Central limit theorem for linear eigenvalue statistics of orthogonally invariant matrix models

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Abstract

We prove central limit theorem for linear eigenvalue statistics of orthogonally invariant ensembles of random matrices with one interval limiting spectrum. We consider ensembles with real analytic potentials and test functions with two bounded derivatives.

1 Introduction and main result

In this paper we consider ensembles of $n \times n$ real symmetric matrices M with the probability distribution

$$P_n(M)dM = Z_{n,\beta}^{-1} \exp\{-\frac{n\beta}{2}\mathrm{Tr}V(M)\}dM, \tag{1.1}$$

where $Z_{n,\beta}$ is the normalization constant, $V : \mathbb{R} \to \mathbb{R}_+$ is a Hölder function satisfying the condition

$$|V(\lambda)| \geq 2(1+\epsilon)\log(1+|\lambda|). \tag{1.2}$$

and dM means the Lebesgue measure on the algebraically independent entries of M. In the case of real symmetric matrices $\beta = 1$. But since it is interesting to compare the results with the case Hermitian matrix models, where $\beta = 2$, we keep the parameter β in (1.1).

Let $\{\lambda_i\}_{i=1}^n$ be eigenvalues of M. Then it is well known (see [9]) that the joint distribution of $\{\lambda_i\}_{i=1}^n$ has the density

$$p_n(\lambda_1, ...\lambda_n) = Q_{n,\beta}^{-1} \exp\{-\frac{n\beta}{2}\sum_{j=1}^n V(\lambda_j)\} \prod_{1\leq j<k\leq n} |\lambda_j - \lambda_k|^\beta, \tag{1.3}$$

where $Q_{n,\beta}$ is the normalizing constant.

The Normalized Counting Measure (NCM) of eigenvalues for any interval $\Delta \subset \mathbb{R}$ is defined as

$$\mathcal{N}_n(\Delta) = \sharp\{\lambda_l \in \Delta\}/n, \tag{1.4}$$

It is known [3, 8] that for any β $\mathcal{N}_n(\Delta)$ converges weakly in probability to a non random measure $N(\Delta)$, and the limiting measure N can be found as a unique minimum of some functional on the set of non negative unit measures. The extremum point equation for this functional gives us in the case of Hölder V'

$$V'(\lambda) = 2\int_\sigma \frac{\rho(\mu)d\mu}{\lambda - \mu}, \quad \lambda \in \sigma, \tag{1.5}$$

where ρ is the density of N and σ is the support of N.

For all $\varphi : \mathbb{R} \to \mathbb{R}$ consider a linear statistics

$$N_n[\varphi] = \varphi(\lambda_1) + \cdots + \varphi(\lambda_n).$$

It follows from the results of [3, 8] that if V is a Hölder function, then

$$\lim_{n\to\infty} n^{-1} N_n[\varphi] = \int \varphi(\lambda) N(d\lambda).$$

Consider the fluctuation of linear eigenvalue statistics

$$\dot{N}_n[\varphi] = N_n[\varphi] - E\{N_n[\varphi]\}. \tag{1.6}$$

For Hermitian matrix models it was proved by Johansson [8] that if V is a real analytic function and the limiting spectrum $\sigma = [-2, 2]$, then for any $\varphi \in C_1[-d-2, 2+d]$ $\dot{N}_n[\varphi]$ converges in distribution, as $n \to \infty$, to a Gaussian random variable. The limiting variance is the limit, as $n \to \infty$, of

$$\begin{aligned}\mathbf{Var}_n[\varphi; V] = E\{\dot{N}_n^2[\varphi]\} = n(n-1) \int d\lambda_1 d\lambda_2 p_{2,\beta}^{(n)}(\lambda_1, d\lambda_2)\varphi(\lambda_1)\varphi(\lambda_2) \\ + n \int d\lambda_1 p_{1,\beta}^{(n)}(\lambda_1)\varphi^2(\lambda_1) - n^2 \Big(\int d\lambda_1 p_{1,\beta}^{(n)}(\lambda_1)\varphi(\lambda_1) \Big)^2.\end{aligned} \tag{1.7}$$

Here and below we denote by $p_{l,\beta}^{(n)}$ the lth marginal density

$$p_{l,\beta}^{(n)}(\lambda_1, ..., \lambda_l) = \int d\lambda_{l+1} \dots d\lambda_n p_n(\lambda_1, \dots, \lambda_n). \tag{1.8}$$

A key role in the proof of CLT and also in the most of studies of Hermitian matrix models belongs to the orthogonal polynomials technics, which allows to write all marginal densities as

$$p_{l,2}^{(n)}(\lambda_1, ..., \lambda_l) = \frac{(n-l)!}{n!} \det\{K_n(\lambda_j, \lambda_k)\}_{j,k=1}^l, \tag{1.9}$$

where

$$K_n(\lambda, \mu; V) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu). \tag{1.10}$$

is a reproducing kernel of the orthonormal system,

$$\psi_l^{(n)}(\lambda) = w_n^{1/2}(\lambda) p_l^{(n)}(\lambda), \;\; l = 0, ..., \tag{1.11}$$

$p_l^{(n)}$, $l = 0, ...$ are orthogonal polynomials on $\mathbb{R}$ associated with the weight $w_n(\lambda) = e^{-nV(\lambda)}$

$$\int p_l^{(n)}(\lambda) p_m^{(n)}(\lambda) w_n(\lambda) d\lambda = \delta_{l,m}.$$

In the Hermitian case it can be proved that

$$\begin{aligned}\frac{d^2}{dt^2} \log E\{e^{t\dot{N}_n[\varphi]}\} = \mathbf{Var}\{N_n[\varphi; V + t\varphi/n]\} \\ = \int d\mu_1 d\mu_2 (\varphi(\mu_1) - \varphi(\mu_2))^2 K_n^2(\mu_1, \mu_2; V + t\varphi/n).\end{aligned} \tag{1.12}$$

Hence, to prove CLT we are faced with the problem to study the last integral or to prove that K_n does not depend on the "small perturbation" $t\varphi/n$ in the limit $n\to\infty$. For unitary matrix models it is true only in the case (see [8]), when the support of N (limiting NCM) consists of one interval. If the limiting support consists of two or more intervals, then the r.h.s. of (1.12) has no limit, as $n\to\infty$ (see [11]).

In the case of real symmetric matrix models the situation is more complicated. According to the result of [18], to study the marginal densities we need to study a matrix kernel of the form

$$\widehat{K}_{n,1}(\lambda,\mu)=\left(\begin{array}{cc} S_n(\lambda,\mu) & S_nd(\lambda,\mu)\\ -IS_n(\lambda,\mu) & S_n(\mu,\lambda)\end{array}\right), \tag{1.13}$$

where

$$S_n(\lambda,\mu)=-\sum_{i,j=0}^{n-1}\psi_i^{(n)}(\lambda)(\mathcal{M}^{(0,n)})^{-1}_{i,j}(n\varepsilon\psi_j^{(n)})(\mu), \tag{1.14}$$

with

$$\mathcal{M}^{(0,n)}=\{M_{j,l}\}_{j,l=0}^{n-1},\quad M_{j,l}=n(\psi_j^{(n)},\epsilon\psi_l^{(n)}). \tag{1.15}$$

Here and below we denote

$$\epsilon(\lambda)=\frac{1}{2}\mathrm{sign}(\lambda);\quad \epsilon f(\lambda)=\int\epsilon(\lambda-\mu)f(\mu)d\mu. \tag{1.16}$$

If we know $\widehat{K}_n(\lambda,\mu)$, then

$$p_{l,1}^{(n)}(\lambda_1,...,\lambda_l)=\frac{(n-l)!}{n!}\frac{\partial^l}{\partial\varphi(\lambda_1)\dots\partial\varphi(\lambda_l)}\det{}^{1/2}\{I+\widehat{K}_n\widehat{\varphi}\},$$

where $\widehat{\varphi}$ is the operator of multiplication by φ and $\widehat{K}_n:L_2[\mathbb{R}]\oplus L_2[\mathbb{R}]\to L_2[\mathbb{R}]\oplus L_2[\mathbb{R}]$ is an integral operator with the matrix kernel $\widehat{K}_n(\lambda,\mu)$.

In particular,

$$\begin{array}{rcl} p_{1,1}^{(n)}(\lambda) &=& \dfrac{1}{2n}\mathrm{Tr}\widehat{K}_n(\lambda,\lambda),\\ p_{2,1}^{(n)}(\lambda,\mu) &=& \dfrac{1}{4n(n-1)}\left[\mathrm{Tr}\widehat{K}_n(\lambda,\lambda)\mathrm{Tr}\widehat{K}_n(\mu,\mu)-2\mathrm{Tr}\widehat{K}_n(\lambda,\mu)\widehat{K}_n(\mu,\lambda))\right].\end{array} \tag{1.17}$$

Below we will use also the following representation of the variance $\mathbf{Var}\{N_n[\varphi_1;V]\}$

Proposition 1

$$\mathbf{Var}\{N_n[\varphi_1];V\}=\frac{1}{4}\int d\mu_1d\mu_2(\varphi_1(\mu_1)-\varphi_1(\mu_2))^2\mathrm{tr}\left(\widehat{K}_n(\mu_1,\mu_2)\widehat{K}_n(\mu_2,\mu_1)\right) \tag{1.18}$$

The structure of the matrix kernel $\widehat{K}_n$ is studied only for a few particular ensembles. The case of GOE it was considered in [18]. The case $V(\lambda)=\lambda^{2m}$ for natural m was studied in [6]. The case $V(\lambda)=\frac{1}{4}\lambda^4-\frac{a}{2}\lambda^2$ was studied in [17].

Let us set our main conditions.

C1. *$V(\lambda)$ satisfies (1.2) and is an even analytic function in*

$$\Omega[d,d_1]=\{z:-2-2d\le\Re z\le 2+2d,\ |\Im z|\le d_1\},\quad d,d_1>0. \tag{1.19}$$

C2. *The support σ of IDS of the ensemble consists of a single interval:*

$$\sigma = [-2, 2].$$

C3. *DOS $\rho(\lambda)$ is strictly positive in the internal points $\lambda \in (-2, 2)$ and $\rho(\lambda) \sim |\lambda \mp 2|^{1/2}$, as $\lambda \sim \pm 2$.*

C4. *The function*

$$u(\lambda) = 2 \int \log |\mu - \lambda| \rho(\mu) d\mu - V(\lambda) \tag{1.20}$$

achieves its maximum if and only if $\lambda \in \sigma$.

It is proved in [2] that these conditions imply that

$$\rho(\lambda) = \frac{1}{\pi} P(\lambda) \sqrt{4 - \lambda^2} \mathbf{1}_\sigma, \tag{1.21}$$

where

$$P(z) = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{V'(z) - V'(\zeta)}{z - \zeta} \frac{d\zeta}{(\zeta^2 - 4)^{1/2}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{V'(z) - V'(2\cos y)}{z - 2\cos y} dy. \tag{1.22}$$

Here the contour $\mathcal{L} \subset \Omega[d, d_1]$ and $\mathcal{L}$ contains inside the interval $(-2, 2)$. It is evident that P is an analytic function in $\Omega[2d/3, 2d_1/3]$ and $P(\lambda) \ge \delta > 0$, $\lambda \in \sigma$.

Under these conditions it was proved in [16] that there exists an n- independent C such that for even n $||(M^{(0,n)})^{-1}|| \le C$ and

$$S_n(\lambda, \mu) = K_n(\lambda, \mu) + r_n(\lambda, \mu) + \tilde{r}_n(\lambda, \mu), \tag{1.23}$$

where

$$r_n(\lambda, \mu) = n \sum_{|k|, |j| \le 2 \log^2 n} A_{j,k}^{(n)} \psi_{n+j}^{(n)}(\lambda) \epsilon \psi_{n+k}^{(n)}(\mu), \tag{1.24}$$

$$\tilde{r}_n(\lambda, \mu) = \sum_{j,k=0}^{n-1} \mathcal{E}_{j,k}^{(n)} \psi_j^{(n)}(\lambda) \epsilon \psi_k^{(n)}(\mu), \quad ||\mathcal{E}_{j,k}^{(n)}|| \le e^{-c \log^2 n}. \tag{1.25}$$

Here and below we denote by $c, C, C_0, C_1, ...$ positive n-independent constants (different in different formulas).

Besides,

$$IS_n(\lambda, \mu) = \int \epsilon(\lambda - \lambda') K_n(\lambda', \mu) d\lambda' + I r_n(\lambda, \mu), + I \tilde{r}_n(\lambda, \mu), \tag{1.26}$$

where

$$I r_n(\lambda, \mu) = \int \epsilon(\lambda - \lambda') r_n(\lambda', \mu) d\lambda', \quad I \tilde{r}_n(\lambda, \mu) = \int \epsilon(\lambda - \lambda') \tilde{r}_n(\lambda', \mu) d\lambda', \tag{1.27}$$

and

$$S_n d(\lambda, \mu) = -\frac{\partial}{\partial \mu} K_n(\lambda, \mu) + \frac{\partial}{\partial \mu} r_n(\lambda, \mu) + \frac{\partial}{\partial \mu} \tilde{r}_n(\lambda, \mu). \tag{1.28}$$

The main result of the present paper is

Theorem 1 *Consider the orthogonally invariant ensemble of random matrices defined by (1.1)-(1.3) with V satisfying conditions C1-C4. Then for any $\varphi \in C_1[-2 - \varepsilon, 2 + \varepsilon]$, growing not faster than polynomial at infinity, fluctuations of linear statistics (1.6) converge in distribution, as $n \to \infty$, to a Gaussian random variable with zero mean and the variance* $\mathbf{Var}[\varphi; V]$*, where*

$$\mathbf{Var}[\varphi; V] = \lim_{n \to \infty} \mathbf{Var}_n[\varphi; V]. \tag{1.29}$$

2 Proof of the main results

Proof of Proposition 1 . By definition and (1.17) we have

$$\begin{aligned}\mathbf{Var}_n[\varphi;V] &= n(n-1)\int d\lambda d\mu\, p_{2,1}^{(n)}(\lambda,\mu)\varphi(\lambda)\varphi(\mu) + n\int d\lambda\, p_{1,1}^{(n)}(\lambda)\varphi^2(\lambda)\\ &\quad - n^2\int d\lambda d\mu\, p_{1,1}^{(n)}(\lambda)p_{1,1}^{(n)}(\mu)\varphi(\lambda)\varphi(\mu)\\ &= -\frac{1}{2}\int d\lambda d\mu\,\mathrm{tr}\left(\widehat{K}_n(\lambda,\mu)\widehat{K}_n(\mu,\lambda)\right)\varphi(\lambda)\varphi(\mu) + \frac{1}{2}\int d\lambda\,\mathrm{tr}\,\widehat{K}_n(\lambda,\lambda)\varphi^2(\lambda) \quad (2.1)\end{aligned}$$

But since

$$\int d\mu\, p_{1,1}^{(n)}(\mu) = 1, \quad \int d\mu\, p_{2,1}^{(n)}(\lambda,\mu) = p_{1,1}^{(n)}(\lambda),$$

we obtain

$$\frac{1}{2}\int\int d\lambda \mathrm{tr}\,\widehat{K}_n(\lambda,\lambda) = 1, \quad \int d\lambda d\mu\,\mathrm{tr}\left(\widehat{K}_n(\lambda,\mu)\widehat{K}_n(\mu,\lambda)\right) = \mathrm{tr}\,\widehat{K}_n(\lambda,\lambda)$$

Using this expression in (2.1) we get (1.18). □

The proof of Theorem 1 is based on the following lemma

Lemma 1 *Let for any $\varphi \in C_1[\sigma_d]$, where $\sigma_d = [-d-2, 2+d]$*

$$\mathbf{Var}_n[\varphi;V] \le C\max_{\sigma_d}|\varphi'|^2, \quad (2.2)$$

and for any polynomial φ and any $|t| \le A$

$$E\{e^{it\dot{N}_n[\varphi]}\} \to e^{-t^2\mathbf{Var}[\varphi;V]/2}, \quad (2.3)$$

Then for any $\varphi \in C_1[\sigma_d]$ the limit in (1.29) exists and (2.3) is valid.

Proof. Since $\varphi \in C_1[\sigma_d]$, for any $\varepsilon > 0$ there exists φ_1 and φ_2, such that $\varphi = \varphi_1 + \varphi_2$, φ_1 is a polynomial and $|\varphi_2'| \le \varepsilon$, it follows from (2.2) and the Schwarz inequality that there exists $C > 0$ independent of ε and n

$$|\mathbf{Var}_n[\varphi;V] - \mathbf{Var}_n[\varphi_1;V]| \le C\varepsilon$$

Besides, for any other choice $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ such that $\varphi = \tilde{\varphi}_1 + \tilde{\varphi}_2$, $|\tilde{\varphi}_2'| \le \varepsilon_1$, we have

$$|\mathbf{Var}_n[\tilde{\varphi}_1;V] - \mathbf{Var}_n[\varphi_1;V]| \le C(\varepsilon + \varepsilon_1)$$

Hence, for any choice of polynomials $\{\varphi_n\}_{n=1}^{\infty}$ such that $\max|\varphi' - \varphi_n'| \to 0$, as $n \to \infty$, the sequence $\mathbf{Var}_n[\varphi_{1,n};V]$ is fundamental and have a limit independent of the choice of $\varphi_{1,n}$. This imply the existence of the limit in (1.29) and that for any $\varphi_1, \varphi_2 \in C_1[\sigma_d]$

$$|\mathbf{Var}[\varphi_1;V] - \mathbf{Var}[\varphi_2;V]| \le C\max_{\sigma_d}|\varphi_1' - \varphi_2'| \quad (2.4)$$

To prove (2.3) for any φ we fixe any $\varepsilon >$, choose φ_1 and φ_2 like above and write by the final increments formula and the Schwarz inequality

$$|E\{e^{it\dot{N}_n[\varphi_1+\varphi_2]} - E\{e^{it\dot{N}_n[\varphi_1]}\}| \le |t|E\{\dot{N}_n[\varphi_2]e^{it\dot{N}_n[\varphi_1+\xi\varphi_2]}\} \le A\mathbf{Var}_n^{1/2}[\varphi_2;V] \le CA\varepsilon$$

Hence, taking the limit $n \to \infty$, we get

$$e^{-t^2 \mathbf{Var}[\varphi_1;V]/2} - CA\varepsilon \le \liminf_{n\to\infty} E\{e^{it\dot{N}_n[\varphi]}\} \le \limsup_{n\to\infty} E\{e^{it\dot{N}_n[\varphi]}\} \le e^{-t^2 \mathbf{Var}[\varphi_1;V]/2} + CA\varepsilon$$

Thus, using (2.4) we get (2.3) for any $\varphi \in C_1[\sigma_d]$. □

The next lemma will help us to prove (2.3) for polynomial φ.

Lemma 2 *Let $\{\phi_n(t)\}_{n=1}^\infty$ be a sequence of analytic uniformly bounded functions in the circle $B_A = \{t : |t| \le A\}$. Assume also that $\phi_n(t) \to \phi(t)$ for any real t, and $\phi(t)$ is also analytic function in B_A. Then $\phi_n(t) \to \phi(t)$ for all $t \in B_A$.*

Proof. The proof of the lemma is very simple. According to the Arcella theorem, the sequence $\{\varphi_n(t)\}$ is weakly compact in B_A. But according to the uniqueness theorem, the limit of any convergent in B_A subsequence $\{\varphi_{n_k}(t)\}$ must coincide with $\varphi(t)$. Hence we obtain the assertion of the lemma.□

Proof of Theorem 1 According to the results of [2] and [13], if we restrict the integration in (1.3) by $|\lambda_i| \le 2 + d$, consider the polynomials $\{p_k^{(n,d)}\}_{k=0}^\infty$ orthogonal on the interval $\sigma_d = [-2-d, 2+d]$ with the weight e^{-nV} and set $\psi_k^{(n,d)} = e^{-nV/2} p_k^{(n,d)}$, then for $k \le n(1+\varepsilon)$ with some $\varepsilon > 0$

$$\sup_{\lambda\in\sigma_d} |\psi_k^{(n,d)}(\lambda) - \psi_k^{(n)}(\lambda)| \le e^{-nC}, \qquad \sup_{|\lambda|\ge 2+d/2} |\psi_k^{(n)}(\lambda)| \le e^{-nC}. \tag{2.5}$$

Hence, if $\mathcal{M}_d^{(0,n)}$ and $S_{n,d}$ are constructed as in (1.15) and (1.14) for σ_d, then

$$||\mathcal{M}_d^{(0,n)} - \mathcal{M}^{(0,n)}|| \le e^{-nC}, \quad \max_{\sigma_d} |S_{n,d}(\lambda,\mu) - S_{n,d}(\lambda,\mu)| \le e^{-nC}.$$

Therefore from the very beginning we can take all integrals in (1.3), (1.8), (1.18), (1.16) and (1.15) over the interval σ_d and then we can study $\mathcal{M}_d^{(0,n)}$ and $S_{n,d}(\lambda,\mu)$ instead of $\mathcal{M}^{(0,n)}$ and $S_n(\lambda,\mu)$. But to simplify notations we omit below the index d. Besides, everywhere below integrals without limits mean the integrals in σ_d and the symbols $(.,.)_2$ and $||.||_2$ mean the standard scalar product in $L_2[\sigma_d]$ and the correspondent norm.

We use Lemma 2 to prove that for polynomial φ

$$\phi_n(t) = E\{e^{t\dot{N}_n[\varphi]}\} \to e^{t^2 \mathbf{Var}[\varphi;V]/2}, \quad n \to \infty,$$

where $\mathbf{Var}[\varphi; V]$ is defined in (1.29).

It is evident that

$$|\phi_n(t)| \le |\phi_n(|t|)| + |\phi_n(-|t|)|.$$

Hence to obtain the uniform bound for $\{\phi_n(t)\}_{n=1}^\infty$ for $t \in B_A$ it is enough to find the uniform bound for $\{\phi_n(t)\}_{n=1}^\infty$ with $t \in [-A, A]$. And to find the last bound and also to prove the convergence of $\{\phi_n(t)\}_{n=1}^\infty$ for real t it is enough to prove that the sequence $\{\phi_n''(t)\}_{n=1}^\infty$ is uniformly bounded for $t \in [-A, A]$ and that

$$\lim_{n\to\infty} \phi_n''(t) = \mathbf{Var}[\varphi; V], \quad t \in [-A, A]. \tag{2.6}$$

But it is easy to see that

$$\phi_n''(t) = \mathbf{Var}_n[\varphi; V + t\varphi/n]. \tag{2.7}$$

By another words, for our goal it is enough to prove that under conditions of Theorem 1

$$\lim_{n\to\infty} \mathbf{Var}_n[\varphi; V + t\varphi/n] = \mathbf{Var}_n[\varphi; V]. \qquad (2.8)$$

Let us first to transform the expression for $\mathbf{Var}_n[f; V + t\varphi/n]$ given by Proposition 1. Using (1.23)-(1.28) and integrating by parts in terms, containing $\frac{\partial}{\partial\mu}K(\lambda,\mu)$, we get

$$\begin{aligned}
2\mathbf{Var}_n[f; V+t\varphi/n] &= \int d\lambda d\mu\, S_n(\lambda,\mu)S_n(\mu,\lambda)\Delta_f^2 - \int d\lambda d\mu\, \frac{\partial}{\partial\mu}S_n(\lambda,\mu)(IS_n(\mu,\lambda)-\epsilon(\mu-\lambda))\Delta_f^2 \\
&= 2\int d\lambda d\mu\, K_n^2(\lambda,\mu)\Delta_f^2 + 3\int d\lambda d\mu\, K_n(\lambda,\mu)r_n(\mu,\lambda)\Delta_f^2 + \int d\lambda d\mu\, r_n(\lambda,\mu)r_n(\mu,\lambda)\Delta_f^2 \\
&- \int d\lambda d\mu\, \frac{\partial}{\partial\mu}r_n(\lambda,\mu)(IK_n(\mu,\lambda)-\epsilon(\mu-\lambda))\Delta_f^2 - \int d\lambda d\mu\, \frac{\partial}{\partial\mu}r_n(\lambda,\mu)Ir_n(\mu,\lambda)\Delta_f^2 \\
&- 2\int d\lambda d\mu\, K_n(\lambda,\mu)(IK_n(\mu,\lambda)-\epsilon(\mu-\lambda))\Delta_f f'(\mu) - 2\int d\lambda d\mu\, K_n(\lambda,\mu)Ir_n(\mu,\lambda)\Delta_f f'(\mu) \\
&+ O(\max|f|^2 e^{-c\log^2 n}) = 2I_1 + 3I_2 + I_3 - I_4 - I_5 - 2I_6 - 2I_7 + O(\max|f| e^{-c\log^2 n}), \qquad (2.9)
\end{aligned}$$

where

$$\Delta_f = f(\lambda) - f(\mu). \qquad (2.10)$$

and $O(\max|f|^2 e^{-c\log^2 n})$ is a contribution of the terms containing integrals of $\tilde{r}_n(\mu,\lambda)$ of (1.25). Note that all integrated terms here contain $\psi_k^{(n)}(\pm 2 \pm d) = O(e^{-nc})$ (see (2.5)). Hence their contribution is $O(e^{-nc})$.

To proceed further let us recall that, by standard arguments, $\{\psi_l^{(n)}\}$ satisfy the recursion formula

$$\lambda\psi_l^{(n)}(\lambda) = J_l^{(n)}\psi_{l+1}^{(n)}(\lambda) + q_l^{(n)}\psi_l^{(n)}(\lambda) + J_{l-1}^{(n)}\psi_{l-1}^{(n)}(\lambda), \quad l = 0,1,\ldots \quad J_{-1}^{(n)} = 0. \qquad (2.11)$$

The Jacobi matrix $\mathcal{J}^{(n)}$ defined by this recursion plays an important role in our proof.

Lemma 3 *Consider $\psi_j^{(n)}$ and $J_j^{(n)}, q_j^{(n)}$ defined by (2.11) for the potential $V + t\varphi/n$. Under conditions of Theorem 1 there exists $\tilde{\varepsilon} > 0$, such that for all $|j| \le \tilde{\varepsilon}n$*

$$J_{n+j}^{(n)} = 1 + \frac{c^{(1)}t + j}{2P(0)n} + r_j^{(1)}, \quad q_{n+j}^{(n)} = \frac{c^{(0)}t}{2P(0)n} + r_j^{(0)}, \quad |r_j^{(\alpha)}| \le C\Big(\frac{j^2}{n^2} + n^{-4/3}\Big), \quad \alpha = 0,1, \qquad (2.12)$$

for $|j| \le n^{1/5}$

$$\epsilon\psi_{n+j-1}^{(n)} - \epsilon\psi_{n+j+1}^{(n)} = 2n^{-1}\sum_{k>0} R_{j-k}\psi_k^{(n)} + n^{-1}\varepsilon_k^{(n)}, \quad ||\varepsilon_k^{(n)}||_2 \le n^{-1/9}, \qquad (2.13)$$

where

$$R_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{ijx}dx}{P(2\cos x)}. \qquad (2.14)$$

and the function P is defined in (1.22). Moreover, there exists $M^*_{n-j,n-k}$ such that for any $|j|, |k| \le n^{1/5}$

$$M_{n-j,n-k} = M^*_{n-j,n-k} + O(n^{-1/9}), \quad M^*_{n-j,n-k} = M_{k-j+1} - \frac{1}{2}(1 + (-1)^j)M_{-\infty} \qquad (2.15)$$

with

$$M_k = (1 + (-1)^k) \sum_{j=k}^{\infty} R_j, \quad M_{-\infty} = 2 \sum_{j=-\infty}^{\infty} R_j, \tag{2.16}$$

The proof of the lemma is given in the next section.

On the basis of the lemma we can prove now that the last two integrals in the r.h.s. of (2.9) (I_6 and I_7) disappear in the limit $n \to \infty$. Using the Christoffel-Darboux formula it is easy to see that for this goal it is enough to prove that for any polynomial f, g and any $|j|, |k| \le \log^2 n$

$$\begin{aligned} &\int d\lambda d\mu \left(\psi_n^{(n)}(\lambda)\psi_{n-1}^{(n)}(\mu) - \psi_n^{(n)}(\mu)\psi_{n-1}^{(n)}(\lambda)\right) (IK_n(\mu,\lambda) - \epsilon(\lambda-\mu)) f(\lambda)g(\mu) \to 0 \\ &n \int d\lambda d\mu \left(\psi_n^{(n)}(\lambda)\psi_{n-1}^{(n)}(\mu) - \psi_n^{(n)}(\mu)\psi_{n-1}^{(n)}(\lambda)\right) \epsilon\psi_{n+k}^{(n)}(\lambda)\epsilon\psi_{n+j}^{(n)}(\mu)f(\lambda)g(\mu) \to 0 \end{aligned} \tag{2.17}$$

We use that

$$IK_n(\mu,\lambda) - \epsilon(\lambda-\mu) = \sum_{k=n}^{\infty} \epsilon\psi_k^{(n)}(\mu)\psi_k^{(n)}(\lambda) \tag{2.18}$$

in the weak sense. Besides, using the recursion formula (2.11) we obtain easily that for polynomial f of the degree l

$$f(\lambda)\psi_{n-\alpha}^{(n)}(\lambda) = \sum_{k=n+\alpha-l}^{j=n+\alpha+l} f_{n-\alpha,j}\psi_{n-\alpha+j}^{(n)}(\lambda), \quad \alpha = 0, 1, \tag{2.19}$$

where, according to (2.12), the coefficients $f_{n+\alpha,j}$ have finite limits, as $n \to \infty$. Using (2.18) and (2.19) in the first integral of (2.17) and integrating with respect to λ, we obtain that the first integral is equal to a finite sum of the terms

$$\int d\mu\, \epsilon\psi_{n+j}^{(n)}(\mu)\psi_{n-\alpha}^{(n)}(\mu)g(\mu). \tag{2.20}$$

But using the representation of the type (2.19) for the polynomial g we obtain easily that every term of the type (2.20) is equal to a finite sum of the terms

$$\int d\mu\, \epsilon\psi_{n+j}^{(n)}(\mu)\psi_{n+j'}^{(n)}(\mu) = n^{-1} M_{n+j',n+j}. \tag{2.21}$$

Since by (2.15) $M_{n+j',n+j}$ have finite limits as $n \to \infty$ we obtain the first line of (2.17).

To prove that the second integral in (2.17) tends to zero, we also use (2.19) and its analog for g. Then we obtain that the second integral is a finite sum with convergent coefficients of the terms

$$n \int d\lambda d\mu\, \epsilon\psi_{n+k}^{(n)}(\lambda)\psi_{n+k'}^{(n)}(\lambda)\epsilon\psi_{n+j}^{(n)}(\mu)\psi_{n+j'}^{(n)}(\mu) = n^{-1} M_{n+k',n+k} M_{n+j',n+j}.$$

Similarly to the above we conclude that all these terms tend to zero and so the second integral in (2.17) tends to zero.

Lemma 4 *Consider the coefficients $A_{j,k}^{(n)}$ from (1.24) defined for the potential $V + t\varphi/n$. Under conditions of Theorem 1 for any $|j|, |k| \le \log^2 n$ there exists $A_{j,k}$ independent of t and such that*

$$|A_{j,k}^{(n)} - A_{j,k}| \le Cn^{-1/9}. \tag{2.22}$$

Moreover, there exists an n-independent c, C such that

$$|A_{j,k}| \le Ce^{-c(|j|+|k|)}. \tag{2.23}$$

We prove this lemma in the next section.

According to the above arguments it is clear now that to prove Theorem 1 it is enough to prove that for any polynomial f there exist limits for all integral I_α, ($\alpha = 1, \dots, 5$) from (2.9). The existence of the limit of I_1 follows from the result of [8]. Using representation (1.24) and the Christoffel-Darboux formula it is easy to understand that I_2 can be represented as a sum of the terms

$$T_{j,k} := n \int d\lambda d\mu \, \left(\psi_n^{(n)}(\lambda)\psi_{n-1}^{(n)}(\mu) - \psi_n^{(n)}(\mu)\psi_{n-1}^{(n)}(\lambda) \right) \psi_{n-j}^{(n)}(\lambda)\epsilon\psi_{n+k}^{(n)}(\mu) \frac{\Delta_f^2}{\lambda - \mu}. \tag{2.24}$$

It is evident that if f is a polynomial of the lth degree, then

$$\frac{\Delta_f^2}{\lambda - \mu} = \sum_{|p|,|q| \le 2l-1} \tilde{f}_p(\lambda)\tilde{g}_q(\mu),$$

where $\tilde{f}_p$ and $\tilde{g}_q$ are some fixed polynomial of the degree less than $2l$. Since we have the bound (2.23), it is enough to prove that the limit exists for any fixed j, k, as $n \to \infty$. But using for (2.19) for $\tilde{f}_p$ and $\tilde{g}_q$ and integrating with respect to λ, we reduce the existence of the limit of $T_2(j,k)$ to the existence of the limits of $M_{n-j',n+k}$ for any fixed j', k, which follows from Lemma 3.

The existence of the limits of I_3 and I_5 can be obtained by the same way. To find the limit of I_4 we use first the relation (2.18), then (2.19) for f and observe that after integration with respect to λ only the finite number of k in the r.h.s. of (2.18) give us nonzero contribution. Hence, as above, we reduce the problem to the existence of the limits $M_{n-j,n+k}$, which follows from Lemma 3.

To complete the proof of the theorem we are left to prove the estimate (2.2). It is clear that for this goal it is enough to prove similar estimates for all terms I_α $\alpha = 1, \dots 7$ in (2.9). For I_1 we have by the Christoffel-Darboux formula

$$\int d\lambda d\mu \, K_n^2(\lambda,\mu)\Delta_f^2 \le \max_{\lambda\in\sigma_d} |f'|^2 \int d\lambda d\mu \, K_n^2(\lambda,\mu)(\lambda-\mu)^2 = 2(J_n^{(n)})^2 \max_{\lambda\in\sigma_d} |f'|^2.$$

To prove the estimates for others I_α let us prove first the following auxiliary statement

Proposition 2 *For any g with g' bounded in σ_d and any $|j|, |k| \le 2\log^2 n$*

$$\left| n \int d\mu \, g(\mu)\psi_{n+j}^{(n)}(\mu)\epsilon\psi_{n+k}^{(n)}(\mu) \right| \le C(\max_{\sigma_d} |g'| + \max_{\sigma_d} |g|). \tag{2.25}$$

Proof of Proposition 2. We start from a simple relation, which follows from the definition of the operator ϵ (see 1.16). For any integrable f, g

$$\int d\lambda \epsilon f(\lambda)\epsilon g(\lambda) = \frac{1}{4}(\mathbf{1}_{\sigma_d}, f)_2(\mathbf{1}_{\sigma_d}, g)_2 - \frac{1}{2}\int_{\sigma_d} d\lambda d\mu \, |\lambda - \mu| f(\lambda)g(\mu). \tag{2.26}$$

In particular, using a simple observation that $\frac{1}{2}|\lambda-\mu| = (\lambda-\mu)\epsilon(\lambda-\mu)$ and then the definition (1.15), we get

$$\int d\lambda \epsilon\psi_j^{(n)}(\lambda)\epsilon\psi_k^{(n)}(\lambda) = \frac{1}{4}(\mathbf{1}_{\sigma_d}, \psi_j^{(n)})_2(\mathbf{1}_{\sigma_d}, \psi_k^{(n)})_2 \\ - \frac{1}{n}\left(J_j^{(n)}M_{j+1,k} + J_{j-1}^{(n)}M_{j-1,k} - J_k^{(n)}M_{j,k+1} - J_{k-1}^{(n)}M_{j,k-1}\right). \quad (2.27)$$

Since for odd k $(\mathbf{1}_{\sigma_d}, \psi_j^{(n)})_2 = 0$, this relation and (2.15) gives us immediately that for odd $|k| \le n^{1/5}$

$$\int d\lambda (\epsilon\psi_{n+k}^{(n)}(\lambda))^2 \le \frac{C}{n}. \quad (2.28)$$

For even k the same relation can be obtained if we apply the analog of (2.27) to $f(\lambda) = \lambda\psi_{n+k}^{(n)}(\lambda) = J_{n+k}^{(n)}\psi_{n+k+1}^{(n)}(\lambda) + J_{n+k-1}^{(n)}\psi_{n+k-1}^{(n)}(\lambda)$ and then use (2.13). Remark also that since (2.5) yield

$$|\epsilon\psi_{n+k}^{(n)}(2+\lambda) - \epsilon\psi_{n+k}^{(n)}(2+d/2)| \le e^{-nc}, \quad d/2 \le \lambda \le d,$$

by (2.28), we have

$$n(\epsilon\psi_{n+k}^{(n)}(2+d))^2 d/2 \le n\int d\mu\, (\epsilon\psi_{n+k}^{(n)}(\mu))^2 + o(1) \le C. \quad (2.29)$$

The last bound and (2.28) imply one more useful estimate, valid for any f with bounded derivative

$$\int d\lambda \left(\epsilon(f\psi_{n+k}^{(n)})(\lambda)\right)^2 \le \frac{C}{n}(\max_{\sigma_d}|f| + \max_{\sigma_d}|f'|)^2. \quad (2.30)$$

Indeed, using that $\psi_{n+k}^{(n)} = (\epsilon\psi_{n+k}^{(n)})'$ and integrating by parts, it is easy to obtain

$$\epsilon(f\psi_{n+k}^{(n)}) = f(\lambda)\epsilon\psi_{n+k}^{(n)} - \frac{1}{2}f(2+d)\psi_{n+k}^{(n)}(2+d) - \frac{1}{2}f(-2-d)\psi_{n+k}^{(n)}(-2-d) - \epsilon\left(f'\epsilon\psi_{n+k}^{(n)}\right).$$

Now, taking the square of the r.h.s. and using (2.29) and (2.28), we obtain (2.30).

To prove Proposition 2 we consider 3 cases:

(a) $j-k$ is even;

(b) k is even and j is odd;

(c) k is odd and j is even.

(a) Using (2.13), it is easy to get that

$$\left|n\int d\mu\, g(\mu)\psi_{n+j}^{(n)}(\mu)\epsilon\psi_{n+k}^{(n)}(\mu) - n\int d\mu\, g(\mu)\psi_{n+k}^{(n)}(\mu)\epsilon\psi_{n+k}^{(n)}(\mu)\right| \le C|k-j|\max_{\sigma_d}|g(\lambda)|.$$

Then, integrating by parts in the second integral we obtain

$$n\int d\mu\, g(\mu)\psi_{n+k}^{(n)}(\mu)\epsilon\psi_{n+k}^{(n)}(\mu) = \frac{n}{2}g(\mu)(\epsilon\psi_{n+k}^{(n)}(\mu))^2\Big|_{-2-d}^{2+d} - \frac{n}{2}\int d\mu\, g'(\mu)(\epsilon\psi_{n+k}^{(n)}(\mu))^2.$$

Relation (2.25) follows now from (2.29) and (2.28).

(b) Since for even k $\epsilon\psi_{n+k}^{(n)}(0) = 0$, using the result of [4] on the asymptotic of orthogonal polynomials, it is easy to get that for any $|\mu| \le 1$

$$|\epsilon\psi_{n+k}^{(n)}(\mu)| = \left|\int_0^\mu \psi_{n+k}^{(n)}(\lambda)d\lambda\right| \le \frac{C}{n}.$$

Hence, if we define

$$\tilde{g}(\mu) = g(\mu)\mu^{-1}\mathbf{1}_{|\mu|>1} + \frac{1}{2}\left[g(1)(1+\mu) + g(-1)(1-\mu)\right]\mathbf{1}_{|\mu|\le 1},$$

so that $g(\mu) = \tilde{g}(\mu)\mu$ for $|\mu| \ge 1$, then

$$n\left|\int d\mu\, g(\mu)\psi^{(n)}_{n+j}(\mu)\epsilon\psi^{(n)}_{n+k}(\mu) - \int d\mu\, \mu\tilde{g}(\mu)\psi^{(n)}_{n+j}(\mu)\epsilon\psi^{(n)}_{n+k}(\mu)\right| \le C\max_{\sigma_d}|g|. \tag{2.31}$$

It is evident that $|\tilde{g}'(\mu)| \le |g'(\mu)| + |g(\mu)|$. Thus, using the recursion relations (2.11), we replace the last integral by

$$n\int d\mu\, \tilde{g}(\mu)\left(J^{(n)}_{n+j}\psi^{(n)}_{n+j+1}(\mu) + J^{(n)}_{n+j-1}\psi^{(n)}_{n+j-1}(\mu)\right)\right)\epsilon\psi^{(n)}_{n+k}(\mu)d\mu.$$

Hence, we obtain again the case (a).

(c) Integrating by parts, we get

$$n\int d\mu\, g(\mu)\psi^{(n)}_{n+j}(\mu)\epsilon\psi^{(n)}_{n+k}(\mu) = ng(\mu)\epsilon\psi^{(n)}_{n+k}(\mu)\epsilon\psi^{(n)}_{n+j}(\mu)\Big|_{-2-d}^{2+d}$$
$$- n\int d\mu\, g'(\mu)\epsilon\psi^{(n)}_{n+j}(\mu)\epsilon\psi^{(n)}_{n+k}(\mu) - n\int d\mu\, g(\mu)\epsilon\psi^{(n)}_{n+j}(\mu)\psi^{(n)}_{n+k}(\mu).$$

The bounds for first two terms in the r.h.s. were found before, and the last integral corresponds to the case (b). Thus we have proved (2.25).□

To find the bound for I_2 in (2.9) we use the Christoffel-Darboux formula. Then we are faced with a problem to find the bounds for the terms $T_{j,k}$ of (2.24). But since the function $\Delta_f^2(\lambda-\mu)^{-1}$ for any λ has a derivative, bounded uniformly with respect to λ, μ, we can apply the bound (2.25) for any fixed λ. We get

$$T_{j,k} \le C\max_{\sigma_d}|f'|^2\int d\lambda|\psi^{(n)}_n(\lambda)||\psi^{(n)}_{n-k}(\lambda)| \le C\max_{\sigma_d}|f'|^2,$$

where the last bound is valid because of the Schwarz inequality.

The estimates for I_3 and I_5 follow directly from (2.25) and (2.23). For I_6 we use the Christoffel-Darboux formula and then the Schwarz inequality. Then we get

$$|I_6|^2 \le C\max_{\sigma_d}|f'(\mu)|^4\cdot\left(\int d\mu\sum_{k=0}^{n-1}(\epsilon\psi^{(n)}_{n-1}(\mu))^2 + C\right).$$

Here the sum with respect to k appears because of integration with respect to λ of $IK^2(\mu,\lambda)$ and C appears because of integration of $\epsilon^2(\mu-\lambda)$. But from (2.27) it is easy to see that

$$\int d\mu\sum_{k=0}^{n-1}(\epsilon\psi^{(n)}_k(\mu))^2 = \frac{1}{4}\sum_{k=0}^{n-1}(\mathbf{1}_{\sigma_d},\psi^{(n)}_k)^2 - \int d\lambda d\mu\, K_n(\lambda,\mu)(\lambda-\mu)\epsilon(\lambda-\mu).$$

It follows from the Bessel inequality that the sum in the r.h.s. is bounded by $(\mathbf{1}_{\sigma_d},\mathbf{1}_{\sigma_d})$. In the second integral we apply the Christoffel-Darboux formula and then (2.15).

For I_7 we apply Christoffel-Darboux formula and then the Schwarz inequality. We obtain

$$|I_7| \le nC \max_{\sigma_d} |f'|^2 \left(\sum_{j,k,j',k'} A_{j,k} A_{j',k'} \int d\lambda d\mu \, \epsilon\psi^{(n)}_{n+j}(\lambda)\epsilon\psi^{(n)}_{n+k'}(\lambda)\epsilon\psi^{(n)}_{n+k}(\mu)\epsilon\psi^{(n)}_{n+k'}(\mu) \right)^{1/2} \le \max_{\sigma_d} |f'|^2, \quad (2.32)$$

where the last inequality follows from (2.28).

Now we are left to prove the bound for I_4 (see (2.9)). Remark, that because of (2.5) and (1.13)-(1.17) the integrals in $[2+d/2, 2+d]$ and from $[-2-d, -2-d/2]$ in (2.9) give us $O(e^{-nc})$ terms. Hence, without loss of generality we can replace the function f in these intervals by a linear one in order to have a new function being continuous with a bounded derivative and such that $f(2+d) = f(-2-d) = 0$. Then, integrating by parts with respect to μ, we need to control only the terms, which do not contain $f(\mu)$. But for odd k $\epsilon\psi^{(n)}_k(\pm 2 \pm d) = 0$, and if j and k are even, then $\epsilon\psi^{(n)}_k(\mu)\epsilon\psi^{(n)}_j(\mu)$ is an even function and so $\epsilon\psi^{(n)}_k(\mu)\epsilon\psi^{(n)}_j(\mu)\big|^{2+d}_{-2-d} = 0$. Hence, integrating by parts in I_4, we obtain that all integrated terms disappear. Thus,

$$I_4 = -I_2 + 2 \int d\lambda d\mu \, r_n(\lambda,\mu)(IK_n(\mu,\lambda) - \epsilon(\mu-\lambda)) f'(\mu)\Delta_f = -I_2 + 2I_{4,1}.$$

The bound for I_2 was found before. Hence, we need to find the bound for $I_{4,1}$. From definitions (1.15) it is evident that $M_{j,k} = -M_{k,j}$ and therefore from (1.14) we derive

$$IS_n(\lambda,\mu) = -IS_n(\mu,\lambda) \Leftrightarrow IK_n(\mu,\lambda) = -IK_n(\lambda,\mu) - Ir_n(\lambda,\mu) - Ir_n(\mu,\lambda).$$

Now if we replace $IK_n(\mu,\lambda)$ by the above expression, then the terms containing $Ir_n(\lambda,\mu)$ and $Ir_n(\mu,\lambda)$ can be easily estimated by using (2.25) and (2.23). Hence we are left to prove the bound for

$$\left| \int d\lambda d\mu \, r_n(\lambda,\mu) IK_n(\mu,\lambda)\tilde{f}(\lambda)\tilde{g}(\mu) \right| = n \left| \sum_{j,k} A_{j,k} \sum_{l=0}^{n-1} (\tilde{f}\psi^{(n)}_{n-j}, \epsilon\psi^{(n)}_l)(\tilde{g}\epsilon\psi^{(n)}_{n+k}, \psi^{(n)}_l) \right|$$
$$\le n \sum_{j,k} |A_{j,k}| \cdot ||\epsilon(\tilde{f}\psi^{(n)}_{n-j})||_2 ||\tilde{g}\epsilon\psi^{(n)}_{n+k}||_2 \le C(\max_{\sigma_d}|\tilde{f}| + \max_{\sigma_d}|\tilde{f}'|) \cdot \max_{\sigma_d}|\tilde{g}|,$$

where the last bound follows from (2.28), (2.30 and (2.22)-(2.23). The term with $\epsilon(\lambda-\mu)$ can be estimated similarly. This completes the proof of Theorem 1.

3 Auxiliary results

Proof of Lemma 3. It is proved in [16], that for $t = 0$, representation (2.12) implies (2.13) and (2.15). If we know (2.12) for $t \ne 0$, then the proofs of (2.13) and (2.15) coincides with that of [16]. Hence we need only to prove (2.12).

The idea is to use the perturbation expansion of the string equations:

$$\begin{aligned} &V_t'(\mathcal{J}^{(n)})_{k,k} = 0, \\ &J^{(n)}_k V_t'(\mathcal{J}^{(n)})_{k,k+1} = \frac{k+1}{n}. \end{aligned} \qquad (3.1)$$

Here and below in the proof of Lemma 3 we denote $V_t = V + t\varphi$ and by $\mathcal{J}^{(n)}$ a semi-infinite Jacobi matrix, defined in (2.11). Relations (3.1) can be easily obtained from the identity

$$\int \left(e^{-nV_t(\lambda)}(P_k^{(n)}(\lambda))^2\right)' d\lambda = 0,$$
$$\int \left(e^{-nV_t(\lambda)}P_{k+1}^{(n)}(\lambda)P_k^{(n)}(\lambda)\right)' d\lambda = 0.$$

We consider (3.1) as a system of nonlinear equations with respect to the coefficients $J_k^{(n)}, q_k^{(n)}$. To have zero order expression for $J_{n+k}^{(n)}$ we use the following lemma, proven in [15]:

Lemma 5 *Under conditions C1-C3 for small enough $\tilde{\varepsilon}$ uniformly in $k : |k| \le \tilde{\varepsilon}n$*

$$\left|q_{n+k}^{(n)}\right|, \left|J_{n+k}^{(n)} - 1\right| \le C\left(n^{-1/4}\log^{1/2} n + (|k|/n)^{1/2}\right). \tag{3.2}$$

Denote $\mathcal{J}^{(0)}$ an infinite Jacobi matrix with constant coefficients

$$\mathcal{J}_{k,k+1}^{(0)} = \mathcal{J}_{k+1,k}^{(0)} = 1, \quad \mathcal{J}_{k,k}^{(0)} = 0 \tag{3.3}$$

and for any positive $n^{1/3} << N < n$ define an infinite Jacobi matrix $\tilde{\mathcal{J}}(N)$ with the entries

$$\tilde{J}_k = \begin{cases} J_{n+k}^{(n)} - 1, & |k| < N, \\ 0, & \text{otherwise}. \end{cases} \quad \tilde{q}_k = \begin{cases} q_{n+k}^{(n)}, & |k| < N, \\ 0, & \text{otherwise}. \end{cases} \tag{3.4}$$

Define a periodic function $\tilde{v}_t(\lambda) = \tilde{v}_t(\lambda+4+2d)$ with $\tilde{v}_t^{(4)} \in L_2[\sigma_d]$, and such that $\tilde{v}(\lambda) = V'(\lambda)$ for $|\lambda| \le 2 + d/2$. Consider the standard Fourier expansion for the function $\tilde{v}_t$

$$\tilde{v}_t(\lambda) = \sum_{j=-\infty}^{\infty} v_{tj} e^{ij\kappa\lambda}, \quad \kappa = \frac{\pi}{2+d}, \tag{3.5}$$

The first step in the proof of (2.12) is the lemma

Lemma 6 *If V satisfies conditions C2-C3 and $V^{(4)} \in L_2[\sigma_d]$, then for any $n^{1/3} << N < n$ and any $|k| \le N/2$*

$$\begin{aligned} V_t'(\mathcal{J}^{(n)})_{n+k,n+k} &= \frac{t}{n}\varphi(\mathcal{J}^{(0)})_{k,k} + \sum \mathcal{P}_{k-l}(t)\tilde{q}_l + \tilde{r}_k^{(0)} + O(\|\tilde{\mathcal{J}}\|/n) + O(N^{-7/2}), \\ V_t'(\mathcal{J}^{(n)})_{n+k,n+k+1} &= 1 - \tilde{J}_k + \frac{t}{n}\varphi(\mathcal{J}^{(0)})_{k,k+1} + \sum \mathcal{P}_{k-l}(t)\tilde{J}_l + \tilde{r}_k^{(1)} \\ &\qquad + O(\|\tilde{\mathcal{J}}\|/n) + O(N^{-7/2}), \end{aligned} \tag{3.6}$$

where for $\alpha = 0, 1$

$$\tilde{r}_k^{(\alpha)} = \sum_{j=-\infty}^{\infty} v_{tj}(ij\kappa)^2 \int_0^1 ds_1 \int_0^{1-s_1} ds_2 \left(e^{ij\kappa s_1\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ij\kappa s_2\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ij\kappa(1-s_1-s_2)(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})}\right)_{k,k+\alpha} \tag{3.7}$$

with v_j, d defined in (3.5), and

$$\mathcal{P}_l(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} (P(2\cos(x/2)) + t\tilde{\varphi}(2\cos(x/2))/n)e^{ilx}dx, \tag{3.8}$$

with P defined in (1.22) and $\tilde{\varphi}$-some polynomial with coefficients depending on φ.

Proof of Lemma 6 By Proposition 1 of [16] it is enough to obtain (3.6) for $\tilde{v}_t(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})_{n+k,n+k+\alpha}$. Using the spectral theorem, we have

$$\tilde{v}_t(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})_{k,k+\alpha}=\sum_{j=-\infty}^{\infty}\left(v_{tj}e^{ij\kappa(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})}\right)_{k,k+\alpha}.$$

Applying the Duhamel formula two times we get for $\alpha=0,1$

$$\begin{aligned}\tilde{v}_t(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})_{k,k+\alpha}&=\tilde{v}_t(\mathcal{J}^{(0)})_{k,k+\alpha}\\&+\sum_{j=-\infty}^{\infty}v_{tj}(ij\kappa)\int_0^1 ds\left(e^{ij\kappa s\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ij\kappa(1-s)\mathcal{J}^{(0)}}\right)_{k,k+\alpha}+r_k^{(\alpha)}.\end{aligned}\tag{3.9}$$

To find the the first term in (3.9) we use the relation, which follows from coincidence $\tilde{v}(\lambda)=V'(\lambda)$, $\lambda\in[-2,2]$ and (1.5)

$$\begin{aligned}\tilde{v}_t(\mathcal{J}^{(0)})_{n+k,n+k+\alpha}&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{v}_t(2\cos x)\cos^\alpha x\,dx\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}(V'(2\cos x)+t\varphi'(2\cos x)/n)\cos^\alpha x\,dx\\=\frac{1}{\pi}\int_{-\pi}^{\pi}dx\int_{-2}^{2}\cos^\alpha x\frac{\rho(\lambda)d\lambda}{2\cos x-\lambda}&+\frac{t}{2\pi n}\int_{-\pi}^{\pi}\varphi'(2\cos x)/n)\cos^\alpha x\,dx=\alpha+\frac{tc^{(\alpha)}}{n}.\end{aligned}\tag{3.10}$$

Besides, since by the spectral theorem

$$(e^{ij\kappa s\mathcal{J}^{(0)}})_{k,l}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ij\kappa s\cos x}e^{i(k-l)x}dx=\mathbf{J}_{k-l}(j\kappa s),\tag{3.11}$$

where $\mathbf{J}_k(s)$ is the Bessel function, and since V' is an odd function, we get for any l and integer α

$$\begin{aligned}&\sum_{j=-\infty}^{\infty}v_{0j}(ij\kappa)\int_0^1 ds\left(e^{ij\kappa s\mathcal{J}^{(0)}}\right)_{k,l}\left(e^{ij\kappa(1-s)\mathcal{J}^{(0)}}\right)_{l\pm\alpha,k+1-\alpha}\\&=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}dxdy\frac{V'(2\cos x)-V'(2\cos y)}{2\cos x-2\cos y}\cos\left((k-l)(x-y)+(\alpha(1\mp1)+1)y\right)=0,\end{aligned}$$

Hence, the linear terms with respect to $\tilde{J}_k$ in the first equation of (3.6) and the linear terms with respect to $\tilde{q}_k$ in the second equation give us only the contribution of the order $tn^{-1}||\tilde{\mathcal{J}}||$. Besides, we derive from (3.9) that the operator $\mathcal{P}$ from the second line of (3.6) can be represented in the form

$$\mathcal{P}_{k-l}(t)=\delta_{k,l}+\int ds\sum_{j=-\infty}^{\infty}v_{tj}(ij\kappa)\left(e^{ij\kappa s\mathcal{J}^{(0)}}E^{(n+l)}e^{ij\kappa(1-s)\mathcal{J}^{(0)}}\right)_{k,k+1},$$

where we denote by $E^{(l)}$ a matrix with entries:

$$E^{(l)}_{k,m}=\delta_{k,l}\delta_{m,l+1}+\delta_{k,l+1}\delta_{m,l}.$$

It is easy to see that $\mathcal{P}(t)$ is a Toeplitz matrix, so its entries can be represented in the form

$$P_{l,k}(t) = P_{l-k}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ilx}F(x,t)dx, \quad F(x,t) = \sum \mathcal{P}_l(t)e^{ilx}.$$

Thus, we obtain

$$\begin{aligned} F(x,1) &= 1 + \sum_j (ij\kappa)v_{tj}\int_0^1 ds_1 \sum_l \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{il(-x_1+x_2+x)}(1+e^{-i(x_1+x_2)}) \\ &\quad \cdot \exp\{2ij\kappa[s_1\cos x_1 + (1-s_1)\cos x_2]\}dx_1dx_2 \\ &= 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{v_t(2\cos x_1) - v_t(2\cos(x_1-x))}{\cos x_1 - \cos(x_1-x)}(1+\cos(2x_1-x))dx_1 \\ &= 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi} v_t(2\cos x_1)\Big(\frac{1+\cos(2x_1-x)}{\cos x_1 - \cos(x_1-x)} + \frac{1+\cos(2x_1+x)}{\cos x_1-\cos(x_1+x)}\Big)dx_1 \\ &= P(2\cos(x/2)) + P(-2\cos(x/2)) + t\tilde{\varphi}(2\cos(x/2))/n, \end{aligned} \tag{3.12}$$

where in the last line we have used (3.10) and (1.22). For the linear operator in the first line of (3.6) the calculations are similar. Lemma 6 is proved.□

Let us use (3.6) in (3.1). We obtain for $k \le N/2$

$$\begin{aligned} \sum \mathcal{P}_{k-l}(t)\tilde{q}_l &= -\frac{tc^{(0)}}{n} - \tilde{r}_k^{(0)} + O(||\tilde{\mathcal{J}}||/n) + O(N^{-7/2}), \\ \sum \mathcal{P}_{k-l}(t)\tilde{J}_l &= \frac{k+1}{n} - \frac{tc^{(1)}}{n} + \tilde{J}_k^2 - \tilde{r}_k^{(1)} + O(||\tilde{\mathcal{J}}||/n) + O(N^{-7/2}), \end{aligned}$$

where $c^{(0)}$ and $c^{(1)}$ are defined in (3.10). We would like to consider this system of equations like two linear equations in l_2. To this end we set for $|k| > N/2$

$$\begin{aligned} \tilde{r}_k^{(0)} &= \sum \mathcal{P}_{k-l}(t)q_l, \\ \tilde{r}_k^{(1)} &= \sum \mathcal{P}_{k-l}(t)\tilde{J}_l - \frac{k+1}{n} - \tilde{J}_k^2. \end{aligned}$$

It follows from (3.8) that the operator $\mathcal{P}$ has a bounded inverse operator whose entries can be represented in the form

$$(\mathcal{P}^{-1})_{k-l} = \frac{1}{4\pi}\int_{-\pi}^{\pi}(P(2\cos(x/2)) + t\tilde{\varphi}(2\cos(x/2))/n)^{-1}e^{i(k-l)x}dx. \tag{3.13}$$

Then

$$\begin{aligned} q_l &= -\sum \mathcal{P}^{-1}_{l-k}(0)\Big(\frac{tc^{(0)}}{n} + O(||\tilde{\mathcal{J}}||/n) + \tilde{r}_k + O(N^{-7/2})\Big), \\ \tilde{J}_l &= \sum \mathcal{P}^{-1}_{l-k}(0)\Big(\frac{k+1}{n} + \tilde{J}_k^2 - \frac{tc^{(1)}}{n} + O(||\tilde{\mathcal{J}}||/n) - \tilde{r}_k + O(N^{-7/2})\Big). \end{aligned} \tag{3.14}$$

Moreover, since by assumption v' has fourth derivative from $L_2[-2,2]$, P also does (see [10]). Therefore, using a standard bound for the tails of the Fourier expansion of the function f with $f^{(p)} \in L_2[-\pi,\pi]$

$$\sum_{j>M}|f_k| \le M^{-p+1/2}\Big(\sum |f_k|^2k^{2p}\Big)^{1/2} \le CM^{-p+1/2}, \tag{3.15}$$

we have for any M

$$\sum_{|l|>M}|\mathcal{P}_l^{-1}|\le M^{-7/2},\quad \sum_{|l|>M}|l||\mathcal{P}_l^{-1}|\le M^{-5/2},\quad \sum_{|l|>M}|l|^2|\mathcal{P}_l^{-1}|\le M^{-3/2}. \tag{3.16}$$

Besides, since $\mathcal{P}_l^{-1}=\mathcal{P}_{-l}^{-1}$, we have

$$\sum_{l-k}\mathcal{P}_{l-k}^{-1}\frac{k+1}{n}=\frac{l+1}{n}\sum_{l-k}\mathcal{P}_{l-k}^{-1}=\frac{1}{2P(2)}\frac{l+1}{n}. \tag{3.17}$$

Using a trivial bound

$$\left|\left(e^{ij\kappa s_1\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ij\kappa s_2\mathcal{J}^{(0)}}\tilde{\mathcal{J}}e^{ij\kappa(1-s_1-s_2)(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})}\right)_{k,k+1}\right|\le||\tilde{\mathcal{J}}||^2 \tag{3.18}$$

and (3.2), we obtain first a rather crude bound

$$|\tilde{r}_k^{(\alpha)}|\le C\left(|k|/n+n^{-1/2}\log^2 n\right),\quad \alpha=0,1. \tag{3.19}$$

This bound combined with (3.14) and (3.15) give us

$$|\tilde{q}_k|,|\tilde{\mathcal{J}}_k|\le C\left(|k|/n+n^{-1/2}\log^2 n+N^{-7/2}\right). \tag{3.20}$$

Now we use the bound, valid for any Jacobi matrix $\mathcal{J}$ with coefficients $J_{k,k+1}=J_{k+1,k}=a_k\in\mathbb{R}$, $|a_k|\le A$. Then there exist positive constants C_0, C_1,C_2, depending on A such that the matrix elements of $e^{it\mathcal{J}}$ satisfy the inequalities:

$$|(e^{it\mathcal{J}})_{k,j}|\le C_0e^{-C_1|k-j|+C_2t}. \tag{3.21}$$

This bound follows from the representation

$$(e^{it\mathcal{J}})_{k,j}=-\frac{1}{2\pi i}\oint_l e^{itz}R_{k,j}(z)dz,$$

where $R=(\mathcal{J}-z)^{-1}$, and from the Comb-Thomas type bound on the resolvent of the Jacobi matrix (see [14])

$$|\mathcal{R}_{k,j}(z)|\le\frac{2}{|\Im z|}e^{-C_1'|\Im z||k-j|}+\frac{8}{|\Im z|^2}e^{-C_1'|\Im z|(M-1)}. \tag{3.22}$$

Let us choose

$$M=\frac{C_1}{4C_2\kappa}n^{1/3}, \tag{3.23}$$

where C_1 and C_2 are the constants from (3.21) and $\kappa=\pi(2+\varepsilon)^{-1}$. Then (3.21) guarantee that for any $l,l':|l-l'|>n^{1/3}$ and any $j:|j|<M$, $|t|\le 1$

$$|(e^{itdj\mathcal{J}^{(0)}})_{l,l'}|,|(e^{itdj(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})})_{l,l'}|\le Ce^{dC_2M-C_1|l-l'|}\le Ce^{-C_1n^{1/3}/3}e^{-C_1|l-l'|/3}. \tag{3.24}$$

Now we split the sum in (3.7) in two parts $|j|<M$ and $|j|\ge M$.

$$\tilde{r}_k^{(\alpha)}=\sum_{j=-\infty}^{\infty}v_j(ij\kappa)^2\sum_{l_1,l_2}\int_0^1 ds_1\int_0^{1-s_1}ds_2$$
$$\left(e^{ij\kappa s_1\mathcal{J}^{(0)}}\tilde{\mathcal{J}}\right)_{k,l_1}\left(e^{ij\kappa s_2\mathcal{J}^{(0)}}\right)_{l_1,l_2}\left(\tilde{\mathcal{J}}e^{ij\kappa(1-s_1-s_2)(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})}\right)_{l_2,k+1}=\sum_{|j|<M}+\sum_{|j|\ge M}. \tag{3.25}$$

Then (3.24) allows us to write

$$\sum_{|j|<M} = \sum_{|j|<M} v_j(ij\kappa)^2 \sum_{l_1,l_2=k-[n^{1/3}]}^{k+[n^{1/3}]} \int_0^1 ds_1 \int_0^{1-s_1} ds_2$$
$$\left(e^{ij\kappa s_1 \mathcal{J}^{(0)}} \tilde{\mathcal{J}}\right)_{k,l_1} \left(e^{ij\kappa s_2 \mathcal{J}^{(0)}}\right)_{l_1,l_2} \left(\tilde{\mathcal{J}} e^{ij\kappa(1-s_1-s_2)(\mathcal{J}^{(0)}+\tilde{\mathcal{J}})}\right)_{l_2,k+1} + O(e^{-Cn^{1/3}/3}).$$

Hence using (3.18) we obtain now

$$\Big|\sum_{|j|<M}\Big| \le C \max_{l:|l-k-n|\le n^{1/3}} |\tilde{J}_l|^2. \tag{3.26}$$

For $\sum_{|j|>M}$ we use (3.18) combined with (3.20) and (3.15) for the function V'. Then we get

$$\Big|\sum_{|j|\ge M}\Big| \le CM^{-3/2}\left((N/n)^2 + n^{-1}\log^4 n\right) \le Cn^{-1/2}(N/n)^2 \tag{3.27}$$

and therefore

$$|\tilde{r}_k^{(\alpha)}| \le C\left(\left((|k|+n^{1/3})/n\right)^2 + n^{-1}\log^4 n + N^{-7/2} + n^{-1/2}(N/n)^2\right). \tag{3.28}$$

Using this bound in (3.14) we obtain (2.12), but the bound for $r_k^{(\alpha)}$ now has the form

$$|r_k^{(\alpha)}| \le C\left((k/n)^2 + n^{-1}\log^4 n + N^{-7/2} + n^{-1/2}(N/n)^2\right) \tag{3.29}$$

Now, using (2.12) with (3.29) in (3.26), and setting $N = 2[n^{1/2}]$ we obtain the bound from (2.12) for $|k| \le n^{1/2}$. Then, setting $N = 2[n^{3/4}]$ and again using (2.12) with (3.29) in (3.26), we obtain the bound from (2.12) for $n^{1/2} < k \le n^{3/4}$. And finally setting $N = 2[\tilde{\varepsilon}n]$, we obtain the bound from (2.12) for $n^{3/4} < k \le \tilde{\varepsilon}n$.□

Proof of Lemma 4 The relation (2.22) is proved in [16]. To prove (2.23) we need some extra definitions. We denote by $\mathcal{H} = l_2(-\infty,\infty)$ a Hilbert space of all infinite sequences $\{x_i\}_{i=-\infty}^{\infty}$ with a standard scalar product $(.,.)$ and a norm $||.||$. Let also $\{e_i\}_{i=-\infty}^{\infty}$ be a standard basis in $\mathcal{H}$ and $I^{(-\infty,n)}$ be an orthogonal projection operator defined as

$$I^{(-\infty,n)} e_i = \begin{cases} e_i, & i<n, \\ 0, & otherwise. \end{cases} \tag{3.30}$$

For any infinite matrix $\mathcal{A} = \{A_{i,j}\}$ we will denote by

$$\begin{aligned} &\mathcal{A}^{(-\infty,n)} = I^{(-\infty,n)}\mathcal{A}I^{(-\infty,n)}, \\ &(\mathcal{A}^{(-\infty,n)})^{-1} = I^{(-\infty,n)}\left(I - I^{(-\infty,n)} + \mathcal{A}^{(-\infty,n)}\right)^{-1} I^{(-\infty,n)}, \end{aligned} \tag{3.31}$$

so that $(\mathcal{A}^{(-\infty,n)})^{-1}$ is a block operator which is inverse to $\mathcal{A}^{(-\infty,n)}$ in the space $I^{(-\infty,n)}\mathcal{H}$ and zero on the $(I - I^{(-\infty,n)})\mathcal{H}$.

Besides, we will say that the matrix $\mathcal{A}^{(-\infty,n)}$ *is of the exponential type*, if there exist constants C and c, such that

$$|A_{n-j,n-k}| \le Ce^{-c(|j|+|k|)}. \qquad (3.32)$$

Define infinite Toeplitz matrices $\mathcal{P}$ and $\mathcal{V}^*$ by their entries

$$P_{j,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(j-k)x}dxP(2\cos x), \quad V^*_{j,k} = \frac{\mathrm{sign}(k-j)}{2\pi}\int_{-\pi}^{\pi} e^{i(j-k)x}dxV'(2\cos x), \qquad (3.33)$$

and let the entries $\mathcal{R}$ be defined in (2.14). Then it is proved in [16] that for $|j|,|k| \le 2\log^2 n$

$$(\mathcal{M}^{(0,n)})^{-1}_{n-j,n-k} = (\mathcal{R}^{(-\infty,n)})^{-1}\mathcal{D}^{(-\infty,n)})_{n-j,n-k} + b_{n-j}a_{n-k} + O(n^{-1/10}), \qquad (3.34)$$

where

$$a_k = ((\mathcal{R}^{(,n)})^{-1}e_{n-1})_k, \quad b_j = ((\mathcal{R}^{(-\infty,n)})^{-1}r^*)_j,$$

and the vector $r^* \in \mathcal{I}^{(0,n)}\mathcal{H}$ has components $r^*_{n-i} = R_i$ $(i = 2,4,\dots)$ with R_i defined by (2.14) Let us prove that

$$\mathcal{F}^{(-\infty,n)} := (\mathcal{R}^{(-\infty,n)})^{-1}\mathcal{D}^{(-\infty,n)} - \mathcal{V}^{*(-\infty,n)} \qquad (3.35)$$

is of the first type. It is proved in [16] (see Proposition 1) that

$$\begin{aligned} &|\mathcal{R}^{-1}_{n-j,n-k}| \le Ce^{-c|j-k|} \\ &|(\mathcal{R}^{(-\infty,n)})^{-1}_{n-j,n-k} - \mathcal{R}^{-1}_{n-j,n-k}| \le C\min\{e^{-c|j|}; e^{-c|k|}\} \le Ce^{-c(|j|+|k|)/2}. \end{aligned} \qquad (3.36)$$

Hence,

$$\begin{aligned} |\mathcal{F}^{(-\infty,n)}_{n-j,n-k}| &\le \Big|\sum_{l\ge1}\mathcal{P}_{n-j,n}\mathcal{D}_{n-l,n-k} - \mathcal{V}^*_{n-j,n-k}\Big| + Ce^{-c|j|}\sum_{l\ge1}e^{-c|l|}e^{-c|l-k|} \\ &\le \Big|\sum_{l\ge0}\mathcal{P}_{n-j,n}\delta_{k,1}\Big| + C'e^{-c(|j|+|k|)/2} \le C_1e^{-c(|j|+|k|)/2}. \end{aligned}$$

Besides, (3.36) imply

$$|a_k| \le Ce^{-c|k|}, \quad |b_j| \le Ce^{-c|j|}. \qquad (3.37)$$

It is easy to see that

$$-\frac{1}{2}\sum_k \mathcal{V}^{(n)}_{k,j}\epsilon\psi^{(n)}_k = \frac{1}{n}(\epsilon\psi^{(n)}_j)' = \frac{1}{n}\psi^{(n)}_j,$$

where we denote $\mathcal{V}_{j,k} = \mathrm{sign}(k-j)V'(J^{(n)})_{j,k}$, and that for $j,k \ge 2\log^2 n$

$$(\mathcal{M}^{(-\infty,n)})^{-1}_{n-j,n-k} = \mathcal{V}_{n-j,n-k} + O(e^{-c\log^2 n}).$$

Hence, if we denote

$$A^{(n)}_{j,k} = (\mathcal{M}^{(-\infty,n)})^{-1}_{n-j,n-k} - \mathcal{V}_{n-j,n-k}, \quad A_{j,k} = \mathcal{F}^{(0,n)})_{n-j,n-k} + b_{n-j}a_{n-k},$$

then S_n is indeed represented in the form (1.23),(2.22) is valid because of (2.12) and (3.34), and (2.23) is valid because we have proved that $\mathcal{F}^{(0,n)}$ is of the first type and because of (3.37).

□

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