Central limit theorem for fluctuations of linear eigenvalue statistics of large random graphs

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Abstract

We consider the adjacency matrix A of a large random graph and study fluctuations of the function $f_n(z,u)=\frac{1}{n}\sum_{k=1}^n \exp\{-uG_{kk}(z)\}$ with $G(z)=(z-iA)^{-1}$. We prove that the moments of fluctuations normalized by $n^{-1/2}$ in the limit $n\to\infty$ satisfy the Wick relations for the Gaussian random variables. This allows us to prove central limit theorem for $\mathrm{Tr}\,G(z)$ and then extend the result on the linear eigenvalue statistics $\mathrm{Tr}\,\varphi(A)$ of any function $\varphi:\mathbb{R}\to\mathbb{R}$ which increases, together with its first two derivatives, at infinity not faster than an exponential.

1 Introduction

Random graphs appear in different branches of mathematics and physics (see monographs [4, 12] and references there in). It is well known that they are closely connected with the theory of random matrices, since there is one to one map between graphs with n vertices and their adjacency matrices (recall that by the definition the entries a_{ij} of the adjacency matrix are 1 if the vertices i and j are connected and $a_{ij}=0$ otherwise). Commonly, the set of n eigenvalues of the adjacency matrix is referred to as the spectrum of the graph. The limit when the dimension of the matrix n (the number of the vertexes of the graphs) tends to infinity provides a natural approximation for the spectral properties of random graphs.

One of the classes of the prime reference in the theory of random graphs is the *binomial random graph* originating by P. Erdős (see, e.g. [12]). Given a number $p_n\in(0,1)$, this family of graphs $\mathbf{G}(n,p_n)$ is defined by taking the set of all graphs on n vertices as the space of events with probability

$$P(G)=p_n^{e(G)}(1-p_n)^{\binom{n}{2}-e(G)}, \tag{1.1}$$

where $e(G)$ is the number of edges of G. Most of the random graphs studies are devoted to the cases where $p_n\to 0$ as $n\to\infty$.

Ensemble of random symmetric $n\times n$ adjacency matrices A corresponding to (1.1) can be represented as $A=\{a_{ij}\}_{i,j=1}^n$ with $a_{ii}=0$, and i.i.d.

$$a_{ij}=\begin{cases} 1, & \text{with probability } p_n, \\ 0, & \text{with probability } 1-p_n, \end{cases} \tag{1.2}$$

For any measurable function f we denote $\mathbf{E}\{f(A)\}$ the averaging with respect to all random variables $\{a_{ij}\}_{1\le i<j\le n}$ and

$$\mathbf{Var}\{f(A)\} := \mathbf{E}\{|f(A) - \mathbf{E}\{f(A)\}|^2\}. \tag{1.3}$$

The normalized eigenvalue counting measure of A is defined by the formula

$$N_n(\lambda) = n^{-1}\sharp\{j : \lambda_j^{(n)} < \lambda\}.$$

The ensemble of adjacency matrices (1.2) is a particular case of the random matrix theory, where the limiting transition $n \to \infty$ is intensively studied during half of century since the pioneering works by E. Wigner [23]. Spectral properties of random adjacency matrix (1.2) were examined in the limit $n \to \infty$ both in numerical and theoretical physics studies [7, 8, 9, 18, 19, 20]. There are two major asymptotic regimes: $p_n \gg 1/n$ and $p_n = O(1/n)$ and corresponding models can be called *dilute random matrices* and *sparse random matrices*, respectively. The first studies of spectral properties of sparse and dilute random matrices in the physical literature are related with the works [19], [20], [18], where equations for the limiting density of states of sparse random matrices were derived. In papers [18] and [10] a number of important results on the universality of the correlation functions and the Anderson localization transition were obtained. Unfortunately these results were obtained with the non rigorous replica and super symmetry methods.

On mathematical level of rigor the eigenvalue distribution of dilute random matrices was studied in [14]. It was shown that the normalized eigenvalue counting measure of $(np_n)^{-1/2}A$ converges in the limit $np_n \to \infty$ to the distribution of explicit form known as the semicircle, or Wigner law [23]. In the paper [5] the adjacency matrix of random graphs (1.1) with $p_n = pn^{-1}$ was studied. It was shown that for any m there exist non random limiting moments $\lim_{n\to\infty} n^{-1}\mathrm{Tr}\, A_n^m$ and these moments can be found from the system of certain recurrent relations. The results of [5] was generalized to the case of weighted random graphs in [15], where the resolvent of the adjacency matrix was studied and equations for the Stieltjes transform $g(z)$ of the limiting eigenvalue distribution were derived rigorously (note, that the same equation for gaussian weights were obtained in [19], [20], [18] by using the replica and the super symmetry approaches.) It was shown in [15] that to prove the existence of the limit $\lim_{n\to\infty} g_n(z) = g(z)$, where $g_n(z)$ is the Stieltjes transform of the normalized counting function $N_n(\lambda)$

$$g_n(z) = \int \frac{dN_n(\lambda)}{\lambda - z} \tag{1.4}$$

we need to study the behavior of the function

$$f_n(z,u) = \frac{1}{n}\sum_{k=1}^{n} e^{-uG_{kk}(z)}, \tag{1.5}$$

where

$$G_{jk}(z) = (z - iA)^{-1}_{jk}, \tag{1.6}$$

The function $f_n(z,u)$ is defined for any u,z such that $\Re z \neq 0$. In what follows it will be important for us that

$$||G|| \leq |\Re z|^{-1}, \quad \sum_{j=1}^{n} |G_{ij}|^2 = (GG^*)_{ii} \leq ||G||^2 \leq |\Re z|^{-2} \tag{1.7}$$
$$\Re(Ge,e)\Re z \geq 0, \quad \forall e \in \mathbb{R}^n \ \Rightarrow \ |e^{-u(Ge,e)}| \leq 1, \ \text{ if } \ u\,\Re z > 0.$$

Here and everywhere below $||A||$ means the operator norm of the matrix A.

The following theorem (proven in [15]) gives us the limiting properties of $f_n(z,u)$ of (1.5)

Theorem 1 *Consider the adjacency matrix (1.2) with $p_n = p/n$. Then for any u,z such that $u\Re z > 0$ we have:*
(i) the variance of the function $f_n(z,u)$ defined by (1.5) vanishes in the limit $n \to \infty$:

$$\mathbf{Var}\{f_n(z,u)\} \leq C/(\Re z)^2 n, \tag{1.8}$$

(ii) there exists the limit

$$\lim_{n\to\infty} \mathbf{E}\{f_n(z,u)\} = f(z,u), \quad |\mathbf{E}\{f_n(z,u)\} - f(z,u)| \leq Cu^{1/2}/|\Re z|n^{1/2} \tag{1.9}$$

(iii) if we consider a class $\mathcal{H}$ of functions which are analytic in $z: \ \Re z > 0$ and for any fixed $z: \ \Re z > 0$ possessing the norm

$$||f(z)|| = \max_{u>0} \frac{|f(z,u)|}{\sqrt{1+u}}, \tag{1.10}$$

then the limiting function is the unique solution in $\mathcal{H}$ of the functional equation

$$f(z,u) = 1 - u^{1/2}e^{-p} \int_0^\infty dv \frac{\mathcal{J}_1(2\sqrt{uv})}{\sqrt{v}} \exp\{-zv + pf(z,v)\}, \tag{1.11}$$

where $\mathcal{J}_1(\zeta)$ is the Bessel function

$$\mathcal{J}_1(\zeta) = \frac{\zeta}{2} \sum_{k=0}^{\infty} \frac{(-\zeta^2/4)^k}{k!(k+1)!}. \tag{1.12}$$

One can easily see that

$$-\frac{\partial}{\partial u} f_n(z,u)\Big|_{u=0} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{E}\{G_{kk}(z)\} = \frac{1}{n}\mathbf{E}\{\mathrm{Tr}\ G(z)\} = \mathbf{E}\{ig_n(-iz)\},$$

where $g_n(z)$ is the Stieltjes transform (1.4) of the normalized counting measure $N_n(\lambda)$. Hence, Theorem 1 implies that for any $z: \ \Im z \neq 0$

$$\lim_{n\to\infty} \mathbf{E}\{|g_n(z) - \mathbf{E}\{g_n(z)\}|^2\} = 0, \tag{1.13}$$

i.e., the fluctuations of $g_n(z)$ vanish in the limit $n \to \infty$. And (1.9) implies that

$$g(z) = \lim_{n\to\infty} \mathbf{E}\{g_n(z)\} = -\frac{\partial}{\partial u} f(z,u)\Big|_{u=0} \tag{1.14}$$

Since the Stieltjes transform uniquely determines the measure, it follows from Theorem 1 that there exists the weak limit $N(\lambda)$ of the normalized counting measure $N_n(\lambda)$ and the Stieltjes transform $g(-iz)$ can be obtained as the first derivative of the solution of (1.11). Using Theorem 1 it is not difficult to obtain the asymptotic expansions for $g(z)$ with respect to z^{-k}. Since it is well known that the coefficients of this expansion are the moments of the limiting normalized counting measure of eigenvalues, we obtain the recurrent formulas for the moments. Besides, constructing the asymptotic expansion of $g(z)$ with respect to p^k, it is easy to show that this expansion is convergent for $p < 1$. Since in the case $a_{ij} = 0, 1$ the coefficients of this expansion are rational functions of z, we can conclude that the limiting spectrum is pure point and consists of the spectra of finite blocks only.

Results of [15] described above can be viewed as the analogs of the Law of Large Numbers for linear eigenvalue statistics

$$\mathcal{N}_n[\varphi] = \sum_{i=1}^{n} \varphi(\lambda_i) = \mathrm{Tr}\ \varphi(A) \tag{1.15}$$

corresponding to continuous test functions. Indeed, it follows from (1.13) – (1.14) that for any continuous test function there exists

$$\lim_{n\to\infty} n^{-1}\mathcal{N}_n[\varphi] = \int \varphi(\lambda)dN(\lambda),$$

where N is the limiting normalized counting measure of eigenvalues. In the present paper we consider the central limit theorem, the second element of the standard probabilistic analysis of linear statistics. Similar questions for other ensembles of random matrices were studied in [2, 3, 11, 13, 16, 21, 22]. Note, however, that for almost all ensembles studied in the random matrix theory, like the Wigner ensemble, the Marchenko-Pastur ensemble, the matrix models, etc the variance of linear statistics for smooth functions is bounded (see [2, 3, 11, 13, 16, 21, 22]). Thus, for these ensembles, one expects the Central Limit Theorem to be valid for statistics themselves, i.e., without an n-dependent normalization factor in front. This has to be compared with the case of i.i.d. random variables with finite second moment, where the variance of linear statistics is always of the order $O(n)$, $n \to \infty$ and the Central Limit Theorem is valid for linear statistics divided by $n^{1/2}$. As we will see below this is the case also for the ensemble of sparse adjacency matrices (1.2) with $p_n = p/n$.

The aim of the present paper is to study the fluctuations of linear eigenvalue statistics for different classes of test functions. Following the method of [15] we study first the functions $f_n(z, u)$ (defined in (1.5)) and prove that its fluctuations converges in distribution to the complex Gaussian random variables.

Define the m-th generalized moment of the fluctuations of $f_n(z, u)$:

$$\begin{aligned} M_{m,n}(z_1, u_1; \ldots; z_m, u_m) &:= n^{-m/2}\mathbf{E}\left\{\prod_{j=1}^{m}\left(\sum_{k=1}^{n} \overset{\circ}{e}^{-u_j G_{kk}(z_j)}\right)\right\} \\ &= n^{m/2}\mathbf{E}\left\{\prod_{j=1}^{m} \overset{\circ}{f}_n(z_j, u_j)\right\}, \quad \Re z_i \neq 0. \end{aligned} \tag{1.16}$$

Here and below for any random variable ξ we denote

$$\overset{\circ}{\xi} = \xi - \mathbf{E}\{\xi\}$$

Theorem 2 *Consider the adjacency matrix ensemble (1.2) with $p_n = p/n$. Let $M_{m,n}(z_1, u_1; \dots; z_m, u_m)$ $(m = 2, 3, \dots\ )$ of (1.16) be the "moments" of the fluctuations of $f_n(z, u)$ of (1.5). Then for any $m > 2$ and $z_1, \dots, z_m : \Re z_j > 0$ there exists*

$$M_m(z_1, u_1; \dots; z_m, u_m) := \lim_{n\to\infty} M_{m,n}(z_1, u_1; \dots; z_m, u_m). \tag{1.17}$$

Moreover, the following recursion equations hold:

$$\begin{aligned} &M_m(z_1, u_1; \dots; z_m, u_m) \\ &\quad = \sum_{j=2}^{m} M_2(z_1, u_1; z_j, u_j) M_{m-2}(z_2, u_2; \dots; z_{j-1}, u_{j-1}; z_{j+1}, u_{j+1}; \dots; z_m, u_m). \end{aligned} \tag{1.18}$$

Theorem 2 can be used to prove the central limit theorem for fluctuations of the trace of $G(z)$ of (1.6). Indeed, if we denote

$$M^*_{m,n}(z_1, \dots, z_m) := n^{-m/2}\mathbf{E}\left\{\mathrm{Tr}\ \overset{\circ}{G}(z_1)\dots \mathrm{Tr}\ \overset{\circ}{G}(z_m)\right\}, \tag{1.19}$$

then it is easy to see that

$$M^*_{m,n}(z_1, \dots, z_m) = \frac{\partial^m}{\partial u_1 \dots \partial u_m} M_m(z_1, u_1; \dots; z_m, u_m)\Big|_{u_1=\dots=u_m=0}.$$

Since $M_m(z_1, u_1; \dots; z_m, u_m)$ are evidently analytic in each u_i in some neighborhood of $u_i = 0$ and bounded uniformly in n for any fixed $z_1, \dots, z_m$ ($\Re z_i \neq 0$) (see Lemma 1 below), we pass to the limit $n \to \infty$ in the above relations and obtain the following theorem:

Theorem 3 *Let $G(z)$ be the resolvent (1.6) of the sparse adjacency matrix (1.2) with $p_n = p/n$. Then for any $m > 2$ and $z_1, \dots, z_m : \Re z_j > 0$ there exists*

$$M^*_m(z_1, \dots, z_m) := \lim_{n\to\infty} M^*_{m,n}(z_1; \dots; z_m) \tag{1.20}$$

and the following recursions hold:

$$M^*_m(z_1, \dots, z_m) = \sum_{j=2}^{m} M^*_2(z_1, z_j) M^*_{m-2}(z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_m) \tag{1.21}$$

Theorem 3 by a standard way implies the central limit theorem for $v_n(z) = n^{-1/2}\mathrm{Tr}\ \overset{\circ}{G}(z)$. Indeed, if we put in (1.20) – (1.21) $z_1 = z_2 = ... = z_m = z$, then Theorem 3 yields that there exist limits of all moments of the complex random variable $v_n(z)$ and

$$M_{2m}(z) := \lim \mathbf{E}\{(n^{-1/2}\mathrm{Tr}\ \overset{\circ}{G}(z))^{2m}\} = (2m-1)!!(M_2(z))^m$$

This means that $v_n(z)$ converges in distribution to a complex Gaussian random variable with zero mean and variance $M_2(z)$.

It is possible also to derive from Theorem 3 the central limit theorem for the linear eigenvalue statistics of any function φ which grows not faster than an exponent at infinity and possesses two derivatives with the same property, i.e. there exists a constant $c > 0$ such that φ, φ', $\varphi'' \in L^2(\mathbb{R}, \cosh^{-2}(c\lambda))$. Here and below

$$L^2(\mathbb{R}, w(\lambda)) = \left\{ f : \int_{\mathbb{R}} |f(\lambda)|^2 w(\lambda) d\lambda < \infty \right\} \tag{1.22}$$

Theorem 4 *Consider the adjacency matrix (1.2) with $p_n = p/n$ and take any function φ which possesses two derivatives such that φ, φ', $\varphi'' \in L^2(\mathbb{R}, \cosh^{-2}(c\lambda))$ with some constant $c > 0$. Then the random variable $n^{-1/2}\overset{\circ}{\mathcal{N}}_n[\varphi]$ converges in distribution to a Gaussian random variable with zero mean and variance $V[\varphi] := \lim_{n\to\infty} \mathbf{Var}\{n^{-1/2}\mathcal{N}_n[\varphi]\}$.*

It is clear from the above discussion that Theorem 2 plays a key role in the paper, because Theorems 3 and 4 are in fact corollaries of Theorem 2. The proof of Theorem 2 is based on a version of the cavity method which has been used many times for proving different limiting relations of statistical mechanics and random matrices. The idea is to compare the behavior of the object function (e.g. free-energy, resolvent, etc.) for the complete system of random variables of the problem and the one with some subset of random variables replaced by 0.

Let us try to explain the connections among the lemmas and propositions which are necessary for the proof of Theorem 2. The proof should be seen as a logical sequence of the following steps:

- We prove first bounds on $M_{m,n}(z_1, u_1; \ldots; z_m, u_m)$ uniform in $(z_1, u_1; \ldots; z_m, u_m)$ ($\Re z_j \geq C > 0$) (see Lemma 1). One uses the norm estimates of the martingale theory (Proposition 1), identities for the resolvent and the cavity method consisting in studying the difference of the resolvent of the full matrix and the same matrix without the first line and the first column.

- To prove the convergence of the variance of the sums of exponentials we need to generalize Theorem 1 and to show the existence of the limits of exponentials multiplied by some entire functions (cf Lemma 2 and Theorem 1). The proof of Lemma 2 is based on the relations for some functions of A given by Proposition 2.

- Lemma 3 proves the self averaging properties and the existence of the limits for the terms which will appear in the proof of CLT.

- Finally we prove that the "moments" (1.16) as functions of u_i satisfy the linear integral equations with the kernel defined in terms of the function f of (1.11) (see (2.54)). Since we are able to prove that these equations are uniquely solvable for $\Re z > M_0$ with some fixed M_0, we finish the proof of Theorem 2.

2 Proofs

We start from the lemma which gives bounds for $M_{m,n}$.

Lemma 1 *For any $m \in \mathbb{N}$ and $z_1, \dots, z_m : \Re z_j > 0$ there exists a constant C_m such that uniformly in $u_1 \dots, u_m > 0$*

$$|M_{m,n}(z_1, u_1; \dots; z_m, u_m)| \leq C_m \tag{2.1}$$

The proof is based on the martingale property of the sequence of averages of the functions of the random matrix A with respect to its rows or columns. The sequence is ordered with respect to the index of the rows and the proposition below is based on the sequence of the conditional expectations like in the proof of self-averaging of the free-energy for disordered systems.

Proposition 1 *Let ξ_α, $\alpha = 1, ..., \nu$ be independent random variables, assuming values in $\mathbb{R}^{m_\alpha}$ and having probability laws P_α, $\alpha = 1, \dots, \nu$ and let $\Phi : \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_\nu} \to \mathbb{C}$ be a Borelian function. Set*

$$\Phi_\alpha(\xi_1, \dots, \xi_\alpha) = \int \Phi(\xi_1, \dots, \xi_\alpha, \xi_{\alpha+1}, \dots, \xi_\nu) P_{\alpha+1}(d\xi_{\alpha+1}) \dots P_\nu(d\xi_\nu) \tag{2.2}$$

so that $\Phi_\nu = \Phi$, $\quad \Phi_0 = \mathbf{E}\{\Phi\}$, where $\mathbf{E}\{\dots\}$ denotes the expectation with respect to the product measure $P_1 \dots P_\nu$.

Then for any positive $p \geq 1$ there exists C'_p, independent of ν and such that

$$\mathbf{E}\{|\Phi - \mathbf{E}\{\Phi\}|^{2p}\} \leq C'_p \nu^{p-1} \sum_{\alpha=1}^{\nu} \mathbf{E}\{|\Phi_\alpha - \Phi_{\alpha-1}|^{2p}\}. \tag{2.3}$$

Moreover, if for every $\alpha = 1, \dots, \nu$ there exists a ξ_α-independent $\Psi^{(\alpha)} : \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_\nu} \to \mathbb{C}$ such that

$$\mathbf{E}\{|\Phi - \Psi^{(\alpha)}|^{2p}\} \leq C < \infty, \quad \alpha = 1, \dots, \nu, \tag{2.4}$$

then

$$\mathbf{E}\{|\Phi - \mathbf{E}\{\Phi\}|^{2p}\} \leq 2C'_p C \nu^p. \tag{2.5}$$

Proof. The proof of (2.3) is given in [6]. Hence, we show only how to derive (2.4) from (2.3). It follows from (2.2) and (2.4) that the integrals of $\Psi^{(\alpha)}$ with respect to $P_{\alpha+1}...P_\nu$ and $P_\alpha P_{\alpha+1}...P_\nu$ coincide and we obtain

$$\begin{aligned} \mathbf{E}\{|\Phi_\alpha - \Phi_{\alpha-1}|^{2p}\} &\leq 2^{2p-1}(\mathbf{E}\{|(\Phi - \Psi^{(\alpha)})_{\alpha-1}|^{2p}\} + \mathbf{E}\{|(\Phi - \Psi^{(\alpha)})_\alpha|^{2p}\}) \\ &\leq 2^{2p}\mathbf{E}\{|\Phi - \Psi^{(\alpha)}|^{2p}\}. \end{aligned}$$

This and (2.3) prove (2.5). $\square$

Proof of Lemma 1 The Hölder inequality yields

$$|M_{m,n}(z_1, u_1; \dots; z_m, u_m)| \leq n^{m/2} \prod_{j=1}^{m} \mathbf{E}\left\{ \left| \overset{\circ}{f}_n(z_j, u_j) \right|^m \right\}^{1/m}.$$

Hence, it suffices to prove the bound for the r.h.s. of the above inequality. For this we use Proposition 1 for the function $\Phi = nf_n(z,u)$ with $f_n(z,u)$ of (1.5). According to (2.5) and the approach of the cavity method for our purposes it is enough to choose the functions Ψ_i independent from $\xi_i = a^{(i)} := (a_{i1},\dots,a_{ii-1},0,a_{ii+1},\dots,a_{in})$ and prove (2.4). Set

$$\begin{aligned} A^{(i)} &= A\Big|_{a_{ij}=0,j=1,\dots,n}, \quad G^{(i)}(z) = (z - iA^{(i)})^{-1}, &(2.6)\\ \Psi^{(i)} &= nf_n^{(i)}(z,u) := \sum_{k\neq i} e^{-uG_{kk}^{(i)}(z)}. &(2.7)\end{aligned}$$

By the symmetry reason it suffices to prove (2.4) for $i=1$. We use the representations:

$$\begin{aligned} G_{ij}(z) &= G_{ij}^{(1)}(z) - \frac{(G^{(1)}a^{(1)})_i(G^{(1)}a^{(1)})_j}{z+(G^{(1)}a^{(1)},a^{(1)})}, \quad i,j\neq 1,\\ G_{1j}(z) &= \frac{i(G^{(1)}a^{(1)})_j}{z+(G^{(1)}a^{(1)},a^{(1)})}, \quad j\neq 1, \qquad (2.8)\\ G_{11}(z) &= (z+(G^{(1)}a^{(1)},a^{(1)}))^{-1}, \end{aligned}$$

where $a^{(1)} = (0,a_{12},\dots,a_{1n})$. The inequality $|e^x - e^y| \le |x-y|\max\{|e^x|,|e^y|\}$ and (1.7) imply

$$\begin{aligned} \left|\sum_{k=2}\left(e^{-uG_{kk}} - e^{-uG_{kk}^{(1)}}\right)\right| &\le u\sum_{k=2}\left|G_{kk} - G_{kk}^{(1)}\right| \qquad (2.9)\\ &\le u\sum_k \frac{(G^{(1)}a^{(1)})_k\overline{(G^{(1)}a^{(1)})_k}}{|z+(G^{(1)}a^{(1)},a^{(1)})|} = u\frac{(G^{(1)}(z)a^{(1)},G^{(1)}(z)a^{(1)})}{|z+(G^{(1)}a^{(1)},a^{(1)})|}. \end{aligned}$$

But the spectral theorem yields

$$(G^{(1)}(z)a^{(1)},G^{(1)}(z)a^{(1)}) = \sum_{j=1}^n \frac{|(\psi^{(j)},a^{(1)})|^2}{(\lambda^{(j)}-\Im z)^2+(\Re z)^2} = \frac{1}{\Re z}\Re(G^{(1)}a^{(1)},a^{(1)}),$$

where $A^{(1)}\psi^{(j)} = \lambda^{(j)}\psi^{(j)}$. Thus, since by (1.7) $\Re z\,\Re(G^{(1)}a^{(1)},a^{(1)})>0$, we have

$$\frac{(G^{(1)}(z)a^{(1)},G^{(1)}(z)a^{(1)})}{|z+(G^{(1)}a^{(1)},a^{(1)})|} \le (\Re z)^{-1}. \qquad (2.10)$$

Inequality (2.4) for our choice of Φ and $\Psi^{(i)}$ follows from (2.9) and (2.10). $\square$

In the proof of Theorem 2 we will replace sometimes $M_{m,n}$ by the moments independent of $\{a_{1,j}\}_{j=2}^n$. Set

$$M_{m,n}^{(1)}(z_1,u_1;\dots;z_m,u_m) := n^{m/2}\mathbf{E}\left\{\prod_{j=1}^m \overset{\circ}{f}{}_n^{(1)}(z_j,u_j)\right\} \qquad (2.11)$$

with $f_n^{(1)}$ of (2.7). Note that (2.9) yields that for any $m \in \mathbb{N}$ and $z_1, \dots, z_m : \Re z_j > 0$ there exists constants C_m, C_m' such that uniformly in $u_1 \dots, u_m$

$$
\begin{aligned}
&|M_{m,n}(z_1,u_1;\dots;z_m,u_m) - M_{m,n}^{(1)}(z_1,u_1;\dots;z_m,u_m)| \le C_m n^{-1/2}, \\
&|M_{m,n}^{(1)}(z_1,u_1;\dots;z_m,u_m)| \le C_m'. \qquad (2.12)
\end{aligned}
$$

To study the behavior of some functions, depending on $\{a_{1,j}\}_{j=2}^n$, we use the proposition:

Proposition 2 *Let* $\mathbf{E}_1\{\dots\}$ *be the averaging with respect to* $\{a_{1k}\}_{k=2}^n$*. Then we have for any* $u, v > 0$ *and* $\Re z > 2$

$$
\begin{aligned}
e^{-v(G^{(1)}a^{(1)},a^{(1)})} &= e^{-v\sum_k G_{kk}^{(1)} a_{1k}} + r_v, \qquad (2.13)\\
r_v &= v\sum_{i\ne j} G_{ij}^{(1)} a_{1i}a_{1j} + O\biggl(v^2\Bigl|\sum_{i\ne j} G_{ij}^{(1)} a_{1i}a_{1j}\Bigr|^2\biggr), \quad \mathbf{E}_1^{1/2}\{|r_v|^2\} \le Cvn^{-1/2}.
\end{aligned}
$$

Moreover, denoting $Z = z + (G^{(1)}a^{(1)}, a^{(1)})$*, we have*

$$
\begin{aligned}
\mathbf{E}_1\Bigl\{\sum_{j=2}^n (e^{-uG_{jj}} - e^{-uG_{jj}^{(1)}})\Bigr\} &= \mathbf{E}_1\Bigl\{\sum_{j,k=2}^n e^{-uG_{jj}^{(1)}}(e^{u(G_{jk}^{(1)})^2/Z} - 1)a_{1,k}\Bigr\} \qquad (2.14)\\
&\quad + O\Bigl(\frac{(u+u^2)e^u}{n}\Bigr)
\end{aligned}
$$

Proof. Note that since $|\Re z|^{-1} \le 1/2$, everywhere below we will replace $|\Re z|^{-1}$ by a constant. We need below the trivial bounds:

$$
|e^a - e^b| \le |a-b|\max\{|e^a|,|e^b|\}, \quad |e^a - e^b - (a-b)| \le |a-b|^2 \max\{|e^a|,|e^b|\}. \quad (2.15)
$$

The first bound and the second line of (1.7) combined with (2.8) imply

$$
|e^{-v(G^{(1)}a^{(1)},a^{(1)})} - e^{-v\sum_k G_{kk}^{(1)} a_{1k}}| \le v\Bigl|\sum_{k_1\ne k_2} G_{k_1k_2}^{(1)} a_{1k_1}a_{1k_2}\Bigr|
$$

Averaging the square of the bound we obtain

$$
\begin{aligned}
\mathbf{E}_1\{|r_v|^2\} &= v^2\mathbf{E}_1\Bigl\{\sum_{k_1\ne k_2, k_3\ne k_4} G_{k_1k_2}^{(1)}\overline{G_{k_3k_4}^{(1)}} a_{1k_1}a_{1k_2}a_{1k_3}a_{1k_4}\Bigr\} \qquad (2.16)\\
&\le \frac{C_1v^2}{n^2}\sum_{k_1,k_2}|G_{k_1k_2}^{(1)}|^2 + \frac{C_2v^2}{n^3}\sum_{k_1,k_2,k_3} G_{k_1k_2}^{(1)}\overline{G_{k_1k_3}^{(1)}} + \frac{C_3v^2}{n^4}\Bigl|\sum_{k_1,k_2} G_{k_1k_2}^{(1)}\Bigr|^2 \le \frac{C_4v^2}{n}.
\end{aligned}
$$

Here we used the bounds valid for any matrix A:

$$
\Bigl|\sum_{j,k} A_{jk}\Bigr| \le n||A||, \quad \sum_k |A_{jk}| \le n^{1/2}\Bigl(\sum_k |A_{jk}|^2\Bigr)^{1/2} \le n^{1/2}||A||. \qquad (2.17)
$$

To prove (2.14) we show first that

$$\Big|\sum_{j=2}^n \mathbf{E}_1\{(\exp\{-uG_{jj}\} - \exp\{-uG_{jj}^{(1)} + \sum (G_{jk}^{(1)})^2 a_{1k}/Z\}\Big|$$

$$:= \Big|\sum_{j=2}^n \mathbf{E}_1\{e^{a_j} - e^{b_j}\}\Big| \le Cn^{-1/2}e^u. \tag{2.18}$$

The second inequality of (2.15) and the bounds that $|e^{a_j}| \le 1$ and $|e^{b_j}| \le e^{u/|\Re z|^3} \le e^u$ yield

$$\Big|\sum_{j=2}^n \mathbf{E}_1\{e^{a_j} - e^{b_j}\}\Big| \le \Big|\sum_{j=2}^n \mathbf{E}_1\{a_j - b_j\}\Big| + \sum_{j=2}^n \mathbf{E}_1\left\{\big|(e^{a_j} - e^{b_j}) - (a_j - b_j)\big|\right\}$$
$$\le \Big|\sum_{j=2}^n \mathbf{E}_1\{a_j - b_j\}\Big| + e^u \sum_{j=2}^n \mathbf{E}_1\left\{|a_j - b_j|^2\right\}$$

Then, similarly to (2.16) we have

$$\begin{aligned}\Big|\sum_{j=2}^n \mathbf{E}_1\{a_j - b_j\}\Big| &= u\Big|\sum_{j=2}^n \mathbf{E}_1\Big\{\sum_{k_1\ne k_2} G_{jk_1}^{(1)}G_{jk_2}^{(1)}a_{1k_1}a_{1k_2}Z^{-1}\Big\}\Big| \\ &\le u\mathbf{E}_1^{1/2}\Big\{\Big|\sum_{k_1\ne k_2}(G^{(1)}G^{(1)})_{k_1,k_2}a_{1k_1}a_{1k_2}\Big|^2\Big\}\mathbf{E}_1^{1/2}\{Z^{-2}\} \le Cun^{-1/2}.\end{aligned}$$

Here we used also that in view of (1.7) $\Re Z > \Re z \ge 2$. Moreover, similarly to (2.16), we obtain

$$\mathbf{E}_1\left\{|a_j - b_j|^2\right\} \le u^2\mathbf{E}_1\Big\{\Big|\sum_{k_1\ne k_2, k_3\ne k_4} G_{jk_1}^{(1)}G_{jk_2}^{(1)}a_{1k_1}a_{1k_2}\overline{G_{jk_3}^{(1)}G_{jk_4}^{(1)}}a_{1k_3}a_{1k_4}\Big|\Big\} \le Cu^2n^{-2}.$$

Summing with respect to j, we get (2.18). Besides, we have

$$\begin{aligned}&\sum_{j=2}^n \Big(\exp\{-uG_{jj}^{(1)} + \sum (G_{jk}^{(1)})^2 a_{1k}/Z\} - \exp\{-uG_{jj}^{(1)}\}\Big) \\ &= \sum_{j=2}^n e^{-uG_{jj}^{(1)}} \sum_{m=1}^\infty \frac{u^m(\sum_k (G_{jk}^{(1)})^2 a_{1k})^m}{m!Z^m} \\ &= \sum_{j,k=2}^n e^{-uG_{jj}^{(1)}} \sum_{m=1}^\infty \frac{u^m (G_{jk}^{(1)})^{2m} a_{1k}}{m!Z^m} + r_n,\end{aligned}$$

where the remainder term r_n admits the bound

$$\mathbf{E}_1\{|r_n|\} \le \sum_{j=2}^n \sum_{k_1\ne k_2} \mathbf{E}_1\left\{|G_{jk_1}^{(1)}|^2|G_{jk_2}^{(1)}|^2 a_{1k_1}a_{1k_2}\right\}\sum_{m=2}^\infty \frac{u^m(\sum_k |G_{jk}^{(1)}|^2)^{m-2}}{2(m-2)!Z^m} \le \frac{Cu^2e^u}{n}.$$

The averaging here is similar to (2.16). Thus, we have proved (2.14)□

Set (cf (1.12))

$$\widetilde{\mathcal{J}}_1(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^{k+1}}{k!(k+1)!} = -2i\zeta^{1/2}\mathcal{J}_1(2i\zeta^{1/2}) \tag{2.19}$$

Below we will need the following properties of $\widetilde{\mathcal{J}}_1(\zeta)$

$$\sup_{|\zeta|\le r}|\widetilde{\mathcal{J}}_1(\zeta)| \le \widetilde{\mathcal{J}}_1(r), \quad |\widetilde{\mathcal{J}}_1(\zeta)| \le |\zeta|(1+\widetilde{\mathcal{J}}_1(|\zeta|)), \quad \sup_{|\zeta|\le r}|\widetilde{\mathcal{J}}_1''(\zeta)| \le (1+\widetilde{\mathcal{J}}_1(r)) \tag{2.20}$$

The following lemma is the analog of Theorem 1 for the function which will appear in the proof of Theorem 3.

Lemma 2 *For any $u > 0$, $v \in \mathbb{C}$, $\Re z > 2$, and $\mathcal{J}_1$ of (1.12 the random variable*

$$V_{J,n}(z,u,v) = n^{-1}\sum_{j,k=1}^{n} e^{-uG_{kk}}\widetilde{\mathcal{J}}_1(vG_{kj}^2) \tag{2.21}$$

possesses the property:

$$\mathbf{Var}\{V_{J,n}(z,u,v)\} \le n^{-1}q(u,|v|)(1+\widetilde{\mathcal{J}}_1^2(|v|)) \tag{2.22}$$

where $q(u,v)$ is a fixed polynomial. Moreover, there exists

$$V_J(z,u,v) := \lim_{n\to\infty}\mathbf{E}\{V_{J,n}(z,u,v)\}. \tag{2.23}$$

and

$$|r_{J,n}(z,u,v)| := |V_{J,n}(z,u,v) - V_J(z,u,v)| \le Cn^{-1/2}(1+\widetilde{\mathcal{J}}_1(|v|)). \tag{2.24}$$

Proof. According to Proposition 1 to prove (2.22) it is enough to prove that

$$\Delta^{(1)} := \left|\sum_{j,k}\left(e^{-uG_{kk}}\widetilde{\mathcal{J}}_1(vG_{kj}^2) - e^{-uG_{kk}^{(1)}}\widetilde{\mathcal{J}}_1(v(G_{kj}^{(1)})^2)\right)\right| \le q_1(u,v)(1+\widetilde{\mathcal{J}}_1(|v|)) \tag{2.25}$$

with polynomial q_1. Then $q = q_1^2$. In view of the second bound of (2.20), (1.7), (2.8), (2.9), and (2.10) we have

$$\begin{aligned}\Delta^{(1,1)} \quad &:= \quad \left|\sum_{j,k}\left(e^{-uG_{kk}} - e^{-uG_{kk}^{(1)}}\right)\widetilde{\mathcal{J}}_1(vG_{kj}^2)\right| \\ &\quad\le Cu|v|(1+\widetilde{\mathcal{J}}_1(|v|))\sum_{j,k}\left|G_{kk}-G_{kk}^{(1)}\right||G_{kj}^2| \\ &\le \quad Cu|v|\widetilde{\mathcal{J}}_1(|v|)\frac{(G^{(1)}(z)a^{(1)},G^{(1)}(z)a^{(1)})}{|z+(G^{(1)}a^{(1)},a^{(1)})|} \le Cu|v|(1+\widetilde{\mathcal{J}}_1(|v|)).\end{aligned}$$

Moreover, by the third bound of (2.20), we can write

$$\begin{aligned}\Delta^{(1,2)} \quad &:= \quad \left|\sum_{j,k}e^{-uG_{kk}^{(1)}}\left(\widetilde{\mathcal{J}}_1(vG_{kj}^2) - \widetilde{\mathcal{J}}_1(v(G_{kj}^{(1)})^2)\right)\right| \le \left|v\sum_{j,k}e^{-uG_{kk}^{(1)}}\left(G_{kj}^2 - G_{kj}^{(1)2}\right)\right| \\ &\quad +C|v|^2(1+\widetilde{\mathcal{J}}_1(|v|))\sum_{j,k}|G_{kj}-G_{kj}^{(1)}|(|G_{kj}|^3+|G_{kj}^{(1)}|^3)\end{aligned} \tag{2.26}$$

Then denoting Σ_1 the first sum in the r.h.s., we have in view of the first line of (2.8), (1.7), and (2.10),:

$$\begin{aligned} |\Sigma_1| &= \Big| \sum_{j,k} \frac{(G^{(1)}(z)a^{(1)})_k (G^{(1)}(z)a^{(1)})_j}{|z+(G^{(1)}a^{(1)},a^{(1)})|} \Big(G_{jk}(z) + G^{(1)}_{jk}(z) \Big) \Big| \\ &\le (||G(z)|| + ||G^{(1)}(z)||) \frac{(G^{(1)}(z)a^{(1)}, G^{(1)}(z)a^{(1)})}{|z+(G^{(1)}a^{(1)},a^{(1)})|} \le C. \end{aligned}$$

To estimate Σ_2 – the second sum in the r.h.s. of (2.26) we use that for any matrix M if we consider the matrix $M^{(2)} = \{|M|^2_{i,j}\}^n_{i,j=1}$, then

$$||M^{(2)}|| \le \sup_i \Big(\sum_j |M_{ij}|^2 \Big)^{1/2} \sup_j \Big(\sum_i |M_{ij}|^2 \Big)^{1/2} \le ||M||^2. \qquad (2.27)$$

Hence, the matrix with entries $|G_{kj}|^2$ has the norm bounded by $||G||^2 \le |\Re z|^{-2}$. Then (2.27) and (2.10) imply for Σ_2:

$$\begin{aligned} \Sigma_2 &\le |\Re z|^{-1} \sum_{j,k} \frac{|(G^{(1)}(z)a^{(1)})_k| \, |(G^{(1)}(z)a^{(1)})_j|}{|z+(G^{(1)}a^{(1)},a^{(1)})|} \Big(|G_{jk}(z)|^2 + |G^{(1)}_{jk}(z)|^2 \Big) \\ &\le 2|\Re z|^{-3} \frac{(G^{(1)}(z)a^{(1)}, G^{(1)}(z)a^{(1)})}{|z+(G^{(1)}a^{(1)},a^{(1)})|} \le C. \end{aligned}$$

Thus, we have proved (2.25) and so (2.22).

To prove (2.23) – (2.24) it suffices to prove that for any $m \ge 2$ there exists

$$V_m(u,z) = \lim_{n\to\infty} \mathbf{E}\Big\{ n^{-1} \sum_{j,k} e^{-uG_{kk}} G^m_{k,j} \Big\} = \lim_{n\to\infty} \mathbf{E}\Big\{ \sum_j e^{-uG_{11}} G^m_{1,j} \Big\},$$

and

$$\Big| \mathbf{E}\Big\{ \sum_j e^{-uG_{11}} G^m_{1,j} \Big\} - V_m(u,z) \Big| \le Cm(1+u)/n^{1/2} \qquad (2.28)$$

To average with respect to $a^{(1)}$ we use the second and the third line of (2.8) and the formulas:

$$R^{-m} = \int_0^\infty dv \frac{v^{m-1}}{(m-1)!} e^{-Rv}, \qquad (2.29)$$

$$e^{-uR} = 1 - u^{1/2} \int_0^\infty dv \frac{\mathcal{J}_1(2\sqrt{uv})}{\sqrt{v}} \exp\{-R^{-1}v\}, \qquad (2.30)$$

which are valid for any $\Re R > 0$ and $u \in \mathbb{C}$. Then we get

$$\begin{aligned}
T_m(u) \;:=\; & \mathbf{E}\Big\{ \sum_j e^{-uG_{11}} G^m_{1,j} \Big\} = \mathbf{E}\mathbf{E}_1\Big\{ \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1(z+(G^{(1)}a^{(1)},a^{(1)}))} \Big\} \\
& -u^{1/2}\mathbf{E}\mathbf{E}_1\Big\{ \int_0^\infty \int_0^\infty dv_1 dv_2 \frac{v_1^{m-1}\mathcal{J}_1(2\sqrt{uv_2})}{\sqrt{v}_2(m-1)!} e^{-(v_1+v_2)(z+(G^{(1)}a^{(1)},a^{(1)}))} \Big\} \\
& +\mathbf{E}\mathbf{E}_1\Big\{ \sum_{j>1} (G^{(1)}a^{(1)})^m_j \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1(z+(G^{(1)}a^{(1)},a^{(1)}))} \Big\} \qquad (2.31) \\
& -u^{1/2}\mathbf{E}\mathbf{E}_1\Big\{ \sum_{j>1} (G^{(1)}a^{(1)})^m_j \cdot \int_0^\infty \int_0^\infty \frac{v_1^{m-1}\mathcal{J}_1(2\sqrt{uv_2})}{\sqrt{v}_2(m-1)!} \\
& \times e^{-(v_1+v_2)(z+(G^{(1)}a^{(1)},a^{(1)}))} \Big\} dv_1 dv_2 \\
= \; & I_{1,m} - u^{1/2} I_{2,m}(u) + I_{3,m}(u) - u^{1/2} I_{4,m}(u).
\end{aligned}$$

Using (2.13) and averaging with respect to $\{a_{1,i}\}$, we have

$$\begin{aligned}
I_{1,m} &= \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1 z} \mathbf{E}\mathbf{E_1}\Big\{ \exp\{-v_1 \sum_l G^{(1)}_{ll} a_{1,l}\} \Big\} + O(n^{-1}) \\
&= \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1 z} \mathbf{E} \prod_l \Big(1 - \frac{p}{n} + \frac{p}{n} e^{-v_1 G^{(1)}_{ll}}\Big) + O(n^{-1}) \qquad (2.32) \\
&= \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1 z} \mathbf{E}\{\exp\{-p + p f^{(1)}_n(z,v_1)\}\}(1+O(n^{-1})) + O(n^{-1}) \\
&= \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1 z} e^{-p+pf(z,v_1)} + r_{1,m},
\end{aligned}$$

where

$$|r_{1,m}| \le Cmn^{-1/2},$$

and we used first (2.9)–(2.10) to replace $f^{(1)}_n(z,v_1)$ by $f_n(z,v_1)$, and then (1.9) to replace $f_n(z,v_1)$ by $f(z,v_1)$. Similarly

$$\begin{aligned}
I_{2,m} &= \int_0^\infty \int_0^\infty dv_1 dv_2 \frac{v_1^{m-1}\mathcal{J}_1(2\sqrt{uv_2})}{\sqrt{v_2}(m-1)!} e^{-z(v_1+v_2)} e^{-p+pf(z,v_1+v_2)} + r_{2,m}(u) \qquad (2.33) \\
& |r_{2,m}(u)| \le Cn^{-1/2}.
\end{aligned}$$

Moreover, using (2.14) and (2.13), we obtain

$$\begin{aligned}
I_{3,m} &= \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1 z} \mathbf{E}\mathbf{E_1}\Big\{ \sum_{j,k} (G^{(1)}_{jk})^m a_{1k} \exp\{-v_1 \sum_l G^{(1)}_{ll} a_{1,l}\} \Big\} + r'_{3,m} \\
&= \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1 z} \mathbf{E}\Big\{ \frac{p}{n} \sum_{j,k} e^{-v_1 G^{(1)}_{kk}} (G^{(1)}_{jk})^m \exp\{-p + p f^{(1)}_n(z,v_1)\} \Big\} + r'''_{3,m} \\
&= p \int_0^\infty dv_1 \frac{v_1^{m-1}}{(m-1)!} e^{-v_1 z} e^{-p+pf(z,v_1)} T_m(v_1) + r_{3,m}, \qquad (2.34) \\
& |r_{3,m}(u)| \le Cmn^{-1/2}.
\end{aligned}$$

Here we used also the relation

$$\mathbf{E}\left\{\frac{1}{n}\sum_{j,k} e^{-v_1 G^{(1)}_{kk}} (G^{(1)}_{jk})^m\right\} \quad = \quad T_m(v_1) + O\left(\frac{m}{n}\right).$$

which can be proved similarly to (2.25). Repeating the argument used for $I_{3,m}$, we obtain

$$\begin{aligned} I_{4,m} \quad &= \quad \int_0^\infty \int_0^\infty dv_1 dv_2 \frac{v_1^{m-1}\mathcal{J}_1(2\sqrt{uv_2})}{\sqrt{v_2}(m-1)!} e^{-z(v_1+v_2)} e^{-p+pf(z,v_1+v_2)} T_m(v_1+v_2) + r_{3,m}, \\ & |r_{4,m}(u)|^2 \le Cmn^{-1/2}. \end{aligned} \tag{2.35}$$

Collecting the above relations, we get in view of (2.31) the equation

$$\begin{aligned} T_m(u) \quad &= \quad \varphi_m(u) + \widehat{K}_m(T_m)(u) + r_m(u), \\ |r_m(u)| \quad &\le \quad Cm(1+\sqrt{u})n^{-1/2}, \end{aligned}$$

where the function $\varphi_m(u)$ is defined by the r.h.s. of (2.32) and (2.33) and the integral operator $\widehat{K}_m$ is defined by the r.h.s. of (2.34) and (2.35). It is easy to see that for $\Re z > 2$ the operator norm in the Banach space of the functions with the norm (1.10) satisfies the inequality

$$||\widehat{K}_m|| \le q < 1.$$

Hence, we get (2.28). Then summing with respect to m and taking into account the bounds for the remainder terms, we obtain (2.23). $\square$

The next lemma is a technical one. We will use it in the proof of Theorem 2 below.

Lemma 3 *Set*

$$D^{(1)}(z,u) := n(f_n(z,u) - f_n^{(1)}(z,u)) = e^{-uG_{11}} + \sum_{i=2}^{n}\left(e^{-uG_{ii}} - e^{-uG^{(1)}_{ii}}\right). \tag{2.36}$$

Then for $\Re z > 2$ we have

$$\begin{aligned} &\mathbf{Var}\left\{\mathbf{E}_1\left\{e^{-uG_{11}(z)}\right\}\right\} \le n^{-1}, \quad \mathbf{Var}\left\{\mathbf{E}_1\left\{D^{(1)}(z,u)\right\}\right\} \le e^u q_1(u) n^{-1}, \\ &\mathbf{Var}\left\{\mathbf{E}_1\left\{e^{-u_1 G_{11}(z_1)} D^{(1)}(z_2,u_2)\right\}\right\} \le e^{u_2} q_2(u_1,u_2) n^{-1}. \end{aligned} \tag{2.37}$$

with polynomial q_1, q_2. Moreover, if we denote

$$V_n(z_1,u_1;z_2,u_2) = \mathbf{Cov}_1\left\{e^{-u_1 G_{11}(z_1)}, D^{(1)}(z_2,u_2)\right\}, \tag{2.38}$$

where $\mathbf{Cov}_1\{F_1,F_2\} := \mathbf{E}_1\{F_1F_2\} - \mathbf{E}_1\{F_1\}\mathbf{E}_1\{F_2\}$, then there exists

$$V(z_1,u_1;z_2,u_2) = \lim_{n\to\infty} V_n(z_1,u_1;z_2,u_2) \tag{2.39}$$

and for any fixed $z_1, u_1; z_2, u_2$

$$|V(z_1,u_1;z_2,u_2) - V_n(z_1,u_1;z_2,u_2)| \le q_3(u_1,u_2) e^{u_2} n^{-1/2}, \tag{2.40}$$

with polynomial q_3.

Proof of Lemma 3. The first bound of (2.37) can be proved similarly to (2.31) – (2.34). Indeed, according to (2.30) we have

$$T_0(u) \;:=\; \mathbf{E}_1\Big\{e^{-uG_{11}}\Big\} = 1 - u^{1/2}\mathbf{E}_1\Big\{\int_0^\infty dv \frac{\mathcal{J}_1(2\sqrt{uv})}{\sqrt{v}} e^{-v(z+(G^{(1)}a^{(1)},a^{(1)}))}\Big\}.$$

Then, averaging with respect to $\{a_{1,i}\}$ similarly to (2.32), we get

$$T_0(u) = 1 - \sqrt{u}\int_0^\infty dv \frac{\mathcal{J}_1(2\sqrt{uv})}{\sqrt{v}} e^{-vz} e^{-p+pf(z,v)} + r_0, \quad \mathbf{E}\{|r_0|^2\} \le C/n.$$

To prove the second bound of (2.37) we use (2.14), which gives us that $\mathbf{E}_1\{D^{(1)} - e^{-uG_{11}^{(1)}}\}$ coincides with the r.h.s. of (2.14). Then (2.30) for $\widetilde{u} = i(G_{jk}^{(1)})^2 u$ applied to the r.h.s. of (2.14) yields:

$$\begin{aligned}
\mathbf{E}_1\{D^{(1)} - e^{-uG_{11}^{(1)}}\} \;&=\; \frac{ip\sqrt{u}}{n}\int_0^\infty dv \sum_{j,l} e^{-uG_{jj}^{(1)}} G_{jl}^{(1)} \frac{\mathcal{J}_1\left(2i\sqrt{uv}G_{jl}^{(1)}\right)}{\sqrt{v}} e^{-zv-p+pf_n^{(1)}(z,v)} \\
+O(q(u)e^u/n^{1/2}) \;&=\; \frac{p}{2}\int_0^\infty dv\, v^{-1} V_{J,n}(z,u,uv) e^{-zv-p+pf(z,v)} + O(e^{cu}/n^{1/2})
\end{aligned}$$

with $V_{J,n}$ of (2.21). Now the second inequality of (2.37) follows from Lemma 2, if we use (2.30) to integrate the bound for $r_{J,n}(z,u,uv)$ of (2.24) with respect to v. The third bound of (2.37) follows from the first and the second one.

Relations (2.39) – (2.40) can be proved if we repeat the argument (2.31) – (2.35) and then apply Lemma 2.□

Now we are ready to prove Theorem 2.

Proof of Theorem 2 Fix $z_1, \dots, z_m$ such that $\Re z_i \ge 2$, $i = 1, \dots m$. We find first $M_{2,n}$. Using the symmetry of the problem and Lemma 1 it is easy to see that

$$\begin{aligned}
M_{2,n} \;&=\; n\mathbf{E}\left\{\overset{\circ}{e}{}^{-u_1G_{11}(z_1)} \overset{\circ}{f}{}_n^{(1)}(z_2,u_2)\right\} + \mathbf{E}\left\{\overset{\circ}{e}{}^{-u_1G_{11}(z_1)} \overset{\circ}{D}{}^{(1)}(z_2,u_2)\right\} \\
&=\; T_1 + V_n(z_1,u_1;z_2,u_2), \qquad (2.41)
\end{aligned}$$

where $V_n(z_1,u_1;z_2,u_2)$ is defined in Lemma 3. Relations (2.8), (2.30), and (2.13) yield

$$\begin{aligned}
T_1 \;&=\; -nu_1^{1/2}\int_0^\infty dv \frac{\mathcal{J}_1(2\sqrt{u_1v})}{\sqrt{v}} e^{-z_1v}\mathbf{E}\left\{\overset{\circ}{f}{}_n^{(1)}(z_2,u_2) e^{-v(G^{(1)}a^{(1)},a^{(1)})}\right\} \qquad (2.42) \\
&=\; -nu_1^{1/2}\int_0^\infty dv \frac{\mathcal{J}_1(2\sqrt{u_1v})}{\sqrt{v}} e^{-z_1v}\mathbf{E}\left\{\overset{\circ}{f}{}_n^{(1)}(z_2,u_2)\left(\prod_k e^{-vG_{kk}^{(1)}a_{1k}} + r_v\right)\right\}
\end{aligned}$$

with r_v of (2.13)

Since $\overset{\circ}{f}{}_n^{(1)}(z_2,u_2)$ does not depend on $\{a_{1j}\}_{j=2}^n$ we can average with respect to $a^{(1)}$ and similarly to (2.16) obtain

$$|\mathbf{E}_1\{r_v\}| \le C(v+v^2)/n.$$

We used that (2.17) and the first bound of (1.7) for $||G^{(1)}||$. The bound, the Schwarz inequality, and Lemma 1 yield

$$\left|\mathbf{E}\left\{\overset{\circ}{f}{}^{(1)}_n(z_2,u_2)r_v\right\}\right| \le n^{-1}|\Re z|^{-2}\mathbf{E}\left\{|\overset{\circ}{f}{}^{(1)}_n(z_2,u_2)|\right\} \le C(v+v^2)/n^{3/2}.$$

Then, integrating with respect to v (recall that $|\mathcal{J}_1|\le 1$) and averaging $\prod_k e^{-vG^{(1)}_{kk}a_{1k}}$ over $\{a_{1k}\}$, we get similarly to (2.32):

$$\begin{aligned} T_1 &= -nu_1^{1/2}\int_0^\infty dv\frac{\mathcal{J}_1(2\sqrt{u_1v})}{\sqrt{v}}e^{-z_1v-p}\mathbf{E}\left\{\overset{\circ}{f}{}^{(1)}_n(z_2,u_2)\left(\prod_k e^{-vG^{(1)}_{kk}a_{1k}}\right)\right\}+O\left(\frac{u_1^{1/2}}{n^{1/2}}\right)\\ &= -nu_1^{1/2}\int_0^\infty dv\frac{\mathcal{J}_1(2\sqrt{u_1v})}{\sqrt{v}}e^{-z_1v-p}\mathbf{E}\left\{\overset{\circ}{f}{}^{(1)}_n(z_2,u_2)e^{pf^{(1)}_n(z_1,v)}\right\}+O\left(\frac{u_1^{1/2}}{n^{1/2}}\right). \end{aligned}$$

Writing $f^{(1)}_n(z_1,v)=\mathbf{E}\{f^{(1)}_n(z_1,v)\}+\overset{\circ}{f}{}^{(1)}_n(z_1,v)$, we have

$$\begin{aligned} T_1 &= -nu_1^{1/2}\int_0^\infty dv\frac{\mathcal{J}_1(2\sqrt{u_1v})}{\sqrt{v}}e^{-z_1v-p}\mathbf{E}\left\{\overset{\circ}{f}{}^{(1)}_n(z_2,u_2)e^{p\mathbf{E}\{f^{(1)}_n(z_1,v)\}+p\overset{\circ}{f}{}^{(1)}_n(z_1,v)}\right\}+r_n^{(2)}\\ &= n\int_0^\infty dvK_n(u_1,v;z_1)\mathbf{E}\left\{\overset{\circ}{f}{}^{(1)}_n(z_2,u_2)\left(\overset{\circ}{f}{}^{(1)}_n(z_1,v)+O\left((\overset{\circ}{f}{}^{(1)}_n(z_1,v))^2\right)\right)\right\}+r_n^{(2)}\\ &= \int_0^\infty dvK_n(u_1,v;z_1)M_{2,n}(z_1,v;z_2,u_2)+r_n^{(3)}, \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} K_n(u_1,v;z_1) &:= -\ pu_1^{1/2}\frac{\mathcal{J}_1(2\sqrt{u_1v})}{\sqrt{v}}e^{-z_1v-p}e^{\mathbf{E}\{pf^{(1)}_n(v,z)\}}\\ r_n^{(2)} &= O((u_1/n)^{1/2})\\ r_n^{(3)} &= r_n^{(2)}+n\int_0^\infty dvK_n(u_1,v;z_1)\mathbf{E}\left\{\overset{\circ}{f}{}^{(1)}_n(z_2,u_2)O\left((\overset{\circ}{f}{}^{(1)}_n(z_1,v))^2\right)\right\}\\ &\le Cn^{-1/2}u_1^{1/2}. \end{aligned} \tag{2.44}$$

The last bound follows from (2.12).

Thus, we obtain that

$$\begin{aligned} M_{2,n}(z_1,u_1;z_2,u_2) &= \int_0^\infty dvK_n(u_1,v;z_1)M^{(1)}_{2,n}(z_1,v;z_2,u_2)+\\ &\quad V_n(z_1,u_1;z_2,u_2)+r_n^{(3)}(z_1,u_1;z_2,u_2)+O(n^{-1/2}), \end{aligned} \tag{2.45}$$

where $V_n(z_1,u_1;z_2,u_2)$ is defined in (2.38). Besides, using (1.5) and the inequality $|\mathcal{J}_1(x)|\le 1$, we obtain that uniformly in $u,v>0$

$$\lim_{n\to\infty}K_n(u,v;z)=-pu^{1/2}\frac{\mathcal{J}_1(2\sqrt{uv})}{\sqrt{v}}e^{-zv-p}\exp\{-\mathbf{E}\{pf(z,v)\}\}=:K(u,v;z),\quad \Re z>2,$$

and

$$|K_n(u,v;z)-K(u,v;z)|\le Cuv^{-1/2}e^{-|\Re z|v}n^{-1/2}.$$

Using the above bounds to replace K_n by K in (2.45) and (2.40) to replace V_n by V, we can write (2.45) in the form

$$\begin{aligned}
M_{2,n}(z_1,u_1;z_2,u_2) &= \int_0^\infty dvK(u_1,v;z_1)M_{2,n}^{(1)}(z_1,v;z_2,u_2)+ \\
&\quad V(z_1,u_1;z_2,u_2)+r_n^{(4)}(z_1,u_1;z_2,u_2)+O(n^{-1/2}), \quad (2.46)\\
&\quad |r_n^{(4)}(z_1,u_1;z_2,u_2)|\le q(u_1,u_2)e^{u_2}n^{-1/2}
\end{aligned}$$

with polynomial q. The inequality

$$|K(u,v;z)|\le pu^{1/2}v^{-1/2}e^{-\Re zv} \tag{2.47}$$

implies that there exists $M_0>2$ such that for all z with $\Re z>M_0$ the norm of the integral operator K in the Banach space $\mathcal{H}$ (see (1.10)) satisfy the inequality

$$||K||\le\frac{1}{2}. \tag{2.48}$$

and so there exists the inverse operator $(I-K)^{-1}$. But the problem is that the bound for $r_n^{(4)}$ above does not allow us to conclude that $r_n^{(4)}\in\mathcal{H}$ (recall that we fixe u_2 and consider $r_n^{(4)}$ as a function of u_1). This difficulty can be easily overcome if we consider a new function

$$\widetilde{M}_{2,n}(z_1,u_1;z_2,u_2)=M_{2,n}(z_1,u_1;z_2,u_2)-r_n^{(4)}(z_1,u_1;z_2,u_2).$$

Then (2.46) takes the form

$$\begin{aligned}
\widetilde{M}_{2,n}(z_1,u_1;z_2,u_2) &= \int_0^\infty dvK(u_1,v;z_1)\widetilde{M}_{2,n}^{(1)}(z_1,v;z_2,u_2)+ \\
&\quad V(z_1,u_1;z_2,u_2)+K(r_n^{(4)})(z_1,u_1;z_2,u_2)+O(n^{-1/2}), \quad (2.49)
\end{aligned}$$

and (2.47) yields

$$|K(r_n^{(4)})(z_1,u_1;z_2,u_2)|\le C\sqrt{u}n^{-1/2}.$$

Thus we can apply the $(I-K)^{-1}$ to (2.49) and obtain that for any $z:\Re z>M_0$ there exists the limit

$$M_2(z_1,u_1;z_2,u_2):=\int_0^\infty (I-K)^{-1}(u_1,v;z_1)V(z_1,v;z_2,u_2)\,dv. \tag{2.50}$$

But according to Lemma 1 $M_{2,n}(z_1,u_1;z_2,u_2)$ is an analytic function bounded uniformly in each compact in the right half plane of $\mathbb{C}$. Hence, taking any bounded domain U which contains some z: $\Re z>M_0$, for any fixed u_1,u_2 we can choose a subsequence $M_{2,n_k}(z_1,u_1;z_2,u_2)$ which converges uniformly in $z_1\in U$ to some analytic in U function. But since for z: $\Re z>M_0$ for any convergent subsequence there exists a unique limit of $M_{2,n_k}(z_1,u_1;z_2,u_2)$, defined by (2.50), on the basis of the uniqueness theorem we conclude that for any $z\in U$ there exists a limit of $M_{2,n}(z_1,u_1;z_2,u_2)$ and this limit for $\Re z>M_0$ is defined by (2.50). Hence we have proved (1.17) for $m=2$.

For arbitrary m we have instead of (2.41)

$$\begin{aligned}
M_{m,n} &:= n^{m/2+1/2}\mathbf{E}\left\{\overset{\circ}{e}^{-u_1G_{11}(z_1)}\prod_{j=2}^{m}\left(n^{-1}D^{(1)}(z_j,u_j)+\overset{\circ}{f}{}_n^{(1)}(z_j,u_j)\right)\right\}+O(n^{-1/2})\\
&= n^{m/2+1/2}\mathbf{E}\left\{\overset{\circ}{e}^{-u_1G_{11}(z_1)}\prod_{j=2}^{m}\overset{\circ}{f}{}_n^{(1)}(z_j,u_j)\right\}\\
&\quad+\sum_{j=2}^{m}n^{(m-1)/2}\mathbf{E}\left\{\overset{\circ}{e}^{-u_1G_{11}(z_1)}D^{(1)}(z_j,u_j)\prod_{i\neq j}\overset{\circ}{f}{}_n^{(1)}(z_i,u_i)\right\}\\
&\quad+O(n^{-1/2}) =: T_1+\sum_{j=2}^{m}T_{2j}+O(n^{-1/2}) \qquad (2.51)
\end{aligned}$$

Then, similarly to (2.43), we write T_1 from the r.h.s. of (2.51) as

$$\begin{aligned}
T_1 &= \int_0^\infty dvK_n(u_1,v;z_1)M_{m,n}(z_1,v;\dots;z_m,u_m)\\
&\quad+r_n^{(3)}(z_1,u_1;\dots;z_m,u_m)+O(n^{-1/2}q(u_1,\dots,u_m)), \qquad (2.52)
\end{aligned}$$

where $r_n^{(3)}$ admits the bound (2.44).

Since $f_n^{(1)}$ does not depend on $\{a_{1j}\}_{j=2}^n$ we can average with respect to these variables and, using (2.37) write T_{2j} in the form

$$\begin{aligned}
T_{2j} &= \mathbf{E}\Big\{\left(\mathbf{E}_1\{e^{-u_1G_{11}(z_1)}D^{(1)}(z_j,u_j)\}-\mathbf{E}_1\{e^{-u_1G_{11}(z_1)}\}\mathbf{E}_1\{D^{(1)}(z_j,u_j)\}\right)\\
&\quad\cdot\prod_{i=2,i\neq j}^{m}\overset{\circ}{f}{}_n^{(1)}(z_j,u_j)\Big\} = \mathbf{E}\{V_n(z_1,u_1;z_j,u_j)\}\mathbf{E}\Big\{\prod_{i=2,i\neq j}^{m}\overset{\circ}{f}{}_n^{(1)}(z_j,u_j)\Big\}\\
&\quad+\mathbf{E}\Big\{\overset{\circ}{V}_n(z_1,u_1;z_j,u_j)\prod_{i=2,i\neq j}^{m}\overset{\circ}{f}{}_n^{(1)}(z_j,u_j)\Big\} \qquad (2.53)
\end{aligned}$$

Using the Schwartz inequality and Lemmas 1,2, it is easy to obtain that the last term in the r.h.s. of (2.53) is $O(n^{-1/2})$. Hence, (2.51), (2.54) and (2.53) yield

$$\begin{aligned}
M_{m,n}(z_1,u_1;\dots;z_m,u_m) &= \int_0^\infty dvK_n(u_1,v;z_1)M_{m,n}(z_1,v;\dots;z_m,u_m)\\
+\sum_{j=1}^{m}\mathbf{E}\{V_n(z_1,u_1;z_j,u_j)\}&M_{m-2,n}(z_2,v_2;\dots;z_{j-1},u_{j-1};z_{j+1},u_{j+1},\dots;z_m,u_m)\\
&+O(n^{-1/2}q_1(u_1,\dots,u_m)(e^{u_2}+\dots+e^{u_j})), \quad (2.54)
\end{aligned}$$

Then, using once more the argument, which we applied to (2.50), we can prove (1.17) first for $\Re z>2$ and then extend it to the whole right half plane of $\mathbb{C}$. $\square$

Proof of Theorem 4 We prove Theorem 4 in two steps: first for polynomial φ and then extend the statement to any real valued functions φ, satisfying conditions of the theorem. For polynomial φ we replace in Theorem 3 the product of traces of resolvent of A with different z_j (see (1.19)) by the product of traces of $\varphi_1(A), \dots, \varphi_p(A)$ with $\varphi_1, \dots, \varphi_m$ being some fixed polynomials. More precisely, we consider (cf (1.19))

$$M_{p,n}(\varphi_1, \dots, \varphi_m) := n^{-m/2}\mathbf{E}\Big\{\mathrm{Tr}\ \overset{\circ}{\varphi}_1(A) \dots \mathrm{Tr}\ \overset{\circ}{\varphi}_m(A)\Big\} = \mathbf{E}\Big\{\prod_{j=1}^m n^{-1/2}\overset{\circ}{\mathcal{N}}_n[\varphi_j]\Big\}$$

and prove that for any m and any fixed polynomial $\varphi_1, \dots, \varphi_m$ there exists the limit

$$\lim_{n\to\infty} M_{m,n}(\varphi_1, \dots, \varphi_m) = M_m(\varphi_1, \dots, \varphi_m) \tag{2.55}$$

and

$$M_m(\varphi_1, \dots, \varphi_m) = \sum_{j=2}^m M_2(\varphi_1, \varphi_j)M_{m-2}(\varphi_2, \dots, \varphi_{j-1}, \varphi_{j+1}, \dots, \varphi_m). \tag{2.56}$$

Then taking $\varphi_1 = \dots = \varphi_m = P$ we obtain that there exist the limits of all moments of $n^{-1/2}\overset{\circ}{\mathcal{N}}_n[P]$ and these moments are expressed in terms of the second moment by the same way as for the Gaussian random variable.

Recall that Theorem 3 imply that the (2.55) and (2.56) are valid for $\varphi_{z_j}(\lambda) = (i\lambda - z_j)^{-1}$. We will replace φ_{z_j} by the polynomial φ_j in (2.55) – (2.56) step by step, starting from the last one $\varphi_{z_m}(\lambda)$. To this end we prove by induction with respect to the polynomial degree k that if we replace $\varphi_{z_m}(\lambda)$ by a polynomial $P_k(\lambda)$ of degree not exceeding k, then (2.55) – (2.56) are valid.

For $k = 0, 1$ $\overset{\circ}{\mathcal{N}}_n[P_k] = 0$ (recall that $A_{jj} = 0$), so (2.55) – (2.56) are trivial. Let us assume that that we know (2.55) – (2.56) for $\varphi_m(\lambda) = P_l(\lambda)$ with $l \le k-1$ and prove that they are valid for $l = k$. Consider

$$\begin{aligned} \varphi_m(\lambda) &= \varphi(\lambda, z_m, k) = -z_m\lambda^k(i\lambda - z_m)^{-1} \\ &= -(-i)^k z_m \left(z_m^k(i\lambda - z_m)^{-1} + \sum_{l=1}^k C_k^l (i\lambda - z_m)^{l-1} z_m^{k-l} \right). \end{aligned} \tag{2.57}$$

By the above representation and the induction assumption (2.55) and (2.56) are valid for $\varphi_m(\lambda) = \varphi(\lambda, z_m, k)$ with any z_m. Moreover, if we use the inequalities

$$\begin{aligned} \mathbf{E}\Big\{\Big|n^{-1/2}\overset{\circ}{\mathcal{N}}_n[P_k^*]\Big|^m\Big\} &\le C(m,k), \quad P_k^*(\lambda) = \lambda^k, \quad k, m \in \mathbb{N}, \\ \mathbf{E}\Big\{\Big|n^{-1/2}\overset{\circ}{\mathcal{N}}_n[\varphi_k^*]\Big|^m\Big\} &\le C(m,k)/|\Re z|, \quad \varphi_k^*(\lambda) = \lambda^k(i\lambda - z)^{-1}, \end{aligned} \tag{2.58}$$

combined with the Hölder inequality

$$|M_{m,n}(\varphi_1, \dots, \varphi_m)| \le \prod_{j=1}^m \mathbf{E}^{1/m}\left\{|n^{-1/2}\overset{\circ}{\mathcal{N}}_n[\varphi_j]|^m\right\},$$

then, since $P_k^*(\lambda) - \varphi(\lambda; z_m, k) = -i\varphi_{k+1}^*(\lambda)$, we obtain

$$\begin{aligned} &|M_{m,n}(\varphi_1, \dots, P_k^*) - M_{m,n}(\varphi_1, \dots, \varphi(.\,; z_m, k))| \\ &\qquad = |M_{m,n}(\varphi_1, \dots, \varphi_{k+1}^*)| \le \frac{C}{|\Re z_m|^{1/m}}, \end{aligned} \tag{2.59}$$

where C does not depend on n and z_m. We will prove (2.58) later. Now let us use a simple proposition

Proposition 3 *Let the sequence of the functions $\{u_n(\zeta)\}_{n=1}^{\infty}$ converges point-wise to the function $u(z)$, as $n \to \infty$, in the domain $\Re\zeta > C$, and for any fixed n $u_n(\zeta) \to u_n^*$, as $\Re\zeta \to \infty$, so that*

$$|u_n(\zeta) - u_n^*| \le C_0/|\Re\zeta|^{\alpha}, \quad \alpha > 0. \tag{2.60}$$

Then there exist the limits

$$\lim_{n\to\infty} u_n^* = \lim_{\Re\zeta\to\infty} u(\zeta) \tag{2.61}$$

Proof. Take any $\varepsilon > 0$ and ζ_ε such that $C_0/|\Re\zeta_\varepsilon|^{\alpha} \le \varepsilon/4$. Moreover, choose N such that $|u_n(\zeta_\varepsilon) - u(\zeta_\varepsilon)| \le \varepsilon/4$ for any $n \ge N$. Then for any $n, n' > N$

$$|u_n^* - u_{n'}^*| \le |u_n^* - u_n(\zeta_\varepsilon)| + |u_{n'}^* - u_{n'}(\zeta_\varepsilon)| + |u_n(\zeta_\varepsilon) - u(\zeta_\varepsilon)| + |u_{n'}(\zeta_\varepsilon) - u(\zeta_\varepsilon)| \le \varepsilon.$$

Hence, there exists $u^* = \lim_{n\to\infty} u_n^*$. In addition, for any ζ and any $\varepsilon > 0$ one can choose N such that $|u_N(\zeta) - u(\zeta)| \le \varepsilon/2$ and $|u_N^* - u^*| \le \varepsilon/2$. Then

$$|u(\zeta) - u^*| \le |u(\zeta) - u_N(\zeta)| + |u_N(\zeta) - u_N^*| + |u_N^* - u^*| \le \varepsilon + C_0/|\Re\zeta|^{\alpha}.$$

Thus, there exists the second limit in (2.61) and it coincides with u^*.□

Now if for fixed $z_1, \dots, z_{m-1}$ we consider the functions $u_n(z_m) = M_{m,n}(\varphi_1, \dots, \varphi(.\,; z_m, k))$, then (2.57) gives the point-wise convergence of $u_n(z_m)$ and (2.59) coincides with (2.60) of Proposition 3 with $u_n^* = M_{k,n}(\varphi_1, \dots, P_k^*)$. Applying the proposition we obtain that (2.55) – (2.56) are valid if we replace the last function φ_m by any polynomial of degree k.

Repeating the above procedure we replace step by step all $\varphi_1, \dots \varphi_{m-1}$ by polynomials of any fixed degree. As it was mentioned about this implies that for any polynomial P $n^{-1/2}\overset{\circ}{\mathcal{N}}_n[P]$ converges in distribution to a gaussian random variable with zero mean and the variance from (2.62). Hence, by the standard argument we conclude that uniformly in x varying in any compact of $\mathbb{R}$

$$\mathbf{E}\left\{e^{ixn^{-1/2}\overset{\circ}{\mathcal{N}}_n[P]}\right\} = e^{-x^2/2V(P)}, \quad V(P) = \lim_{n\to\infty} \mathbf{Var}\{n^{-1/2}\mathcal{N}_n[P]\}. \tag{2.62}$$

To finish the proof of CLT for polynomials we are left to prove (2.58). It is done in the further proof of Theorem 4.

To extend CLT to a wider class of functions we use

Proposition 4 *Let $\{\xi_l^{(n)}\}_{l=1}^{n}$ be a triangular array of random variables, $\mathcal{N}_n[\varphi] = \sum_{l=1}^{n} \varphi(\xi_l^{(n)})$ be its linear statistics, corresponding to a test function $\varphi : \mathbb{R} \to \mathbb{R}$, and*

$$V_n[\varphi] = \mathbf{Var}\{n^{-1/2}\mathcal{N}_n[\varphi]\}$$

be the variance of $\mathcal{N}_n[\varphi]$. Assume that *(a) there exists a vector space $\mathcal{L}$ endowed with a norm $||...||$ and such that V_n is defined on $\mathcal{L}$ and admits the bound*

$$V_n[\varphi] \le C||\varphi||^2, \ \forall \varphi \in \mathcal{L}, \tag{2.63}$$

where C does not depend on n;

(b) there exists a dense linear manifold $\mathcal{L}_1 \subset \mathcal{L}$ such that the Central Limit Theorem is valid for $\mathcal{N}_n[\varphi]$, $\varphi \in \mathcal{L}_1$, i.e., if

$$Z_n[x\varphi] = \mathbf{E}\left\{ e^{ixn^{-1/2}\overset{\circ}{\mathcal{N}}_n[\varphi]} \right\}$$

is the characteristic function of $n^{-1/2}\overset{\circ}{\mathcal{N}}_n[\varphi]$, then there exists a continuous quadratic functional $V : \mathcal{L}_1 \to \mathbb{R}_+$ such that we have uniformly in x, varying on any compact interval

$$\lim_{n\to\infty} Z_n[x\varphi] = e^{-x^2 V[\varphi]/2}, \ \forall \varphi \in \mathcal{L}_1; \tag{2.64}$$

Then V admits a continuous extension to $\mathcal{L}$ and Central Limit Theorem is valid for all $\mathcal{N}_n[\varphi]$, $\varphi \in \mathcal{L}$.

Proof. Let $\{\varphi_k\}$ be a sequence of elements of $\mathcal{L}_1$ converging to $\varphi \in \mathcal{L}$. We have then in view of the inequality $|e^{ia} - e^{ib}| \le |a-b|$, the linearity of $\overset{\circ}{\mathcal{N}}_n[\varphi]$ in φ, the Schwarz inequality, and (2.63):

$$\begin{aligned} \left| Z_n(x\varphi) - Z_n(x\varphi)|_{\varphi=\varphi_k} \right| &\le |x|\mathbf{E}\left\{ \left| n^{-1/2}\overset{\circ}{\mathcal{N}}_n[\varphi] - n^{-1/2}\overset{\circ}{\mathcal{N}}_n[\varphi_k] \right| \right\} \\ &\le |x|\mathbf{Var}^{1/2}\{n^{-1/2}\mathcal{N}_n[\varphi - \varphi_k]\} \le C|x| \, ||\varphi - \varphi_k||. \end{aligned}$$

Now, passing first to the limit $n \to \infty$ and then $k \to \infty$, we obtain the assertion. □

Let us show now that hypothesis (a) and (b) of Proposition 4 are fulfilled in some vector space. We fix some $c > 0$ and consider the vector space $\mathcal{L}$ of functions φ such that φ, φ', $\varphi'' \in L^2(\mathbb{R}, \cosh^{-2}(c\lambda))$ (see (1.22)). Denote

$$||\varphi||^2 = \int |\varphi''(\lambda)|^2 \cosh^{-2}(c\lambda)d\lambda + \int |\varphi'(\lambda)|^2 \cosh^{-2}(c\lambda)d\lambda + \int |\varphi(\lambda)|^2 \cosh^{-2}(c\lambda)d\lambda$$

It is evident that the space of all polynomials $\mathcal{L}_\infty$ is dense subspace in $\mathcal{L}$ with respect to the norm $||.||$. Moreover, (2.62) proves (b). Hence we are left to check assumption (a) of Proposition 4.

It is easy to see that if $\varphi \in \mathcal{L}$ then $f(\lambda) = \varphi(\lambda)\cosh^{-1}(c\lambda) \in L_2(\mathbb{R})$ and also $f', f'' \in L_2(\mathbb{R})$ and

$$||f||^2_{L_2(\mathbb{R})} + ||f''||^2_{L_2(\mathbb{R})} \le C||\varphi||^2.$$

Hence it is enough to check that

$$\mathbf{Var}\{n^{-1/2}\mathrm{Tr}\ f(A)e^{\pm cA}\} \le C(||f||^2_{L_2(\mathbb{R})} + ||f''||^2_{L_2(\mathbb{R})}). \tag{2.65}$$

According to Proposition 1 (see (2.4))

$$\mathbf{E}\left\{ \left| n^{-1/2}\overset{\circ}{\mathrm{Tr}}\ f(A)e^{\pm cA} \right|^{2m} \right\} \le C_{2m}\mathbf{E}\left\{ \left| \mathrm{Tr}\ (f(A)e^{\pm cA} - f(A^{(1)})e^{\pm cA^{(1)}}) \right|^{2m} \right\}, \tag{2.66}$$

where $A^{(1)}$ is defined in (2.6). Note that to prove (2.65) it suffices to consider $m = 1$, but we need other m to prove (2.58). Write

$$\mathrm{Tr}\ \left(f(A)e^{cA} - f(A^{(1)})e^{cA^{(1)}}\right) = \int d\xi \widehat{f}(\xi)\mathrm{Tr}\ \left(e^{(i\xi+c)A} - e^{(i\xi+c)A^{(1)}}\right),$$

where $\widehat{f}$ is the Fourier transform of f. Then the Duhamel formula yields

$$\begin{aligned}
\left|\mathrm{Tr}\ \left(e^{(i\xi+c)A} - e^{(i\xi+c)A^{(1)}}\right)\right| &= \left|\int_0^1 dt(i\xi+c)\mathrm{Tr}\ \left(e^{t(i\xi+c)A}(A - A^{(1)})e^{(i\xi+c)A^{(1)}(1-t)}\right)\right| \\
&= \left|\int_0^1 dt(i\xi+c)\sum_{j=1}^n (e^{t(i\xi+c)A})_{j1}(e^{(i\xi+c)A^{(1)}(1-t)}a^{(1)})_j\right| \\
&\le (e^{2tcA}e_1, e_1)^{1/2} e^{2cA^{(1)}(1-t)}a^{(1)}, a^{(1)})^{1/2}.
\end{aligned}$$

Here we used that

$$\begin{aligned}
\mathrm{Tr}\ \left(e^{t(i\xi+c)A}(A - A^{(1)})e^{(i\xi+c)A^{(1)}(1-t)}\right) &= \sum_{j,k}(e^{t(i\xi+c)A})_{j1}a_{1k}(e^{(i\xi+c)A^{(1)}(1-t)})_{kj} \\
&+ \sum_{j,k}(e^{t(i\xi+c)A})_{jk}a_{k1}(e^{(i\xi+c)A^{(1)}(1-t)})_{1j}.
\end{aligned}$$

The first some gives $\sum_{j=1}^n (e^{t(i\xi+c)A}e_1)_j(e^{(i\xi+c)A^{(1)}(1-t)}a^{(1)})_j$ where $e_1 = (1, 0, \dots, 0)$ and the vector $a^{(1)}$ is defined in (2.8). The second sum can be estimated similarly, since relations $A^{(1)}_{i1} = A^{(1)}_{1i} = 0$ $(i = 1, \dots, n)$ imply $(e^{(i\xi+c)A^{(1)}(1-t)})_{1i} = 0$ $(i = 2, \dots, n)$. Then the Schwarz inequality yields

$$\begin{aligned}
\mathbf{Var}\{n^{-1/2}\mathrm{Tr}\ f(A)e^{cA}\} &\le C_2\left(\int |\widehat{f}(\xi)|(|\xi| + c)d\xi\right)^2 \mathbf{E}^{1/2}\{(e^{2tcA}e_1, e_1)^2\} \\
&\quad \mathbf{E}^{1/2}\{(e^{2(1-t)cA^{(1)}}a^{(1)}, a^{(1)})^2\}. \qquad (2.67)
\end{aligned}$$

Using the Schwarz inequality once more and then the symmetry of the problem, we obtain

$$\mathbf{E}\{(e^{2tcA}e_1, e_1)^2\} \le \mathbf{E}\{(e^{4tcA}e_1, e_1)\} = \mathbf{E}\{\mathrm{Tr}\ e^{4tcA}\}.$$

Similarly, using the Schwarz inequality and then the independence $A^{(1)}$ of $a^{(1)}$, we can average with respect to $a^{(1)}$ to obtain

$$\begin{aligned}
&\mathbf{E}\{(e^{2(1-t)cA^{(1)}}a^{(1)}, a^{(1)})^2\} \le \mathbf{E}\{(e^{4(1-t)cA^{(1)}}a^{(1)}, a^{(1)})(a^{(1)}, a^{(1)})\} \\
&\le C(p + p^2)\mathbf{E}\{n^{-1}\mathrm{Tr}\ e^{4(1-t)cA^{(1)}}\} + C(p^2 + p^3)\mathbf{E}^{1/2}\{n^{-1}\mathrm{Tr}\ e^{8(1-t)cA^{(1)}}\}.
\end{aligned}$$

Since all entries of A and $A^{(1)}$ and $A - A^{(1)}$ are positive, we have for any t

$$\mathbf{E}\{\mathrm{Tr}\ e^{4(1-t)cA^{(1)}}\} \le \mathbf{E}\{\mathrm{Tr}\ e^{4(1-t)cA}\} \le \mathbf{E}\{\mathrm{Tr}\ e^{4cA}\}.$$

Moreover, according to the result of [5] we have for any m

$$\mathbf{E}\{n^{-1}\mathrm{Tr}\ A^{2m}\} \le C_0^m m! \quad \Rightarrow \quad \mathbf{E}\{n^{-1}\mathrm{Tr}\ e^{cA}\} \le 2e^{C_0c^2/2}.$$

In addition the Schwarz inequality yields

$$\left(\int |\widehat{f}(\xi)|(|\xi|+c)d\xi\right)^4 \leq \int |\widehat{f}(\xi)|^2(|\xi|+c)^4 d\xi \int (|\xi|+c)^{-2}d\xi \leq C(||f||^2_{L^2(\mathbb{R})}+||f''||^2_{L^2(\mathbb{R})}).$$

Summarizing the above inequalities, we obtain (2.65) and hence the assumption (a) of Proposition 4. Then Theorem 4 follows from Proposition 4.

To prove (2.58) we use again (2.66), where for the first line of (2.66) $f(\lambda) = \lambda^k \cosh^{-1}(c\lambda)$ and for the second line $f(\lambda) = \lambda^k \cosh^{-1}(c\lambda)$. Repeating the above argument we obtain (2.66). □

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