (2.60) and (2.61), we obtain

$$\begin{aligned}
\tfrac{N}{p}I &= a_1^{(1)}\sqrt{N}E\{\langle((R_{1,1}-R)\tilde{P}^-\rangle\}+a_2^{(1)}E\{T_1^kP\}\\
&\quad +(k-1)A_*a_3^{(1)}E\{\langle T_1^{k-2}\rangle\}+O(N^{-1/2+\epsilon}),
\end{aligned} \tag{2.62}$$

where $a_1^{(1)},a_2^{(1)},a_3^{(1)}$ are some algebraic combinations of b_0, b_1, $c_0,\ldots,$ c_4 defined by (2.38). Here, similarly to Lemma 3, we replaced $\varphi_0(\ldots)$ by $E\{\langle\ldots\rangle\}$ and $\tilde{P}^-$ by $T_1^{k-1}P$. Combining with (2.53), (2.54), we get finally

$$\begin{aligned}
E\{T_1^kP\} &= \alpha a_1^{(1)}\sqrt{N}E\{\langle((R_{1,1}-R)\tilde{P}^-\rangle\}+\alpha a_2^{(1)}E\{T_1^kP\}\\
&\quad +(k-1)(b_1+\alpha A_*a_3^{(1)})E\{\langle T_1^{k-2}P\rangle\}+O(N^{-1/2+\epsilon}).
\end{aligned} \tag{2.63}$$

Now we are faced with a problem to compute $E\{\langle(R_{1,1}-R)\tilde{P}^-\rangle\}\sqrt{N}$. By using the above procedure, we get

$$\begin{aligned}
\sqrt{N}E\{\langle(R_{1,1}-R)T_1^{k-1}P\rangle\} &= \alpha a_1^{(2)}\sqrt{N}E\{\langle(R_{1,1}-R)T_1^{k-1}P\rangle\}+\alpha a_2^{(2)}E\{T_1^kP\}\\
&\quad +(k-1)(2b_1+\alpha a_3^{(2)})E\{T_1^{k-2}P\}+O(N^{-1/2+\epsilon}),
\end{aligned} \tag{2.64}$$

where $a_1^{(2)},\ldots,a_3^{(2)}$ are some algebraic combinations of b_0, b_1, $c_0,\ldots,$ c_4 defined by (2.38). Hence, we have the system of two equations with respect to $E\{T_1^kP\}$ and $E\{\langle(R_{1,1}-R)T_1^{k-1}P\rangle\}\sqrt{N}$. This system gives us

$$E\{T_1^kP\}=(k-1)B_*E\{T_1^{k-2}P\},\quad E\{\langle(R_{1,1}-R_{2,2})T_1^{k-1}P\rangle\}\sqrt{N}=(k-1)\tilde{B}_*E\{T_1^{k-2}P\}\sqrt{N} \tag{2.65}$$

with some B_* and $\tilde{B}_*$, depending only on the coefficients b_0, b_1, $c_0,\ldots,$ c_4 defined by (2.38).

Proof of Lemma 5

Similarly to Lemmas 3 and 4 we use the representation (2.40) to write

$$\begin{aligned}
E\{\dot{q}^k\} &= \sqrt{N}\nu_1((s_1s_2-q)\tilde{P}^-)\\
&\quad +\sum_{i=2}^k\nu_1((s_1s_2-q)^2\tilde{P}_i^-)+O(N^{-1/2})\\
&= I+II+O(N^{-1/2}),
\end{aligned} \tag{2.66}$$

where

$$\tilde{P}^-=\prod_{l=2}^k N^{1/2}(R_{2l-1,2l}^--q)$$

By the same way as in Lemmas 3 and 4 we obtain

$$II = (k-1)q^2 E\{\dot{q}^{k-2}\} + O(N^{-1/2}) \tag{2.67}$$

Calculating I with (2.22), and then substituting $\tilde{R}^-_{l,l'}$ by $\tilde{R}_{l,l'}$ and ν_0 by ν_1 (see the proof of Lemma 3 for details), we get

$$\begin{aligned} I &= \sqrt{N}\Big[q^2 \textstyle\sum_{l<l'} \nu_1((\tilde{R}_{l,l'} - \tilde{R}_{l,n+1} - \tilde{R}_{l',n+2} + \tilde{R}_{n+1,n+2})\tilde{P}^-) \\ &\quad -(2b_1 - q^2)\sum_{l\geq 3} \nu_1((\tilde{R}_{1,2} - \tilde{R}_{l,2})\tilde{P}^-) - \frac{q^2}{2}\sum_{l\geq 3}\nu_1((\tilde{R}_{1,1} - \tilde{R}_{l,l})\tilde{P}^-) \\ &\quad +(b_0 - 2b_1)\nu_1((\tilde{R}_{1,2} - d)\tilde{P}^-) + 2b_1\nu_1((\tilde{R}_{1,1} - U)\tilde{P}^-)\Big] + O(N^{-1/2+\epsilon}). \end{aligned} \tag{2.68}$$

Here we have used that since $\tilde{P}$ does not contain replicas 1 and 2 we can replace $n+1, n+2 \to 1, 2$ Now we apply formula (2.33) using the following relations:

$$\begin{aligned} E\{\langle T_{l,l'}\tilde{P}\rangle\} &= A_* \sum_{k=2} \delta_{l,2k-1}\delta_{l',2k} E\{\dot{q}^{k-2}\} \ (l < l'), \\ E\{\langle T_l \tilde{P}\rangle\} &= B_* E\{\dot{q}^{k-2}\}, \\ \sqrt{N} E\{\langle (R_{1,1} - R_{l,l})\tilde{P}\rangle\} &= -\tilde{B}_* E\{\dot{q}^{k-2}\}. \end{aligned}$$

These relations follows from Lemmas 3, 4.

$$\begin{aligned} \tfrac{N}{p} I &= a_1^{(3)} E\{\langle (R_{1,1} - R)\dot{q}^{k-2}\rangle\}\sqrt{N} \\ &\quad + a_2^{(3)} E\{\dot{q}^k\} + (k-1)a_3^{(3)} E\{\dot{q}^{k-2}\} + O(N^{-1/2+\epsilon}). \end{aligned} \tag{2.69}$$

So we have got the equation

$$\begin{aligned} E\{\dot{q}^k\} &= (k-1)(q^2 + \alpha a_3^{(3)}) E\{\dot{q}^{k-2}\} \\ &\quad + \alpha a_1^{(3)} E\{\langle (R_{1,1} - R)\dot{q}^{k-2}\rangle\}\sqrt{N} + \alpha a_2^{(3)} E\{\dot{q}^k\} + O(N^{-1/2+\epsilon}), \end{aligned} \tag{2.70}$$

where $a_1^{(3)}, \dots, a_3^{(3)}$ are some algebraic combinations of b_0, b_1, $c_0, \dots, c_4$ defined by (2.38).

By the same way, studying $\sqrt{N} E\{\langle (R_{1,1} - R)\rangle \dot{q}^{k-1}\}$, we get the equation

$$\begin{aligned} \sqrt{N} E\{\langle (R_{1,1} - R)\rangle \dot{q}^{k-1}\} &= (k-1)(q^2 + \alpha a_3^{(4)}) E\{\dot{q}^{k-2}\} \\ &\quad + \alpha a_1^{(4)} E\{\langle (R_{1,1} - R)\rangle \dot{q}^{k-2}\}\sqrt{N} \\ &\quad + \alpha a_2^{(4)} E\{\dot{q}^k\} + O(N^{-1/2+\epsilon}), \end{aligned} \tag{2.71}$$

where $a_1^{(4)}, \dots, a_3^{(4)}$ are some algebraic combinations of b_0, b_1, $c_0, \dots, c_4$, defined by (2.38).

Considering (2.70) and (2.71) as a system of equations with respect to $E\{\dot{q}^k\}$ and $\sqrt{N} E\{\langle (R_{1,1} - R)\rangle \dot{q}^{k-1}\}$, we finish the proof of Lemma 5.

3 Auxiliary results

Proof of Proposition 2 Consider any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^N$ and write

$$(\mathcal{A}^{(f)}\mathbf{x}, \mathbf{y}) = \langle (\dot{\boldsymbol{J}}, \mathbf{x})(\dot{\boldsymbol{J}}, \mathbf{y}) f\rangle \leq \langle (\dot{\boldsymbol{J}}, \mathbf{x})^2 (\dot{\boldsymbol{J}}, \mathbf{y})^2\rangle^{1/2} \langle f^2\rangle^{1/2} \leq \frac{3\langle f^2\rangle^{1/2}}{z} |\mathbf{x}|\,|\mathbf{y}|,$$

where we have used inequality (1.16). Similarly, using inequalities (1.16) and (1.17), for any $\mathbf{x} \in \mathbf{R}^p, \mathbf{y} \in \mathbf{R}^p$

$$\begin{aligned}(\mathcal{B}^{(f)}\mathbf{x}, \mathbf{y}) &= \langle (\dot{\boldsymbol{J}}, \mathbf{x})^4 \rangle^{1/4} \langle f^4 \rangle^{1/4} \Big\langle \sum_\mu (g'(S_\mu) - \langle g'(S_\mu) \rangle)^2 y_\mu^2 \Big\rangle^{1/2} \\ &\leq \frac{\langle 3f^4 \rangle^{1/4} ||\mathcal{A}_*||^{1/2} \langle |g''|^2 \rangle^{1/2}}{z} |\mathbf{x}|\,|\mathbf{y}|.\end{aligned}$$

To prove the inequality for the matrix $\mathcal{C}$ let us take $\mathbf{y} \in \mathbf{R}^p$ and apply the inequality (1.17) to the function $f = \sum_\mu g'(S_\mu) y_\mu$. Then

$$\begin{aligned}\sum_{\mu,\nu} \mathcal{C}_{\mu,\nu} y_\mu y_\nu &= \langle (f - \langle f \rangle)^2 \rangle \\ &\leq \frac{1}{Nz} \sum_{i,\mu,\nu} \xi_i^{(\mu)} \xi_i^{(\nu)} \langle g''(S_\mu) g''(S_\nu) \rangle y_\mu y_\nu \\ &\leq \frac{1}{z} ||\tilde{\mathcal{A}}_*| \langle |g''(S_1)|^2 \rangle |\mathbf{y}|^2,\end{aligned}$$

where the matrix $\tilde{\mathcal{A}}_*$ is defined by (2.2). Relations (2.9) follow.

Proof of Proposition 3

From (1.18) one can easily derive that

$$(g'(S))^2 \leq C_1 - C_2 g(S).$$

Thus it is enough to prove that

$$\Big\langle -\frac{1}{N} \sum g(S_\mu) \Big\rangle \leq C.$$

Define the Hamiltonian

$$-\tilde{\mathcal{H}}(\tau) = \tau \sum g(S_\mu) + z(\boldsymbol{J}, \boldsymbol{J}).$$

Let $\langle ... \rangle_\tau$ be a corresponding Gibbs average. One can see easily that

$$\varphi(\tau) \equiv -\Big\langle \frac{1}{N} \sum g(S_\mu) \Big\rangle.$$

is a decreasing function of τ. Thus

$$\begin{aligned}\Big\langle -\frac{1}{N} \sum g(S_\mu) \Big\rangle &= \varphi(1) \leq \varphi(0) = \Big\langle -\frac{1}{N} \sum g(S_\mu) \Big\rangle_0 \\ &\leq \frac{C}{N} \langle \sum \langle S_\mu^2 \rangle_0 \leq \frac{C}{zN^2} \sum (\boldsymbol{\xi}^{(\mu)}, \boldsymbol{\xi}^{(\mu)}).\end{aligned}$$

Proof of Lemma 2

We prove first that

$$\begin{aligned}|\nu_t(\tilde{R}_{1,2}) - \nu_0(\tilde{R}_{1,2})| \leq \frac{C}{\sqrt{N}}, \quad |\nu_t(\tilde{U}_1) - \nu_0(\tilde{U}_1)| \leq \frac{C}{\sqrt{N}}, \\ |\varphi_t(R_{1,2}) - \varphi_0(R_{1,2})| \leq \frac{C}{\sqrt{N}}, \quad |\varphi_t(R_{1,1}) - \varphi_0(R_{1,1})| \leq \frac{C}{\sqrt{N}}.\end{aligned} \tag{3.1}$$

To this end consider the Hamiltonian $H_N(t)$ which has the form (2.17) with d, U substituted by

$$d_N = \nu_0(g'(S_\mu^{(1)}) g'(S_\mu^{(2)})), \quad U_N = \nu_0(g''(S_\mu) + g'^2(S_\mu))$$

respectively. Then we use formula (2.21) for $f = \sqrt{N}\tilde{R}_{1,2}$ and $f = \sqrt{N}\tilde{U}_1$, but we write it in the form:

$$\begin{aligned} \nu_t'(f) &= \frac{1}{2}\sum_{l=1}^{n}\nu_t((f-\langle f\rangle_t)s_l^2(U_l^- - U_N)) + \sum_{l<l'}^{n}\nu_t((f-\langle f\rangle_t)s_l s_{l'}(\tilde{R}_{l,l'}^- - d_N)) \\ &\quad -n\sum_{l=1}^{n}\nu_t((f-\langle f\rangle_t)s_l s_{n+1}(\tilde{R}_{l,n+1}^- - d_N)) \\ &\quad +\frac{n(n+1)}{2}\nu_t((f-\langle f\rangle_t)s_{n+1}s_{n+2}(\tilde{R}_{n+1,n+2}^- - d_N)). \end{aligned} \tag{3.2}$$

Using the Schwartz inequality and (1.17), due to the terms $\nu_t((f-\langle f\rangle_t)^2)$ we obtain the first line of (3.1). But then, on the basis of (3.1) one can derive from (3.2) that the first line of (3.1) is valid even if we replace $CN^{-1/2}$ by CN^{-1}. Now similarly to (2.16) one can conclude that for any $r > 2$

$$E\{\langle|\tilde{R}_{l,l'} - d_N|^r\} \le \frac{C}{N}, \quad E\{\langle|\tilde{U}_l - U_N|^r\rangle\} \le \frac{C}{N}. \tag{3.3}$$

Similarly, using (2.30) for $f = \sqrt{N}R_{1,2}$ and $f = \sqrt{N}R_{1,1}$ with $q_N = \phi_0(s_1 s_2)$, $R_N = \phi_0(s_1^2)$, we prove first the second line of (3.1). Then, by the same way as above, we get

$$E\{\langle|R_{l,l'} - q_N|^2\rangle\} \le \frac{C}{N}, \quad E\{\langle|R_{l,l} - R_N|^2\rangle\} \le \frac{C}{N}. \tag{3.4}$$

Now we remark that since it was proved in [S-T2], that the system (1.11) has a unique solution, to prove (2.11) it is enough to show that our q_N, R_N satisfy this system with the error terms $O(N^{-1+\epsilon})$ and d_N, U_N satisfy relations (2.10) with the same error. Now, on the basis of (3.3) and (3.4) it can be shown easily by formulas (2.21), (2.23) with $f(s) = s_1 s_2$ and $f(s) = s_1^2$ and by formulas (2.30) and (2.33) with $f = \mathcal{D}_1\mathcal{D}_2$ and $f = \mathcal{D}_1^2$.

Proof of Proposition 4
Let us denote

$$\begin{aligned} &\phi(s) = \log\int d\boldsymbol{J}^- \exp\{-H_t(\boldsymbol{J}^-, s)\} = \phi_t(s) + us\sqrt{d(1-t)} - \frac{s^2}{2}(z-(1-t)(U-d)) \\ &\Rightarrow \langle s^n\rangle_t = \langle s^n\rangle_\phi = \frac{\int s^n e^{\phi(s)}ds}{\int e^{\phi(s)}ds}. \end{aligned} \tag{3.5}$$

According to the results [1], $\phi_t(s)$ is a concave function. Besides, there exists $\delta > 0$ such that

$$(z-(1-t)(U-d)) \ge (z-(U-d)) = (R-q)^{-1} > \delta.$$

Then, according to the results [1],

$$\langle|s - \langle s\rangle_\phi|^n\rangle_\phi \le \delta^{-n}2^n\Gamma(n). \tag{3.6}$$

So, to prove Proposition 4 it is enough to estimate $\langle s\rangle_\phi$.

Denote s^* the point of maximum of the function $\phi(s)$. Then it follows from representation (3.5) of the function $\phi(s)$ that

$$|s^*| \le \delta^{-1}|\varphi_t'(0) + u\sqrt{d(1-t)}|.$$

On the other hand, by [1]

$$\langle|s - s^*|\rangle_\phi \le \delta^{-1} \quad \Rightarrow \quad |\langle s\rangle_\phi| \le \delta^{-1}(1 + |\phi_t'(0) + \sqrt{d(1-t)}u|)$$

Now, since

$$\phi_t'(0) = \sum_\mu \frac{\xi_1^{(\mu)}}{\sqrt{N}}\langle g'(S_\mu^-)\rangle_0$$

and $\langle ... \rangle_0$ does not depend on $\xi_1^{(\mu)}$, we have

$$E\{\langle s\rangle^n\} \le 2^n \delta^{-n}\Gamma(n)\Big(d^n + E\Big\{\Big(N^{-1}\sum_{\mu}\langle g'(S_\mu^-)\rangle_0^2\Big)^{n/2}\Big\}\Big).$$

Then using Proposition 3, we obtain the statement of Proposition 4.

Proof of Proposition 5

According to the representation (2.29), $\mathcal{P}_t$ is some polynomial of the derivatives $\frac{\partial^k}{\partial S^k}\log G_t(S_1^{(j)}, u)$ $(k = 1, .., 6,\ j = 1, ..., n)$. But under condition (1.18) for $k \ge 2$ these derivatives are uniformly bounded functions. So we need only to prove that

$$E\left\{\left\langle\left(\frac{\partial}{\partial S}\log G_t(S_1^{(j)}, u)\right)^{2k}\right\rangle_{(t)}\right\} \le C(k). \tag{3.7}$$

Similarly to the proof of Proposition 4 by (1.17) and (1.18) inequality (3.7) can be derived from the inequality

$$E\left\{\left\langle\left(\frac{\partial}{\partial S}\log G_t(S_1^{(j)}, u)\right)^{2}\right\rangle_{(t)}^{k}\right\} \le C(k). \tag{3.8}$$

But similarly to the proof of Proposition 3, since

$$\left(\frac{\partial}{\partial S}\log G_t(S_1^{(j)}, u)\right)^2 \le C_1 - C_2\log G_t(S_1^{(j)}, u),$$

one can get that

$$\left\langle\left(\frac{\partial}{\partial S}\log G_t(S_1^{(j)}, u)\right)^2\right\rangle_{(t)} \le C_1 - C_2\Big\langle \log G_t(S_1^{(j)}, u)\Big\rangle_{(0)}.$$

Now, since $\langle\ldots\rangle_{(0)}$ does not depend on $\{\xi_1^{(\mu)}\}_{\mu=1}^p$, (3.8) follows immediately.

References

[1] Brascamp H.J., Lieb E.H. On the Extension of the Brunn-Minkowsky and Pekoda-Leindler Theorems, Includings Inequalities for Log Concave functions, and with an Application to the Diffusion Equation. J.Func.Anall. **22**,366-389 (1976)

[D-G] B.Derrida, E.Gardner Optimal Stage Properties of Neural Network Models. J.Phys.A: Math.Gen. **21**, 271-284 (1988)

[G] E.Gardner: The Space of Interactions in Neural Network Models. J.Phys.A: Math.Gen. **21**, 257-270 (1988)

[Gu-T] F.Guerra and F.L.Toninelli: Central Limit Theorem for Fluctuations in the High Temperature Region of the Sherrington- Kirkpatrick Spin Glass Model.

[M-P-V] Mezard, M., Parisi, G., Virasoro, M.A.: Spin Glass Theory and Beyond. Singapore: World Scientific, 1987

[S-T1] M. Shcherbina, B. Tirozzi: The Free Energy of a Class of Hopfield Models. J. of Stat. Phys., **72** 1/2, 113-125, 1993

[S-T2] M. Shcherbina, B. Tirozzi: Rigorous Solution of the Gardner Problem. Commun.Math.Phys., (2003)

[S-T3] M.Shcherbina, B.Tirozzi: On the Volume of the Intersection of a Sphere with Random Half Spaces. CRAS Ser.I **334** p.803-806, (2002).

[T2] Talagrand M.: Intersecting Random Half-Spaces: Toward the Gardner-Derrida Problem. Ann.Probab.,**28**, 725-758 (2000)

[T4] Talagrand M.: A New Look at Independence. Ann.Probab. **24**, 1 (1996)

[T5] Talagrand M. Spin glasses: a challenge for mathematicians. Mean field models and cavity method. Springer-Verlag, 2002.