C. R. Acad. Sci. Paris, t. 332, Série I, p. 1–??, 2001 - PXMA.TEX -
Physique mathematique/*Mathematical Physics*
(Geometrie/*Geometry*)

On the Volume of the Intersection of a Sphere with Random Half Spaces

Maria Shcherbina [a], **Brunello Tirozzi** [b]

[a] **Institute for Low Temperatures, Ukr. Ak. Sci., 47 Lenin Av.,Kharkov, Ukraine**
E-mail:shcherbi@ilt.kharkov.ua

[b] **Department of Physics, University "La Sapienza", P.A. Moro 2, 00185 Rome, Italy**
E-mail: tirozzi@krishna.phys.uniroma1.it

Abstract. We find an asymptotic expression of the volume of the intersection of the N dimensional sphere with $p = \alpha N$ random half spaces. This expression coincides with the one found by E. Gardner ([3]) using replica calculations. We get also the same value for α_c. Our proof is rigorous and based on the cavity method. The required decay of correlations is obtained by means of a geometrical argument which holds for general hamiltonians. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur le Volume de l'Intersection d'une Boule avec des Demi Espaces Aléatoires

Résumé. *Nous trouvons une expression asymptotique du volume de l'intersection de une boule á N dimensions avec $p = \alpha N$ demi espaces aléatoires. Cette expression est la mème trouvée par E. Gardner ([3]) en utilisant le calcul de replicas. Nous trouvons aussí la mème valeur de α_c. Notre démonstration est rigoureuse et basée sur la methode de la cavité. La nécessaire décroissance des corrélations est obtenue en utilisant un argument géométrique qui est vrai pour des hamiltoniens générales.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

A very interesting problem was solved by E. Gardner by means of non rigorous replica calculations. She found the volume of the interaction couplings J_{ij} for which p independent patterns of N independent ± 1 bits can be retrieved by the neural dynamics in the limit $N, p \to \infty$, $p/N \to \alpha$. This problem is equivalent to study the volume of the intersection of a N dimensional sphere with p half spaces delimited by planes with random independent ± 1 coeffi cients. She found also the critical value $\alpha_c(k) \equiv (\frac{1}{\sqrt{2\pi}} \int_{-k}^{\infty} (u + k)^2 e^{-u^2/2} du)^{-1}$ where k is a stabilization parameter . In this paper we give a rigorous proof of these results. We solve the problem in three steps shown in the Theorems 2.1, 2.2 and 2.3. Since we apply

Note présentée par Michel TALAGRAND

S0764-4442(00)0-/FLA
© 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. Tous droits réservés.

the cavity method for proving this result we need to show the factorization of the correlation functions. We get this property from a general geometrical statement proved in Theorem 2.1. We consider a general convex hamiltonian and the partition function generated by it. We substitute the integration with respect to "spins" (state variables) by the integration over the energy U (the value of our Hamiltonian). Then the Gibbs average of any linear combination of "spins" can be obtained as a two-dimensional integral of the energy U and the "partial entropy", which is given by the logarithm of the volume of the intersection of the level surfaces with the family of parallel hyperplanes generated by our linear combination. Since the level surfaces of the convex functions are convex, to study such intersection we apply a theorem of classical geometry known since the nineteenth century as the Brunn-Minkowski theorem ([2]). From this theorem we obtain that the "partial entropy" is a concave function. Thus, we can apply the Laplace method to evaluate the Gibbs averages and so we obtain the factorization of correlation functions. We notice that a similar idea was used in [1].

In Theorem (2.1) we show the derivation of self-consistent equations for the order parameters of our model. In order to do this we use a common trick: substitute the θ-functions appearing in the expression of the partition function

$$\Theta_{N,p}(k) = \sigma_N^{-1} \int_{(\boldsymbol{J},\boldsymbol{J})=N} d\boldsymbol{J} \prod_{\mu=1}^{p} \theta(N^{-1/2}(\boldsymbol{\xi}^{(\mu)}, \boldsymbol{J}) - k) \tag{1}$$

by some smooth functions which depend on the small parameter ε and tend, as $\varepsilon \to 0$, to the θ-functions. We choose for this purpose $\mathrm{H}(x\varepsilon^{-1/2})$, where H is the erf-function. The proof of Theorem 2.2 is based on the the application to the Gardner problem of the so-called cavity method, the rigorous version of which was proposed in [6] and developed in [7], [8],[9]. But in the previous papers ([6],[7], [8]) we assumed the self-averaging of the order parameters which allowed us to show the factorisation of the correlation functions. Here we derived the factorisation from the geometrical statement of Brunn-Minkowski theorem which holds for any values of α and k and so we got the rigorous proof of all the result of Gardner. As far as we know this is one of the first problems of spin glass theory completely solved (i.e. for all values of α and k) in a rigorous way. A possible explanation is that in the Gardner problem the so-called replica symmetry solution is true for all α and k, while e.g. in the Hopfield and Sherrington-Kirkpatrick models the replica symmetry solution is valid only for small enough α or for high temperatures (see [5] for the physical theory and [9], [10], [12], [13] for the respective rigorous results). Also only the case of small enough α was studied rigorously for the Gardner-Derrida [4] model (see [14]). In Theorem (2.3) we make our last step studying the limiting transition $\varepsilon \to 0$. We prove that the product of αN θ-functions in (1) can be replaced by the product of $\mathrm{H}(\frac{x}{\sqrt{\varepsilon}})$ with an error going to zero when $\varepsilon \to 0$. This part is the most difficult from the technical point of view.

2. Main Results

As it was mentioned above, we start from the general statement, which allows us to prove the factorisation of all correlation functions for a large class of models.

Let $\{\Phi_N(\boldsymbol{J})\}_{N=1}^{\infty}$ ($\boldsymbol{J} \in \mathbf{R}^N$) be a system of convex functions which have third derivatives bounded in any compact. Consider also a system of convex domains $\{\Gamma_N\}_{N=1}^{\infty}$ ($\Gamma_N \subset \mathbf{R}^N$) whose boundaries consist of a finite number (may be depending on N) of smooth pieces. We remark here that for the Gardner problem we need to study Γ_N which is the intersection of αN half-spaces but in Theorem 2.1 we consider a more general sequence of convex sets. Define the Gibbs measure and the free-energy $\langle . \rangle_{\Phi_N} \equiv \Sigma_N^{-1} \int_{\Gamma_N} d\boldsymbol{J}(.) \exp\{-\Phi_N(\boldsymbol{J})\}$, $\Sigma_N(\Phi_N) \equiv \int_{\Gamma_N} d\boldsymbol{J} \exp\{-\Phi_N(\boldsymbol{J})\}$, $f_N(\Phi_N) \equiv \dfrac{1}{N} \log \Sigma_N(\Phi_N)$.

Denote $\tilde{\Omega}_N(U) \equiv \{\boldsymbol{J} : \Phi_N(\boldsymbol{J}) \leqslant NU\}$, $\Omega_N(U) \equiv \tilde{\Omega}_N(U) \cap \Gamma_N$, $\mathfrak{D}_N(U) \equiv \tilde{\mathfrak{D}}_N(U) \cap \Gamma_N$, where $\tilde{\mathfrak{D}}_N(U)$ is the boundary of $\tilde{\Omega}_N(U)$. Then define $f_N^*(U) = \frac{1}{N} \log \int_{J \in \mathfrak{D}_N(U)} d\boldsymbol{J} e^{-NU}$.

THEOREM 2.1. – *Let the functions* $\Phi_N(\boldsymbol{J})$ *satisfy the conditions:* $\frac{d^2}{dt^2}\Phi_N(\boldsymbol{J}+t\mathbf{e})|_{t=0} \geqslant C_0 > 0$ *for any* $\mathbf{e} \in \mathbf{R}^N$, *(*$|\mathbf{e}| = 1$*) and uniformly in any set* $|\boldsymbol{J}| \leqslant N^{1/2}R_1$, $\Phi_N(\boldsymbol{J}) \geqslant C_1(\boldsymbol{J},\boldsymbol{J})$ *as* $(\boldsymbol{J},\boldsymbol{J}) > NR^2$ *and for any* $U > U_{min} \equiv \min_{J\in\Gamma_N} N^{-1}\Phi_N(\boldsymbol{J}) \equiv N^{-1}\Phi_N(\boldsymbol{J}^*)$ $|\nabla\Phi_N(\boldsymbol{J})| \leqslant N^{1/2}C_2(U)$ *as* $\boldsymbol{J} \in \tilde{\Omega}_N(U)$ *with some positive* N*-independent* $C_0, C_1, C_2(U)$ *and* $C_2(U)$ *continuous in* U.

Assume also, that there exists some finite N*-independent* C_3 *such that* $f_N(\Phi_N) \geqslant -C_3$. *Then*

$$|f_N(\Phi_N) - f_N^*(U_*)| \leqslant O(\frac{\log N}{N}), \quad \left(U_* \equiv \frac{1}{N}\langle\Phi_N\rangle_{\Phi_N}\right). \tag{2}$$

Moreover, for any $\mathbf{e} \in \mathbf{R}^N$ $(|\mathbf{e}| = 1)$ *and any natural* p

$$\langle(\dot{\boldsymbol{J}},\mathbf{e})^p\rangle_{\Phi_N} \leqslant C(p) \quad (\dot{J}_i \equiv J_i - \langle J_i\rangle_{\Phi_N}) \tag{3}$$

with some positive N*-independent* $C(p)$.

Theorem 2.1 has two rather important consequences.

In fact under the conditions of Theorem 2.1 for any $U > U_{min}$ and uniformly in N

$$f_N^*(U) = \min_{z>0}\{f_N(z\Phi_N) + zU\} + O(\frac{\log N}{N}); \quad \frac{1}{N^2}\sum\langle\dot{J}_i\dot{J}_j\rangle_{\Phi_N}^2 \leqslant \frac{C}{N}. \tag{4}$$

We notice also that the the second relation can be obtained from Brascamb-Lieb inequalities ([1]). To found the free energy of the model (1) and to derive the replica symmetry equations for the order parameters we introduce the "regularised" Hamiltonian, depending on the small parameter $\varepsilon > 0$

$$\mathcal{H}_{N,p}(\boldsymbol{J},k,h,z,\varepsilon) \equiv -\sum_{\mu=1}^{p}\log\mathrm{H}\left(\frac{k-(\boldsymbol{\xi}^{(\mu)},\boldsymbol{J})N^{-1/2}}{\sqrt{\varepsilon}}\right) + h(\boldsymbol{h},\boldsymbol{J}) + \frac{z}{2}(\boldsymbol{J},\boldsymbol{J}), \tag{5}$$

where the function $\mathrm{H}(x)$ is defined as $\mathrm{H}(x) \equiv \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt$ and $\boldsymbol{h} = (h_1,...,h_N)$ is an external random field with independent Gaussian h_i with zero mean and variance 1, which we need from the technical reasons.

The partition function and the free energy for this Hamiltonian are

$$Z_{N,p}(k,h,z,\varepsilon) = \sigma_N^{-1}\int d\boldsymbol{J}\exp\{-\mathcal{H}_{N,p}(\boldsymbol{J},k,h,z,\varepsilon)\}, \quad f_{N,p}(k,h,z,\varepsilon) \equiv \frac{1}{N}\log Z_{N,p}(k,h,z,\varepsilon).$$

We denote also by $\langle\ldots\rangle$ the corresponding Gibbs averaging.

THEOREM 2.2. – *For any* $\alpha, k \geqslant 0$ *and* $z > 0$ *the functions* $f_{N,p}(k,h,z,\varepsilon)$ *are self-averaging in the limit* $N,p \to \infty$, $\alpha_N \equiv \frac{p}{N} \to \alpha$*:* $E\left\{(f_{N,p}(k,h,z,\varepsilon) - E\{f_{N,p}(k,h,z,\varepsilon)\})^2\right\} \to 0$ *and, if* ε *is small enough,* $\alpha < 2$ *and* $z \leqslant \varepsilon^{-1/3}$*, then there exists*

$$\begin{aligned}\lim_{N,p\to\infty,\alpha_N\to\alpha} E\{f_{N,p}(k,h,z,\varepsilon)\} &= \max_{R>0}\min_{0\leqslant q\leqslant R} F(R,q;\alpha,k,h,z,\varepsilon),\\ F(R,q;\alpha,k,h,z,\varepsilon) &\equiv \left[\alpha E\left\{\log\mathrm{H}\left(\frac{u\sqrt{q}+k}{\sqrt{\varepsilon+R-q}}\right)\right\}\right.\\ &\left.+\frac{1}{2}\frac{q}{R-q} + \frac{1}{2}\log(R-q) - \frac{z}{2}R + \frac{h^2}{2}(R-q)\right],\end{aligned} \tag{6}$$

where u *is a Gaussian random variable with zero mean and variance 1.*

Let us note that the bound $\alpha < 2$ is not important for us, because for any $\alpha > \alpha_c(k)$ ($\alpha_c(k) < 2$ for any k) the free energy of the partition function (1) tends to $-\infty$, as $N \to \infty$ (see Theorem 2.3 for the exact statement). The bound $z < \varepsilon^{-1/3}$ also is not a restriction for us. We could need to consider $z > \varepsilon^{-1/3}$ only if, applying (4) to the Hamiltonian (5), we obtain that the point of minimum $z_{min}(\varepsilon)$ in (4) does not satisfy this bound. But it is shown in Theorem 2.3 that, for any $\alpha < \alpha_c(k)$, $z_{min}(\varepsilon) < \overline{z}$ with some finite $\overline{z}$ depending only on k and α.

We start the analysis of $\Theta_{N,p}(k)$, defined in (1), from the following remark.

Remark 1. – *Let us note that $\Theta_{N,p}(k)$ can be zero with nonzero probability (e.g., if for some $\mu \neq \nu$ $\boldsymbol{\xi}^{(\mu)} = -\boldsymbol{\xi}^{(\nu)}$). Therefore we cannot, as usually, just take $\log \Theta_{N,p}(k)$. To avoid this difficulty, we take some large enough M and replace below the* log- *function by the function $\log_{(MN)}$, defined as $\log_{(MN)} X = \log \max\left\{X,\ \ e^{-MN}\right\}$.*

THEOREM 2.3. – *For any $\alpha \leqslant \alpha_c(k)$ $N^{-1}\log_{(MN)} \Theta_{N,p}(k)$ is self-averaging in the limit $N,p\to\infty$, $p/N \to \alpha$ $E\left\{\left(N^{-1}\log_{(MN)}\Theta_{N,p}(k) - E\{N^{-1}\log_{(MN)}\Theta_{N,p}(k)\}\right)^2\right\} \to 0$ and for M large enough there exists*

$$\lim_{N,p\to\infty, p/N\to\alpha} E\{N^{-1}\log_{(MN)}\Theta_{N,p}(k)\} = \min_{0\leqslant q<1} F(1,q;\alpha,k,0,0,0) \tag{7}$$

For $\alpha > \alpha_c(k)$ $E\{N^{-1}\log_{(MN)}\Theta_{N,p}(k)\} \to -\infty$, as $N\to\infty$ and then $M\to\infty$.
We would like to mention here that the self-averaging of $N^{-1}\log\Theta_{N,p}(k)$ was proven in ([15]), but our proof of this fact is necessary for the proof of (7).

Acknowledgements. The authors would like to thank Prof. A.D. Milka for a fruitful discussion of the geometrical aspects of the problem.

References

[1] Bovier A., Gayrard V., Hopfield Models as a Generalized Random Mean Field Models, Mathematical Aspects of Spin Glasses and Neuronal Networks, A. Bovier, P. Picco Eds., Progress in Probability, Birkhauser, 41(1998)3-89
[2] Hadwiger H., Vorlesungen uber Inhalt, Oberlache und Isoperimetrie. Springer-Verlag, 1957.
[3] Gardner G., The Space of Interactions in Neural Network Models, J.Phys.A:Math.Gen. 21(1988)271-284.
[4] Gardner E., Derrida B., Optimal Stage Properties of Neural Network Models, J.Phys.A: Math.Gen. 21(1988)257-270.
[5] Mezard, M., Parisi, G., Virasoro, M.A., Spin Glass Theory and Beyond, Singapore, World Scientific, 1987.
[6] Pastur L., Shcherbina M., Absence of Self-Averaging of the Order Parameter in the Sherrington-Kirkpatrick Model. J.Stat.Phys. 62(1991)1-26.
[7] Pastur L., Shcherbina M., Tirozzi B., The Replica-Symmetric Solution Without Replica Trick for the Hopfield Model. J. Stat. Phys. 74 5/6(1994)1161-1183.
[8] Pastur L., Shcherbina M., Tirozzi B., On the Replica Symmetric Equations for the Hopfield Model, J.Math.Phys. 40 8 (1999) 3930-3947.
[9] Shcherbina M., On the Replica Symmetric Solution for the Sherrington-Kirkpatrick Model, Helvetica Physica Acta 70 (1997) 772-797.
[10] Shcherbina M., Some Estimates for the Critical Temperature of The Sherrington-Kirkpatrick Model with Magnetic Field, Mathematical Results in Statistical Mechanics World Scientific, Singapore (1999) 455-474.
[11] Shcherbina M., Tirozzi B., The Free Energy of a Class of Hopfield Models, J. of Stat. Phys. 72 1/2 (1993) 113-125.
[12] Talagrand M., Rigorous Results for the Hopfield Model with Many Patterns. Prob. Theor. Rel. Fields 110 (1998) 176-277.
[13] Talagrand M., Exponential Inequalities and Replica Symmetry Breaking for the Sherrington-Kirkpatrick Model, Ann.Probab. 28 (2000) 1018-1068.
[14] Talagrand M., Intersecting Random Half-Spaces: Toward the Gardner-Derrida Problem, Ann.Probab. 28 (2000) 725-758.
[15] Talagrand M., Self Averaging and the Space of Interactions in Neural Networks, Random Structures and Algorithms 14 (1988) 199-213.