On Asymptotic Properties of Certain Orthogonal Polynomials.

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Abstract

We compute the asymptotic distribution of zeros and the weak limit of orthogonal polynomials on the whole line whose weight contains a large parameter in the exponent. The techniques used and the results are motivated by recent studies on the eigenvalue statistics of random matrices.

1 Introduction and Main Results

The study of asymptotic properties of orthogonal polynomials is a branch of analysis which goes back to classics, has numerous links with various areas of mathematics and related fields, and which is still actively developing, especially for the case of polynomials that are orthogonal on the whole real axis (see e.g. the books [1-6] and references therein).

One newer useful link is with the theory of random matrices where orthogonal polynomials provide a powerful tool for the study of the eigenvalue statistics of random Hermitian matrices whose probability distribution has the form

$$p_n(M)dM = Z_n^{-1} \exp\{-n\mathrm{Tr}V(M)\}dM \tag{1.1}$$

where M is a $n \times n$ Hermitian matrix,

$$dM = \prod_{j=1}^{n} dM_{jj} \prod_{j<k} d\Im M_{jk} d\Re M_{jk}$$

is the "Lebesgue" measure for Hermitian matrices, the symbols $\Re z$ and $\Im z$ denote the real and imaginary parts of z, Z_n is the normalization factor and $V(\lambda)$ is a real valued function (see the Theorems below for explicit conditions and [7] for the physical motivation of (1.1)).

We denote by $p_n(\lambda_1, ..., \lambda_n)$ the joint eigenvalue probability density, which we assume to be symmetric without loss of generality. From random matrix theory [8] it is known that

$$p_n(\lambda_1, ...\lambda_n) = Q_n^{-1} \prod_{1\le j<k\le n} (\lambda_j - \lambda_k)^2 \exp\{-n \sum_{j=1}^{n} V(\lambda_j)\}. \tag{1.2}$$

where Q_n is the respective normalization factor. Let

$$p_k^{(n)}(\lambda_1,...,\lambda_k) = \int p_n(\lambda_1,...,\lambda_k,\lambda_{k+1},...\lambda_n)d\lambda_{k+1}...d\lambda_n \tag{1.3}$$

be the k-th marginal distribution density of (1.2). The link with orthogonal polynomials is provided by the formula (see [8])

$$p_k^{(n)}(\lambda_1,...,\lambda_l) = \frac{(n-k)!}{n!}\det||k_n(\lambda_i,\lambda_j)||_{j,k=1}^{l} \tag{1.4}$$

where

$$k_n(\lambda,\mu) = \sum_{l=0}^{n}\psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu) \tag{1.5}$$

is the reproducing kernel for the orthonormalized system

$$\psi_l^{(n)}(\lambda) = \exp\{-nV(\lambda)/2\}P_{l-1}^{(n)}(\lambda),\ l=0,1,... \tag{1.6}$$

and $P_l^{(n)}(\lambda), l=0,1,...$ are orthogonal polynomials on $\mathbf{R}$ associated with the weight

$$w_n(\lambda) = e^{-nV(\lambda)}, \tag{1.7}$$

i.e.

$$\int P_l^{(n)}(\lambda)P_m^{(n)}(\lambda)w_n(\lambda)d\lambda = \delta_{l,m}. \tag{1.8}$$

Our goal is to study asymptotic properties of the polynomials $P_n^{(n)}(\lambda)$ and related quantities.

Let us notice, that the weight (1.7) has an unusual form from the point of view of the traditional theory of orthogonal polynomials associated with weights on the whole $\mathbf{R}$. Indeed, in that theory the study of asymptotic properties of the orthogonal polynomial P_n for $n\to\infty$ is carried out for a weight of the form

$$w(x) = e^{-Q(x)} \tag{1.9}$$

that does not contain the large parameter n.[1] In this case most nontrivial asymptotic properties of $P_n(x)$ manifest themselves for $x = O(a_n)$ where $a_n\to\infty$ as $n\to\infty$ (a_n are known as the Mhaskar-Rakhmanov-Saff numbers, e.g. $a_n = const\sqrt{n}$ for the Hermite polynomials corresponding to $Q(x) = x^2$). Therefore the parts of $Q(x)$ in (1.9) that contribute to the asymptotic behaviour of $P_n(x)$ are the "tails" of $Q(x)$ and that is why one needs to impose certain regularity conditions on the tails behaviour, requiring roughly their power law form (see e.g. [2]). In our case of (1.7) it suffices to consider $\lambda = O(1)$ as $n\to\infty$ and no conditions on the tails $V(\lambda)$ are needed in order to study the asymptotic properties of $P_n^{(n)}$. Therefore the mathematical mechanisms which determine the asymptotic properties of the orthogonal polynomials associated with (1.7) and (1.9) are different, especially for "strictly" nonpower law (e.g. nonconcave) V's.

There is however a class of weights that belongs to both cases and for which we can reduce (1.7) to (1.9) and vice versa. This class consists of monomial weights

$$Q(x) = |x|^\alpha, \qquad V(\lambda) = |\lambda|^\alpha, \qquad \alpha > 0 \tag{1.10}$$

. In this case the rescaling

$$n^{1/\alpha}\lambda = x \tag{1.11}$$

transforms (1.9) in (1.7) and gives the simple correspondence

$$P_l^{(n)}(\lambda) = P_l(\lambda n^{1/\alpha})n^{1/2\alpha}$$

[1] See, however, the book [5], devoted to polynomial approximations with weights of the form (1.7)

between the orthogonal polynomials associated with the weights (1.9) and (1.7).

We shall now return back to the discussion of random matrices, in particular, to formula (1.4).

The simplest case of $p_n^{(1)}(\lambda_i)$ is already of considerable interest. Indeed, if $\lambda_1^{(n)},...,\lambda_n^{(n)}$ are eigenvalues of a random Hermitian matrix M, then N_n defined by

$$N_n(\Delta) = \frac{1}{n}\sum_{j=1}^{n}\chi_\Delta(\lambda_j^{(n)}), \qquad \Delta=(a,b), \qquad a\le b, \tag{1.12}$$

is their normalized counting function (empirical eigenvalue distribution), and

$$E\{N_n(\Delta)\} = \int_\Delta p_1^{(n)}(\lambda)d\lambda \equiv \int_\Delta \rho_n(\lambda)d\lambda \tag{1.13}$$

where according to (1.4) and (1.5)

$$\rho_n(\lambda) = \frac{1}{n}k_n(\lambda,\lambda) = \frac{1}{n}\sum_{j=0}^{n-1}[\psi_j^{(n)}(\lambda)]^2. \tag{1.14}$$

Here and below the function $\chi_\Delta(\lambda)$ is the indicator of the interval Δ and the symbol $E\{...\}$ denotes the expectation with respect to the probability measure (1.1).

The function

$$\Lambda(\lambda) = [k_n(\lambda,\lambda)]^{-1}$$

is known in the theory of orthogonal polynomials as the Christoffel function.

In the recent paper [9] it was proved that if $V(\lambda)$ is bounded from below for all $\lambda\in\mathbf{R}$ and satisfies the conditions

$$|V(\lambda)| \ge (2+\epsilon)\log|\lambda|,\ |\lambda|\ge L_1 \tag{1.15}$$

for some L_1, $\epsilon>0$, and

$$|V(\lambda_1)-V(\lambda_2)| \le C(L_2)|\lambda_1-\lambda_2|^\gamma,\ \ |\lambda_{1,2}|\le L_2 \tag{1.16}$$

for any $0<L_2<\infty$ and some $\gamma>0$, then:

(i) $\rho_n(\lambda)$ converges to the limiting density $\rho(\lambda)$ (called the density of states of the random matrix ensemble (1.1)) in the Hilbert space defined by the energy norm

$$\left(-\int \log|\lambda-\mu|\rho(\lambda)\rho(\mu)d\lambda d\mu\right)^{1/2}; \tag{1.17}$$

(ii) $\rho(\lambda)$ can be found either as the unique solution of the equation

$$\mathrm{supp}\rho \in \{\lambda : u(\lambda) = \max_\mu u(\mu)\} \tag{1.18}$$

where

$$u(\lambda) = 2\int d\mu\rho(\mu)\log|\lambda-\mu| - V(\lambda), \tag{1.19}$$

or as the density of the unique minimum of the functional (energy) defined on the set of all probability measures ν by the formula

$$U(\nu) = -\int \nu(d\lambda)\nu(d\mu))\log|\lambda-\mu| - \int \nu(d\lambda)V(\lambda). \tag{1.20}$$

Remark. Let us note that equation (1.18) after simple transformations gives us singular integral equation

$$\int\frac{2\rho(\mu)d\mu}{\lambda-\mu} = V'(\lambda) \qquad \lambda\in\mathrm{supp}\rho$$

If we know the support of ρ,to find it we can use well-known inversial formulae of the singular integral equations theory [11]. Unfortunately, information about the support of ρ cannot be obtained easily from equations (1.18)- (1.20). We can say only that the number of intervals of the support is not more then the number of extremal points of function V. To find the endpoints of these intervals we need as a rool to solve the system of nonalgebraic equations.

Theorem 1 *Let function $V(\lambda)$ satisfy condition (1.15), σ be a support of the limiting distribution $\rho(\lambda)$ and one of the following conditions be fulfilled (i) σ consists only of one interval (a,b) (ii) $V(\lambda)$ is an even function and the support σ consists of two intervals $\sigma=(a,b)\cup(-b,-a)$. We assume also that function $V(\lambda)$ has an analytical continuation into some open domain $\mathbf{D}\subset\mathbf{C}$ ($\sigma\subset\mathbf{D}$) and $\rho(\lambda)$ can be represented in the form*

$$\rho(\lambda)=P(\lambda)X(\lambda),\quad X(\lambda)=\begin{cases}\chi_\sigma(\lambda)\sqrt{(b-\lambda)(\lambda-a)}, & \textit{in the case (i)}\\ \chi_\sigma(\lambda)\sqrt{(\lambda^2-a^2)(b^2-\lambda^2)}, & \textit{in the case (ii)},\end{cases}\tag{1.21}$$

where $\chi_\sigma(\lambda)$ is the indicator of σ and the function $P(\lambda)$ is some function, which has no zeros in $\overline{\sigma}$.

Then, for any natural namber m, taking $M_1=o(n^{1/m})$ and $M_1>>\log^2 n$, one can find coefficients $J^{(0)}_{k,k+1},...,J^{(m)}_{k,k+1}$ ($n-M_1\le k\le n+M_1$) and functions $g^{(0)}(z),...,g^{(m)}(z)$ such that for for any $n-M_1\le k\le n+M_1$

$$n^m|J_{k,k+1}-\sum_{j=0}^m n^{-j}J^{(j)}_{k,k+1}|\le\varepsilon_n,\quad(\varepsilon_n\to0,\quad n\to\infty).\tag{1.22}$$

and

$$n^m|g_n(z)-\sum_{j=0}^m n^{-j}g^{(j)}(z)|\to0,\quad(n\to\infty)\tag{1.23}$$

uniformly in any set $\{z:\tilde\delta(z)\ge\tilde d\}$. Here we denote $\tilde\delta(z)$ the distance from the point z to the support σ. In particular,

$$g^{(0)}(z)=g(z),\quad g^{(1)}(z)=0\tag{1.24}$$

in the case (i)

$$J^{(0)}_{k,k+1}=\frac14(b-a),\quad J^{(1)}_{k,k+1}=\frac{2(k-n)}{b-a}[(\frac{1}{P(b)}+\frac{1}{P(a)})+\frac{2(b+a)}{b-a}(\frac{1}{P(b)}-\frac{1}{P(a)})]\tag{1.25}$$

and in the case (ii)

$$\begin{aligned}&J^{(0)}_{2k,2k+1}=a_1, && J^{(0)}_{2k,2k-1}=b_1\\ &a_1J^{(1)}_{2k,2k-1}=\frac{2k-n-1}{(a^2-b^2)}(\frac{a^2}{P(a)}-\frac{b^2}{P(b)}) && b_1J^{(1)}_{2k,2k-1}=\frac{2k-n}{(a^2-b^2)}(\frac{a^2}{P(a)}-\frac{b^2}{P(b)}),\end{aligned}\tag{1.26}$$

where

$$a_1=\frac12|a+b|,\ b_1=\frac12|a-b|\quad\textit{or}\quad a_1=\frac12|a-b|,\ b_1=\frac12|a+b|.$$

2 Proof of the Main Result

Proof of Theorem 1

In addition to distribution (1.2) consider k-dimensional distributions with the densities of the form

$$p_{k,n}(\lambda_1,...\lambda_k)=Z_{k,n}^{-1}\prod_{1\le j<m\le k}(\lambda_j-\lambda_m)^2\exp\{-n\sum_{j=1}^k V(\lambda_j)\},\tag{2.1}$$

where k assumes integer values and $Z_{k,n}$ is a normalizing factor. Let

$$\begin{aligned}\tilde\rho_{k,n}(\lambda_1)&=\int d\lambda_2...d\lambda_k p_{k,n}(\lambda_1,...\lambda_k)\\ \tilde\rho_{k,n}(\lambda_1,\lambda_2)&=\int d\lambda_3...d\lambda_k p_{k,n}(\lambda_1,...\lambda_k)\end{aligned}\tag{2.2}$$

be the first and the second marginal densities of (2.1). Let us take any z with $\Im z\neq 0$ and integrate by parts the expression

$$\int\frac{V'(\lambda)\tilde\rho_{k,n}(\lambda)}{z-\lambda}d\lambda=\frac{1}{n}\int\frac{\tilde\rho_{k,n}(\lambda)}{(z-\lambda)^2}d\lambda+2\frac{k-1}{n}\int\frac{\tilde\rho_{k,n}(\lambda,\mu)}{(z-\lambda)(\lambda-\mu)}d\lambda d\mu.\tag{2.3}$$

By using the simple identity, based on the symmetry $\tilde\rho_{k,n}(\lambda,\mu)=\tilde\rho_{k,n}(\mu,\lambda)$:

$$\int\frac{\tilde\rho_{k,n}(\lambda,\mu)}{(z-\lambda)(\lambda-\mu)}d\lambda d\mu=-\int\frac{\tilde\rho_{k,n}(\lambda,\mu)}{(z-\mu)(\lambda-\mu)}d\lambda d\mu,$$

we get from (2.3)

$$\int\frac{V'(\lambda)\tilde\rho_{k,n}(\lambda)}{z-\lambda}d\lambda=\frac{1}{n}\int\frac{\tilde\rho(\lambda)}{(z-\lambda)^2}d\lambda+\frac{k-1}{n}\int\frac{\tilde\rho_{k,n}(\lambda,\mu)}{(z-\lambda)(z-\mu)}d\lambda d\mu.\tag{2.4}$$

According to standard representation (see [8]), we get (cf. (1.4))

$$\tilde\rho_{k,n}(\lambda)=\tilde K_{k,n}(\lambda,\lambda),\quad \tilde\rho_{k,n}(\lambda,\mu)=\frac{k}{k-1}\det||\tilde K_{k,n}(\lambda_i,\lambda_j)||_{j,k=1}^2,\tag{2.5}$$

where

$$\tilde K_{k,n}(\lambda,\mu)=k^{-1}\sum_{l=1}^k\psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu)\tag{2.6}$$

with $\psi_l^{(n)}(\lambda)$ defined by (1.6). Let us introduce the notations

$$\begin{aligned}K_{k,n}(\lambda,\mu)&\equiv n^{-1}\sum_{l=1}^k\psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu)=\frac{k}{n}\tilde K_{k,n}(\lambda,\mu),\\ \rho_{k,n}(\lambda)&\equiv K_{k,n}(\lambda,\lambda)=\frac{k}{n}\tilde\rho_{k,n}(\lambda).\end{aligned}\tag{2.7}$$

Then, using (2.5)-(2.7), we can rewrite equation (2.4) as

$$\int\frac{V'(\lambda)\rho_{k,n}(\lambda)}{z-\lambda}d\lambda=n^{-1}\int\frac{\rho_{k,n}(\lambda)}{(z-\lambda)^2}d\lambda+\int\frac{\rho_{k,n}(\lambda)\rho_{k,n}(\lambda)-(K_{k,n}(\lambda,\mu))^2}{(z-\lambda)(z-\mu)}d\lambda d\mu.\tag{2.8}$$

We use also the result [13], which implies, in particular, that, choosing any n-independent $\varepsilon>0$ and denoting by $\sigma_\varepsilon\subset\mathbf{R}$ the ε-neigborhood of σ, we have

$$\int_{\mathbf{R}\setminus\sigma_\varepsilon}\rho(\lambda)d\lambda\le e^{-nC(\varepsilon)},\quad \int_{\mathbf{R}\setminus\sigma_\varepsilon}(\psi_{k,n}(\lambda))^2d\lambda\le e^{-nC(\varepsilon)},\tag{2.9}$$

with some positive $C(\varepsilon)$. Let us fix ε smal enough, that $\sigma_\varepsilon\subset\mathbf{D}$. Then (2.9) allows us to replace in (2.8) integrals in the whole line by integrals over σ_ε, Therefore, denoting

$$g_{k,n}(z)\equiv\int_{\sigma_\varepsilon}\frac{\rho_{k,n}(\lambda)d\lambda}{z-\lambda},\quad R_{j,m}(z)\equiv\int_{\sigma_\varepsilon}\frac{\psi_j(\lambda)\psi_m(\lambda)d\lambda}{z-\lambda},\quad R'(z)_{j,m}\equiv-\int_{\sigma_\varepsilon}\frac{\psi_j(\lambda)\psi_m(\lambda)d\lambda}{(z-\lambda)^2}$$
$$\tilde V(z,\zeta)=\frac{V'(\zeta)}{z-\zeta}, \tag{2.10}$$

we get from (2.8):

$$(g_{k,n}(z))^2-\int\tilde V(z,\lambda)\rho_{k,n}(\lambda)d\lambda-\frac{1}{n^2}\sum_{m=1}^{k}R'(z)_{m,m}-\frac{1}{n^2}\sum_{m,j=1}^{k}R^2_{m,j}(z)=e_n(z). \tag{2.11}$$

Here and below we denote by $e_n(z)$ the sequence of functions (may be different in different formulae) which are analytic everywhere in $\mathbf{C}\setminus\sigma_\varepsilon$ and satisfy the estimates

$$e_n(z)\le\frac{C_1e^{-nC_2}}{|\delta(z)|^l}, \tag{2.12}$$

where $\delta(z)$ is the distanse from z to σ_ε, l is some positive integer number and $C_{1,2}$ are some positive n-independent constants.

Let us note, that $\tilde V(z,\zeta)$ in (2.10) is an analytic with respect to ζ inside $\mathbf{D}$, except the point $\zeta=z$. So one can write

$$\int\tilde V(z,\lambda)\rho_{k,n}(\lambda)d\lambda=\frac{1}{2\pi i}\int d\lambda\int_L d\zeta\tilde V(z,\zeta)\frac{\rho_{k,n}(\lambda)}{\zeta-\lambda}=\int_L d\zeta\tilde V(z,\zeta)g_{k,n}(\zeta), \tag{2.13}$$

where $L\subset\mathbf{D}$ is an arbitrary closed contour, which contains σ_ε and does not contain z. Then we can rewrite (2.11) as

$$(g_{k,n}(z))^2-\frac{1}{2\pi i}\int_L\tilde V(z,\zeta)g_{k,n}(\zeta)d\zeta-\frac{1}{n^2}\sum_{m=1}^{k}R'(z)_{m,m}-\frac{1}{n^2}\sum_{m,j=1}^{k}R^2_{m,j}(z)=0. \tag{2.14}$$

Now, subtracting the $(k-1)$th equation from the kth one we obtain:

$$2R_{k,k}(z)g_{k-1,n}(z)-\frac{1}{2\pi i}\int_L\tilde V(z,\zeta)R_{k,k}(\zeta)d\zeta-\frac{1}{n}R'_{k,k}(z)-\frac{2}{n}\sum_{j=1}^{k-1}R^2_{k,j}(z)=e_n(z). \tag{2.15}$$

We consider (2.14) and (2.15) as a system of equations with respect to the functions $g_{k,n}(z)$, $R_{j,m}(z)$ and $R'_{k,k}(z)$ and solve them by the methods of the perturbation theory.

It was proved in [10], that

$$\frac{1}{n}|\sum_{j=1}^{k}[-R'(z)_{j,j}-\sum_{m=1}^{k}R^2_{j,m}(z)]|\le\frac{\mathrm{const}}{n\delta^4(z)}+|e_n(z)|. \tag{2.16}$$

Thus, it follows from (2.14) that for any $k=n(1+o(1))$ there exists the limit

$$\lim_{n\to\infty}g_{k,n}(z)=g(z), \tag{2.17}$$

where $g(z)$ in $\mathbf{D}$ is the solution of the equation

$$g^2(z)-V'(z)g(z)+Q(z)=0, \tag{2.18}$$

with

$$Q(z)=\int Q(z,\lambda)\rho(\lambda)d\lambda, \quad Q(z,\zeta)\equiv\frac{V'(z)-V'(\zeta)}{z-\zeta}. \tag{2.19}$$

Here $\rho(\lambda)$ is the limiting density, which has the Stilties transform $g(z)$.

Thus, zero order terms for functions $g_{k,n}(z)$ are equal to $g(z)$.

We use also

Proposition 1 *Under conditions of the theorem the Stilties transorm $g(z)$ of the limiting density ρ for $z\in\mathbf{D}$ can be represented in the form*

$$g(z)=\frac{1}{2}(V'(z)+X(z)P(z)), \tag{2.20}$$

with

$$X(z)=\begin{cases}\sqrt{(z-a)(z-b)}, & \textit{in the case (i)}\\ \sqrt{(z^2-a^2)(z^2-b^2)}, & \textit{in the case (ii)}\end{cases} \tag{2.21}$$

where we take the branches of the square root, which are analytic everywhere except σ and have the asymptotic $X(z)=z^p(1+O(z^{-1}))$ with $p=1,2$ for the one- and two- interval cases respectively. $P(z)$ in (2.20) is an analytic in $\mathbf{D}$ (including σ) function, which can be represented in the form

$$P(z)=\frac{1}{2\pi i}\int_L Q(z,\zeta)X^{-1}(\zeta)d\zeta, \tag{2.22}$$

where L is any closed contour, encircling σ

The proof of Proposition 1 is given in the next section.

Remarks

1. It follows from representation (2.20), that $P(\lambda)$, defined by (1.21) coincides with $P(z)$ defined by (2.22) for $z=\lambda\in\sigma$.
2. Starting from this moment we assume, that ε is chosen small enough, that σ_ε does not contain zeros of the function $P(z)$.

To find the zero order term for $R_{k,k}(z)$, let us note that

$$|R'_{k,k}(z)|, |\sum_{j=1}^{k-1}R^2_{n+k+1,j}(z)|\le\frac{\text{const}}{\delta^2(z)},$$

where $\delta(z)$ is defined in (2.12). Therefore, the last two terms in the l.h.s. of (2.15) have the order n^{-1} and the zero order equations for $R_{kk}(z)$ have the form

$$2g(z)R_{kk}(z)=\frac{1}{2\pi i}\int_L d\zeta\tilde{V}(z,\zeta)R_{kk}(\zeta)-r^{(0,R)}_{k,n}(z)+e_n(z), \tag{2.23}$$

where the remainder function

$$r^{(0,R)}_{k,n}(z)\equiv-\frac{1}{n}R'_{k,k}(z)-\frac{2}{n}\sum_{j=1}^{k-1}R^2_{k,j}(z)+2R_{kk}(z)(g_{k-1,n}(z)-g(z))$$

is an analytic in $\mathbf{C}\setminus\sigma_\varepsilon$ function, which satisfies the bound

$$|r^{(0,R)}_{k,n}(z)|\le\frac{\text{const}}{n\delta^2(z)}. \tag{2.24}$$

Besides, since by definition

$$\int (\psi_k(\lambda))^2 d\lambda = 1,$$

we have from (2.9), that

$$R_{kk}(z) \sim \frac{1}{z} + e_n(z), \quad z \to \infty. \tag{2.25}$$

The solution of (2.23) was already found in [12]. But since, as usually in the perturbation theory, equations of such form appear at the each step of the expansion procedure, we use here a bit different way to analyse the equation, which is based on the following lemma:

Lemma 1 *Consider the equation*

$$2g(z)R(z) = \frac{1}{2\pi i}\int_L d\zeta \tilde{V}(z,\zeta)R(\zeta), \quad z \in \mathbf{D}\setminus\sigma_\varepsilon \tag{2.26}$$

with closed contour L, that contains σ_ε and does not contain point z.

Under condition of the theorem equation (2.26) has in the case (i) only one solution $R(z) = \Psi(z)$ in the class of functions which have an analitical continuation in $\mathbf{C}\setminus\sigma_\varepsilon$ and behave like $\frac{1}{z}$ as $z \to \infty$. In the case (ii) it has also only one solution $R(z) = \Psi(z)$, if we add an additional symmetry contition $R(-z) = -R(z)$. Here and below

$$\Psi(z) = \begin{cases} X^{-1}(z), & \text{in the case (i)}, \\ & \qquad\qquad (z \notin \sigma), \\ zX^{-1}(z), & \text{in the case (ii)}, \end{cases} \tag{2.27}$$

where $X(z)$ is defined by (2.21). For the case (ii) equation (2.26) has the unique solution $R(z) = \Psi(z)$ in the same class of functions, if we put the additional symmetry condition $R(z) = -R(-z)$.

In both cases equation (2.26) has no solutions in the class of functions $R(z)$ analytic everywhere except σ and satisfying conditions

$$\lim_{|z|\to\infty} |z^2 R(z)| \le \text{ const } < \infty. \tag{2.28}$$

. For any analytic $\mathbf{C}\setminus\sigma_\varepsilon$ function $F(z)$, satisfying condition (2.28), the equation

$$2g(z)R(z) = \frac{1}{2\pi i}\int_L d\zeta \tilde{V}(z,\zeta)R(\zeta) - F(z) \tag{2.29}$$

has a unique solution of the form

$$R(z) = \frac{X^{-1}(z)}{2\pi i}\int_L d\zeta \frac{F(\zeta)}{P(\zeta)(z-\zeta)}, \tag{2.30}$$

where $P(z)$ is defined by (2.22) and a closed contour L should be taken close enough to σ, so that z and all zeros of $P(z)$ are outside of L.

On the basis of this lemma, we obtain

$$R_{k,k}(z) = \Psi(z) + \tilde{r}^{(0,R)}_{k,n}(z) + e_n(z), \tag{2.31}$$

where $\tilde{r}^{(0,R)}_{k,n}(z)$ is obtained by formula (2.30) with $F(z) = r^{(0,R)}_{k,n}(z)$ and therefore admits the bound

$$|\tilde{r}^{(0,R)}_{k,n}(z)| \le \frac{\text{const}}{\delta^2(z)} \max_{\{\zeta:\delta(\zeta)=\delta(z)/2\}} |r^{(0,R)}_{k,n}(\zeta)| \le \frac{\text{const}}{n\delta^4(z)}. \tag{2.32}$$

Thus,

$$R^{(0)}_{k,k} \equiv \lim_{n\to\infty} R_{k,k}(z) = \Psi(z). \tag{2.33}$$

Then, as it was shown in [12], for any $k = n(1 + o(1))$

$$\lim_{n\to\infty} J_{k,k+1} = J^{(0)}_{k,k+1},$$

where in the case (i) $J^{(0)}$ is the symmetric Jacoby matrix with constant coefficients (1.25), and in the case (ii) it is the symmetric two-periodic Jacobi matrix with the coefficients (1.26). Thus, we obtain zero approximation for the coefficients $J_{k,k+1}$ for $k = n(1 + o(1))$.

To get the first order terms for these coefficients, let us study the first order terms of equations (2.14). By the virtue of (2.16) we conclude that the first order equation for the function

$$g^{(1)}_{k,n}(z) \equiv n(g_{k,n}(z) - g(z)) \tag{2.34}$$

has the form

$$2g(z)g^{(1)}_{k,n}(z) = \frac{1}{2\pi i}\int V(z,\zeta)g^{(1)}_{n,k}(\zeta)d\zeta - r^{(1,g)}_{k,n}(z) + e_n(z), \tag{2.35}$$

with

$$r^{(1,g)}_{k,n}(z) \equiv \frac{1}{n}(g^{(1)}_{n,k}(z))^2\frac{1}{n}\sum_{j=1}^{k}[-R'(z)_{j,j} - \sum_{m=1}^{k}(R_{j,m}(z))^2], \quad |r^{(1,g)}_{k,n}(z)| \le \frac{\text{const}}{n\delta^4(z)}$$

and the normalization condition

$$g^{(1)}_{k,n}(z) \sim (k-n)z^{-1} + e_n(z), \tag{2.36}$$

which follows from definition (2.10) of the function $g_{k,n}(z)$. According to Lemma 1, we get then

$$g^{(1)}_{k,n}(z) = (k-n)\Psi(z) + \tilde{r}^{(1,g)}_{k,n}(z) + e_n(z),$$

where the remainder function $\tilde{r}^{(1,g)}_{k,n}(z)$ admits the bound

$$\tilde{r}^{(1,g)}_{k,n}(z) \le \frac{\text{const}}{\delta^2(z)} \max_{\{\zeta:\delta(\zeta)=\delta(z)/2\}} |r^{(1,g)}_{k,n}(\zeta)| \tag{2.37}$$

In other words

$$g^{(1)}_{k}(z) \equiv \lim_{n\to\infty} g^{(1)}_{n,k}(z)) = (k-n)\Psi(z). \tag{2.38}$$

Now we need the lemma, which after p steps of our expansion process allows us to replace in (2.14),(2.15) $R_{k,j}(z)$ by some expression from the coefficients $J^{(l)}_{k,k+1}$ $(l = 0,...p)$ found at the previous steps and to estimate the error of this replacement.

Lemma 2 *Let us take $M = [log^2 n]$ and $M_1 >> M$. Assume that for any $n - M_1 \le k \le n + M_1$ we have found the coefficients $J^{(0)}_{k,k+1}, ..., J^{(p)}_{k,k+1}$ such that (1.22) is fulfilled for $m = p$. Consider $\tilde{J}^{(p)}(s)$-$(2M_1 + 1)$-periodic symmetric Jacobi matrix such that*

$$\tilde{J}^{(p)}_{k,k+1} = \sum_{j=1}^{p} s^j J^{(j)}_{k,k+1} \quad (k = n - M_1, ..., n + M_1) \tag{2.39}$$

and let $\tilde{R}^{(p)}(z,s)$ be a resolvent of $\tilde{J}^{(p)}(s)$. Denote

$$R^{(j)}(z) \equiv \frac{1}{j!}\frac{\partial^j}{\partial s^j}\tilde{R}^{(p)}(z,s)|_{s=0}, \quad S^{(p)}(z) \equiv \sum_{j=1}^{p} n^{-j}R^{(j)}. \tag{2.40}$$

Then there exist some positive n-independent constants C_1 and C_2 such that for any $n-M_1+2M \le k \le n+M_1-2M$ for any $z \notin \sigma_\varepsilon$

$$\begin{gathered}|R_{k,k}(z)-S^{(p)}_{k,k}(z)|,|-R'_{k,k}(z)-(S^{(p)}\cdot S^{(p)})_{k,k}(z)| \le \frac{\varepsilon_n+n^{-1}C_1}{\delta^{p+1}(z)}+\frac{e^{-C_2\delta(z)M}}{\delta^3(z)},\\ |\sum_{m=1}^{k}(R_{k,m}(z))^2-\sum_{m=1}^{k}(S^{(p)}_{k,m}(z))^2| \le \frac{\varepsilon_n+n^{-1}C_1}{\delta^{p+1}(z)n^p}+\frac{e^{-\delta(z)M}}{\delta^3(z)},\\ |\frac{1}{n}\sum_{j=1}^{k}[-R'(z)_{j,j}-\sum_{m=1}^{k}(R_{j,m}(z))^2]-\frac{1}{n}\sum_{j=1}^{k}[(S^{(p)}\cdot S^{(p)})_{j,j}(z)-\sum_{m=1}^{k}(S^{(p)}_{j,m}(z))^2]| \le\\ \frac{M(\varepsilon_n+n^{-1}C_1)}{\delta^{p+1}(z)n^p}+\frac{e^{-\delta(z)M}}{\delta^3(z)},\end{gathered} \tag{2.41}$$

where $\delta(z)$ is ddefined in (2.12).

Consider the function

$$R^{(1,n)}_{k,k}(z) \equiv n(R_{k,k}(z)-R^{(0)}_{k,k}(z)),$$

with $R^{(0)}_{k,k}(z)$ defined in (2.33). ¿From (2.15) we get

$$2g(z)R^{(1,n)}_{k,k}(z)=\frac{1}{2\pi i}\int_L d\zeta\tilde{V}(z,\zeta)R^{(1,n)}_{kk}(\zeta)-F^{(1,R)}_k(z)-r^{(1,R)}_{k,n}(z)+e_n(z), \tag{2.42}$$

where

$$\begin{gathered}F^{(1,R)}_k(z) \equiv 2R^{(0)}_{k,k}(z)g^{(1)}_{k-1}(z)+(R^{(0)}\cdot R^{(0)})_{k,k}(z)-2\sum_{j=1}^{k-1}(R^{(0)}_{k,j}(z))^2=\\ 2R^{(0)}_{k,k}(z)g^{(1)}_{k-1}(z)+(R^{(0)}_{k,k}(z))^2=(2(k-n)+1)\Psi^2(z)^2,\end{gathered}$$

and

$$\begin{gathered}r^{(1,R)}_{k,n}(z) \equiv \frac{1}{n}R_{k,k}(z)\tilde{r}^{(1,g)}_{k,n}(z)+\frac{1}{n}R^{(1,n)}_{k,k}(z)g^{(1)}_{k-1}(z)+[-R'_{k,k}(z)-(R^{(0)}\cdot R^{(0)})_{k,k}(z)]-\\ -2[\sum_{j=1}^{k-1}(R_{k,j}(z))^2-\sum_{j=1}^{k-1}(R^{(0)}_{k,j}(z))^2]\end{gathered}$$

By the vertue of Lemma 2, $r^{(1,R)}_{k,n}(z) \to 0$, as $n \to \infty$. Then, on the basis of Lemma 1, we get for the case (i)

$$R^{(1)}_{k,k}(z)=\frac{2(k-n)+1}{X(z)(b-a)}(\frac{1}{P(b)(z-b)}-\frac{1}{P(a)(z-a)})+\tilde{r}^{(1,R)}_{k,n}(z), \tag{2.43}$$

and for the case (ii)

$$R^{(1)}_{k,k}(z)=\frac{2(k-n)+1}{(a^2-b^2)X(z)}[\frac{a^2}{P(a)(z^2-a^2)}-\frac{b^2}{P(b)(z^2-b^2)}]+\tilde{r}^{(1,R)}_{k,n}(z), \tag{2.44}$$

where for both cases $\tilde{r}^{(1,R)}_{k,n}(z)$ admits the bound

$$|\tilde{r}^{(1,R)}_{k,n}(z)| \le \frac{\mathrm{const}}{\delta^2(z)}\max_{\{\zeta:\delta(\zeta)=\delta(z)/2\}}|r^{(1,R)}_{k,n}(\zeta)| \tag{2.45}$$

Now, since evidently

$$J^2_{k,k+1}+J^2_{k,k-1}=\int\lambda^2\psi^2_k(\lambda)d\lambda=\frac{1}{2\pi i}\int_L\zeta^2R_{k,k}(\zeta)d\zeta, \tag{2.46}$$

we can get in the first order with respect to n^{-1}

$$2(J^{(0)}_{k,k+1}J^{(1)}_{k,k+1}+J^{(0)}_{k,k-1}J^{(1)}_{k,k-1})=\frac{1}{2\pi i}\int_L \zeta^2 R^{(1)}_{k,k}(\zeta)d\zeta+r_k^{(J,n)},$$

where we denote

$$r_k^{(J,n)}\equiv\int_L \zeta^2\tilde r^{(1,R)}_{k,n}(\zeta)d\zeta\to 0,\quad (n\to\infty).$$

In particular, for the case (i) we get

$$\frac{b-a}{4}(J^{(1)}_{k,k+1}+J^{(1)}_{k,k-1})=(2(k-n)+1)I^{(i)},\quad I^{(i)}\equiv\frac{1}{2}(\frac{1}{P(a)}+\frac{1}{P(b)})+\frac{a+b}{b-a}(\frac{1}{P(b)}-\frac{1}{P(a)})]+r_k^{(J,n)} \quad (2.47)$$

and for the case (ii)

$$a_1J^{(1)}_{2k-1,2k}+b_1J^{(1)}_{2k+1,2k}=(2(2k-n)+1)I^{(ii)},\quad I^{(ii)}\equiv\frac{1}{(a^2-b^2)}(\frac{a^2}{P(a)}-\frac{b^2}{P(b)})+r_k^{(J,n)}. \quad (2.48)$$

with a_1 and b_1 defined in (1.26).

Solving this equations, starting from $k=n$, and expressing the next coefficient through $J_{n,n+1}$, is easy to obtain that both of this system have one-parameter families of solutions

$$\begin{aligned}(i)\quad & \tfrac{b-a}{4}J^{(1)}_{k,k+1}=(k-n)I^{(i)}-c(-1)^{k-n}+\tilde r_k^{(J,n)}\\ (ii)\quad & a_1J^{(1)}_{2k-1,2k}=(2k-1-n)I^{(ii)}+c+\tilde r_{2k-1}^{(J,n)},\\ & b_1J^{(1)}_{2k+1,2k}=(2k-n)I^{(ii)}-c+\tilde r_k^{(J,n)},\end{aligned} \quad (2.49)$$

where

$$\tilde r_k^{(J,n)}=\tilde r_n^{(J,n)}+\textstyle\sum_{j=1}^k(-1)^j r_{n+j}^{(J,n)},\quad (k>n)$$

$$\tilde r_k^{(J,n)}=\tilde r_n^{(J,n)}+\textstyle\sum_{j=1}^k(-1)^j r_{n-j}^{(J,n)},\quad (k<n).$$

To choose the value of the parameter c we use well known in the random matrix theory equation (see [8])

$$J_{k,k+1}\int V'(\lambda)\psi_k(\lambda)\psi_{k+1}(\lambda)d\lambda=\frac{k}{n}.$$

We shall use it in the form

$$\frac{J_{k,k+1}}{2\pi i}\int_L V'(\zeta)R_{k,k+1}(\zeta)d\zeta=\frac{k}{n}+O(e^{-nC}). \quad (2.50)$$

The first order equation, which follows from (2.50) has the form

$$\frac{J^{(0)}_{k,k+1}}{2\pi i}\int_L V'(\zeta)R^{(1)}_{k,k+1}(\zeta)d\zeta+\frac{J^{(1)}_{k,k+1}}{2\pi i}\int_L V'(\zeta)R^{(0)}_{k,k+1}(\zeta)d\zeta=k-n$$

Substituting here the solutions (2.49), we get in the case (i) a linear equation with respect to c of the form

$$D^{(i)}c+A_k^{(i)}=(k-n) \quad (2.51)$$

with

$$\begin{aligned}D_k^{(i)}&\equiv J^{\pm}_{k,k+1}\int_L V'(\zeta)R^{(0)}_{k,k+1}(\zeta)d\zeta+\frac{a}{2}\int_L V'(\zeta)(R^{(0)}(\zeta)J^{\pm}R^{(0)}(\zeta))_{k,k+1}d\zeta,\\ A_k^{(i)}&\equiv J^{(1,0)}_{k,k+1}\int_L V'(\zeta)R^{(0)}_{k,k+1}(\zeta)d\zeta+J^{(0)}_{k,k+1}\int_L V'(\zeta)(R^{(0)}(\zeta)\cdot J^{(1,0)}\cdot R^{(0)}(\zeta))_{k,k+1}d\zeta,\end{aligned} \quad (2.52)$$

where $J^{\pm}$ is the symmetric Jacoby matrix with coefficient $J^{\pm}_{k+1,k} = (-1)^{n-k-1}$ and $J^{(1,0)}$ is the symmetric Jacoby matrix with coefficients defined by (2.49).

In the case (ii) we get the pair of linearly depending (see Proposition 2 below) equations with respect to c

$$\begin{aligned} D^{(ii)}_{2k-1,2k}c + A^{(ii)}_{2k-1,2k} &= (2k-1-n), \\ D^{(ii)}_{2k+1,2k}c + A^{(ii)}_{2k+1,2k} &= (2k-n) \end{aligned} \tag{2.53}$$

where

$$\begin{aligned} D^{(ii)}_{2k+1,2k} &\equiv J^{(1,0)}_{2k-1,2k}\int_L V'(\zeta)R^{(0)}_{2k-1,2k}(\zeta)d\zeta + J^{(0)}_{2k-1,2k}\int_L V'(\zeta)(R^{(0)}(\zeta)J^{\pm}R^{(0)}(\zeta))_{2k-1,2k} \\ D^{(ii)}_{2} &\equiv J^{(1,0)}_{2k+1,2k}\int_L V'(\zeta)R^{(0)}_{2k+1,2k}(\zeta)d\zeta + J^{(0)}_{2k-1,2k}\int_L V'(\zeta)(R^{(0)}(\zeta)J^{\pm}R^{(0)}(\zeta))_{2k+1,2k}d\zeta \\ A^{(i)}_{2k-1,2k} &\equiv J^{(1,0)}_{2k-1,2k}\int_L V'(\zeta)R^{(0)}_{2k-1,2k}(\zeta)d\zeta + J^{(0)}_{2k-1,2k}\int_L V'(\zeta)(R^{(0)}(\zeta)*J^{(1,0)}*R^{(0)}(\zeta))_{2k-1,2k}d\zeta, \\ A^{(i)}_{2k+1,2k} &\equiv J^{(1,0)}_{2k+1,2k}\int_L V'(\zeta)R^{(0)}_{2k+1,2k}(\zeta)d\zeta + J^{(0)}_{2k+1,2k}\int_L V'(\zeta)(R^{(0)}(\zeta)*J^{(1,0)}*R^{(0)}(\zeta))_{2k+1,2k}d\zeta, \end{aligned} \tag{2.54}$$

with $J^{\pm}_{2k-1,2k} = a_1^{-1}$, $J^{\pm}_{2k+1,2k} = b_1^{-1}$ and $J^{(1,0)}$ defined by (2.49) for the case (ii).

One can see easily, that to have the unique parameter c- solution of (2.51) or (2.53) it is sufficient in the case (i) to prove that $D^{(i)} \neq 0$, and in the case (ii) to prove that at least one of values $D^{(ii)}_{2k-1,2k}$, $D^{(ii)}_{2k+1,2k}$ is nonzero.

Proposition 2 *Under conditions of the theorem* $D^{(i)} \neq 0$, $(D^{(ii)}_{2k-1,2k})^2 + (D^{(ii)}_{2k+1,2k})^2 \neq 0$ *and equations (2.53) are linearly depending and have the unique solution* $c = 0$.

The proof of Proposition 2 is given in the next section.

Thus, on the basis of this proposition we find the first order terms of our expansion.

Now we shall prove (1.23) and (1.22) by induction. Assume that we have found coefficients $J^{(0)}_{k,k+1}, ..., J^{(p)}_{k,k+1}$ and functions $g^{(0)}_k(z), ..., g^{(p)}_k(z)$ such that for any $n - M_1 \le k \le n + M_1$ (1.22) is fulfilled with $m = p$ and any $\tilde{d} > 0$

$$\max_{z:\delta(z)\le\tilde{d}} n^p|g_{k,n}(z) - \sum_{j=0}^{p} n^{-j}g^{(j)}_k(z)| \le \varepsilon_n(\tilde{d}), \quad (\varepsilon_n(\tilde{d}) \to 0, \quad as \quad n \to \infty). \tag{2.55}$$

Let matrices $R^{(j)}(z)$ $(j = 0, ..., p)$ be defined as in Lemma 2 (see formulae (2.39), (2.40)).

Then, denoting

$$g^{(p+1)}_{k,n}(z) \equiv n^{p+1}(g_{k,n}(z) - \sum_{j=0}^{p} n^{-j}g^{(j)}_k(z))$$

we get from (2.14) the equation of the form

$$2g(z)g^{(p+1)}_{k,n}(z) = \frac{1}{2\pi i}\int \tilde{V}(z,\zeta)g^{(p+1)}_{k,n}(\zeta)d\zeta - F^{(g,p+1)}_k(z) - r^{(g,p+1)}_{k,n}(z) + e_n(z) \tag{2.56}$$

with

$$F_k^{(g,p+1)}(z)=\sum_{l=1}^{p}g_k^{(p+1-l)}(z)g_k^{(l)}(z)+\sum_{m=1}^{k}\sum_{j=k+1}^{\infty}\sum_{l=0}^{p-1}R_{m,j}^{(p-l-1)}(z)R_{m,j}^{(l)}(z)$$

$$r_{k,n}^{(g,p+1)}(z)=n^{-p-1}(g_{k,n}^{(p+1)}(z))^2+2n^{-1}g_{k,n}^{(p+1)}(z)\sum_{l=1}^{p}n^{-(l-1)}g_k^{(l)}(z)+\sum_{l,l'=1,l+l'>p+1}^{p}n^{p+1-l-l'}g_k^{(l)}(z)g_k^{(l')}(z)+$$
$$n^p\{\frac{1}{n}\sum_{j=1}^{k}[-R'(z)_{j,j}-\sum_{m=1}^{k}(R_{j,m}(z))^2]-\frac{1}{n}\sum_{j=1}^{k}[(S^{(p)}\cdot S^{(p)})_{j,j}(z)-\sum_{m=1}^{k}(S_{j,m}^{(p)}(z))^2]\}. \tag{2.57}$$

where $S_{j,m}^{(p)}(z)$ is defined by (2.40). On the basis of (2.55), (1.22) and Lemma 2, we conclude that

$$r_{k,n}^{(p+1,g)}(z)\to 0,\quad as\quad n\to\infty,$$

uniformly in $\{z:\delta(z)\geq\tilde{d}\}$, for any fixed $\tilde{d}>0$. Then, on the basis of Lemma 1, we conclude

$$g_{k,n}^{(p+1)}(z)=g_k^{(p)}(z)+\tilde{r}_{k,n}^{(p+1,g)}(z), \tag{2.58}$$

with

$$g_k^{(p+1)}(z)=\frac{1}{2\pi i}\int_L\frac{F_k^{(p+1,g)}(\zeta)}{P(\zeta)(\zeta-z)}d\zeta$$

and

$$|\tilde{r}_{k,n}^{(p+1,g)}(z)|\leq\frac{\text{const}}{\delta^2(z)}\max_{\{\zeta:\delta(\zeta)=\delta(z)/2\}}|r_{k,n}^{(p+1,g)}(\zeta)|.$$

Now, denoting

$$R_{k,k}^{(p+1,n)}(z)\equiv n^{p+1}(R_{k,k}(z)-\sum_{j=0}^{p}n^{-j}R_{k,k}^{(j)}(z)),$$

we get from (2.15) the equation of the form

$$2g(z)R_{k,k}^{(p+1,n)}(z)=\frac{1}{2\pi i}\int\tilde{V}(z,\zeta)R_{k,k}^{(p+1)}(\zeta)d\zeta-F_k^{(p+1,R)}(z)-r_{k,n}^{(p+1,R)}(z)+e_n(z) \tag{2.59}$$

with

$$F_k^{(p+1,R)}(z)=\sum_{l=0}^{p}g_{k-1}^{(p+1-l)}(z)R_{k,k}^{(l)}(z)+(\sum_{j=1}^{k}-\sum_{j=k+1}^{\infty})\sum_{l=0}^{p}R_{m,j}^{(p-l)}(z)R_{m,j}^{(l)}(z),$$

$$r_{k,n}^{(p+1,R)}(z)=2n^{-1}R_{k,k}^{(p+1)}(z)\sum_{l=1}^{p}n^{-(l-1)}g_{k-1}^{(l)}(z))+\sum_{l,l'=1,l+l'>p+1}^{p}n^{p+1-l-l'}g_{k-1}^{(l)}(z)R_{k,k}^{(l')}(z)+$$
$$n^{p-1}\{[-R'(z)_{k,k}-2\sum_{m=1}^{k}(R_{k,m}(z))^2]-[(S^{(p)}\cdot S^{(p)})_{j,j}(z)-2\sum_{m=1}^{k}(S_{j,m}^{(p)}(z))^2]\}.$$

By the vertue of (2.55), (1.22) and Lemma 2, we conclude that

$$r_{k,n}^{(p+1,R)}(z)\to 0,\quad as\quad n\to\infty,$$

uniformly in $\{z:\delta(z)\geq\tilde{d}\}$, for any fixed $\tilde{d}>0$. Using again Lemma 1, we get

$$R_{k,k}^{(p+1,n)}(z)=R_{k,k}^{(p+1)}(z)+\tilde{r}_{k,n}^{(p+1,R)}(z), \tag{2.60}$$

with

$$R_{k,k}^{(p+1)}(z) = \frac{1}{2\pi i}\int_L \frac{F_k^{(p+1,R)}(\zeta)}{P(\zeta)(\zeta - z)}d\zeta$$

and

$$|\tilde{r}_{k,n}^{(p+1,R)}(z)| \le \frac{\text{const}}{\delta^2(z)} \max_{\{\zeta:\delta(\zeta)=\delta(z)/2\}} |r_{k,n}^{(p+1,R)}(\zeta)|.$$

Then, from equaton (2.46) we get in the case (i)

$$a(J_{k,k+1}^{(p+1)} + J_{k,k-1}^{(p+1)}) = a_k^{(p+1)} + r_{k,n}^{(p+1)}, \quad r_{k,n}^{(p+1)} \to 0, \tag{2.61}$$

where

$$a_k^{(p+1)} \equiv \frac{1}{2\pi i}\int_L \zeta^2 R_{k,k}^{(p+1)}(\zeta)d\zeta,$$

and in the case (ii)

$$a_1 J_{2k-1,2k}^{(p+1)} + b_1 J_{2k+1,2k}^{(p+1)} = a_k^{(p+1)} + r_{k,n}^{(p+1)}, \quad r_{k,n}^{(p+1)} \to 0, \tag{2.62}$$

Solving this equations, starting from $k = n$, and expressing the next coefficient through $J_{n,n+1}^{(p+1)}$ is easy to obtain, that both of this system have one-parameter families of solutions

$$\begin{aligned} &(i) \quad J_{k,k+1}^{(p+1)} = b_k^{(p+1)} - c(-1)^{k-n} + \tilde{r}_k^{(J,n)} \\ &(ii) \quad a_1 J_{2k-1,2k}^{(p+1)} = b_k^{(p+1)} + c + \tilde{r}_{2k-1}^{(J,n)} \\ &\qquad b_1 J_{2k+1,2k}^{(1)}) = b_k^{(p+1)} - c + \tilde{r}_k^{(J,n)}, \end{aligned} \tag{2.63}$$

where

$$b_k^{(p+1)} = a_n^{(p+1)} + \textstyle\sum_{j=1}^k (-1)^j a_{n+j}^{(p+1)}, \quad \tilde{r}_k^{(J,n)} = \tilde{r}_n^{(J,n)} + \sum_{j=1}^k (-1)^j r_{n+j}^{(J,n)}, \quad (k > n)$$

$$b_k^{(p+1)} = a_n^{(p+1)} + \textstyle\sum_{j=1}^k (-1)^j a_{n+j}^{(p+1)}, \quad \tilde{r}_k^{(J,n)} = \tilde{r}_n^{(J,n)} + \sum_{j=1}^k (-1)^j r_{n-j}^{(J,n)}, \quad (k < n).$$

To choose the value of this parameter we use like for $p = 1$ the analogue of equation (2.50), which due to Proposition 2 gives us the unique value of parametr c. Thus, we have finished the proof of Theorem 1.

3 Auxiliary results

Proof of Proposition 1

It follows from equation (2.18) that in $\mathbf{D}$ $g(z)$ could be written in the form (2.20) with $P(z)$ being an analytical function in $\mathbf{D}$, except σ. Thus $P(z)$ is also analytical in $\mathbf{D}$, except σ. But for $\lambda \in \sigma$, according the theory of singular integration equation (see [11]), we have

$$\rho(\lambda) = \frac{1}{2}\tilde{X}(\lambda)\int_\sigma Q(\lambda,\mu)\tilde{X}^{-1}(\mu),$$

where $X(\lambda)$ is defined by (1.21).

On the other hand, since $g(z)$ is the Stilties transform of $\rho(\lambda)$

$$\rho(\lambda) = -\frac{1}{\pi i}\lim_{\varepsilon\to+0} \Im g(\lambda + i\varepsilon) = \frac{1}{2}\tilde{X}(\lambda)P(\lambda)$$

Thus, for $\lambda \in \sigma$

$$P(\lambda) = \int_\sigma Q(\lambda,\mu)\tilde{X}^{-1}(\mu).$$

Then, according to the uniqueness theorem, the integral in the r.h.s. of formula (2.22) coincides with function $P(z)$. Thus, it is analytical in the whole $\mathbf{D}$.

Proof of Lemma 1

Using Proposition 1, one can rewrite equation (2.26) in $\mathbf{D}$ as follows

$$P(z)X(z)R(z) = \frac{1}{2\pi i}\int_L d\zeta Q(z,\zeta)R(\zeta) \tag{3.1}$$

with $Q(z,\zeta)$ defined by (2.19). Then, denoting $\tilde{Q}(z)$ the r.h.s. of (3.1), we get that $\tilde{Q}(z)$ is an analytic function in $\mathbf{D}$. ¿From equation (3.1) we derive that zeros of $P(z)$ in $\mathbf{D}$ coincides with zeros $\tilde{Q}(z)$ and have the same order. Thus, function $R(z)X(z) = \frac{\tilde{Q}(z)}{P(z)}$ is analytical in $\mathbf{D}$. But in the rest of $\mathbf{C}$ it is analytical, because we are looking for the solutions of such a type. Thus it is analytical in the whole $\mathbf{C}$. Besides, if $R(z) \sim \frac{1}{z}$, as $|z| \to \infty$, then in the case (i) we get that $R(z)X(z)$ is bounded, as $|z| \to \infty$. Therefore due to the generalized Liouville theorem [11], $R(z)X(z)$ is a constant. In the case (ii) we get also from the generalized Liouville theorem, that $R(z)X(z) = az + b$. Then, using the symmetry of the function $R(z)$, we get $R(z) = zX^{-1}(z)$. Similarly, under condition (2.28), we get that in the case (i) $R(z)X(z) \to 0$, as $|z| \to \infty$ and thus $D(z) = 0$. In the case (ii) $R(z)X(z) =$ const and we get $R(z)X(z) = 0$ from the symmetry condition.

To prove that (2.30) is indeed solution of equation (2.29) substitute it in the r.h.s. of (2.29). We take the closed contour L which is outside L_1 and write

$$\begin{aligned}
\frac{1}{2\pi i}\int_L Q(z,\zeta)R(\zeta)d\zeta = \frac{1}{(2\pi i)^2}\int_L d\zeta Q(z,\zeta)X^{-1}(\zeta)\int_L d\zeta_1 \frac{F(\zeta_1)}{P(\zeta_1)(\zeta-\zeta_1)} = \\
\frac{1}{(2\pi i)^2}\int_{L_1} d\zeta_1 \int_L d\zeta \frac{Q(z,\zeta)}{X(\zeta)}\frac{F(\zeta_1)}{P(\zeta_1)(z-\zeta_1)} + \\
\frac{1}{(2\pi i)^2}\int_{L_1} d\zeta_1 \int_L d\zeta \frac{Q(z,\zeta)}{X(\zeta)}\frac{F(\zeta_1)}{P(\zeta_1)}\left(\frac{1}{\zeta-\zeta_1} - \frac{1}{z-\zeta_1}\right) = \\
P(z)X(z)R(z) + \frac{1}{(2\pi i)^2}\int_{L_1} d\zeta_1 \int_L d\zeta \frac{V'(z)-V'(\zeta)}{X(\zeta)(\zeta_1-\zeta)}\frac{F(\zeta_1)}{P(\zeta_1)(\zeta_1-z)} = \\
P(z)X(z)R(z) + \frac{1}{(2\pi i)^2}\int_{L_1} d\zeta_1 \int_L d\zeta \frac{V'(\zeta_1)-V'(\zeta)}{X(\zeta)(\zeta_1-\zeta)}\frac{F(\zeta_1)}{P(\zeta_1)(\zeta_1-z)} = \\
P(z)X(z)R(z) + \frac{1}{2\pi i}\int_{L_1} d\zeta_1 \frac{F(\zeta_1)}{(\zeta_1-z)} = P(z)X(z)R(z) - F(z).
\end{aligned} \tag{3.2}$$

Here we have use representation (2.22) for the function $P(z)$ and the fact that for $\zeta_1 \in L_1$ (i.e. inside L)

$$\int_L \frac{d\zeta}{X(\zeta)(\zeta_1-\zeta)} = 0. \tag{3.3}$$

Uniqueness follows from the absence of solutions of the homogeneous equation (2.26).

Proof of Lemma 2

Consider the "blok" symmetric Jacoby matrix $\dot{J}^{(n,M_1)}$ which can be obtained from J if we put $J_{n-M_1-1,n-M_1} = J_{n+M_1+1,n+M_1+2} = 0$. Let $\dot{R}^{(n,M_1)}(z)$ be its resolvent. We use the resolvent identity valid for any two Jacobi matrices $J^{(1,2)}$ with resolvents $R^{(1,2)})$ respectively.

$$R^{(1)}(z) - R^{(2)}(z) = R^{(1)}(z)(J^{(2)} - J^{(1)})R^{(2)}(z) \tag{3.4}$$

Thus, taking $R^{(1)}(z) = \mathcal{R}(z)$-the resolvent of J, and $R^{(2)}(z) = \dot{R}^{(n,M_1)}(z)$, we obtain

$$\mathcal{R}_{k,j}(z) - \dot{R}^{(n,M_1)}_{k,j}(z) = \dot{R}^{(n,M_1)}_{k,n-M_1-1}J_{n-M_1-1,n-M_1}\mathcal{R}_{n-M_1,j}(z) + \dot{R}^{(n,M_1)}_{k,n+M_1+1}J_{n+M_1+1,n+M_1+2}\mathcal{R}_{n+M_1+2,j}(z). \tag{3.5}$$

Due to the standard theory of the Jacobi matrices with bounded coefficients we get

$$|\dot{R}^{(n,M_1)}_{j,k}(z)| \le \frac{1}{\delta(z)} e^{-C_2\delta(z)|j-k|}. \tag{3.6}$$

Thus, for $n-(M_1-M)\le k\le n+(M_1-M)$ we have

$$|\dot{R}^{(n,M_1)}_{n-M_1-1,k}(z)|, |\dot{R}^{(n,M_1)}_{n+M_1+1,k}(z)| \le \frac{1}{\delta(z)} e^{-C_2\delta(z)M}.$$

So, it follows from (3.5) that

$$|\mathcal{R}_{k,j}(z) - \dot{R}^{(n,M_1)}_{k,j}(z)| \le \frac{\text{const}}{|\Im z|\delta(z)} e^{-C_1\delta(z)M}. \tag{3.7}$$

Similarly, if we consider $\tilde{J}$ the $2M_1+1$-periodic symmetric Jacobi matrix such that

$$\tilde{J}_{k,k+1} = J_{k,k+1} \quad (k = n-M_1, ..., n+M_1), \tag{3.8}$$

and denote by $\tilde{R}$ its resolvent, then

$$|\tilde{R}_{kk} - \dot{R}^{(n,M_1)}_{kk}(z)| \le \frac{2}{|\Im z|\delta(z)} e^{-C_2\delta(z)M}. \tag{3.9}$$

Therefore,

$$|\mathcal{R}_{kk}(z) - \tilde{R}_{kk}(z)| \le \frac{\text{const}}{|\Im z|\delta(z)} e^{-C_2\delta(z)M}. \tag{3.10}$$

Applying the resolvent identity to the matrices $\tilde{J}^{(p)}$ and $\tilde{J}$, we get due to the estimate (1.22)

$$|\tilde{R}_{k,j}(z) - \tilde{R}^{(p)}_{k,j}(z,n^{-1})| \le \frac{\varepsilon_n}{n^p|\Im z|^2}. \tag{3.11}$$

Now, expanding $\tilde{R}^{(p)}_{k,k}(z,n^{-1})$ with respect to n^{-1} it is easy to get that

$$|\tilde{R}^{(p)}_{k,j}(z,n^{-1}) - S^{(p)}_{k,j}(z)| \le \frac{C_1}{\delta^{p+1}(z)n^{p+1}}. \tag{3.12}$$

¿From (3.10)-(3.12) we derive that

$$|\mathcal{R}_{k,k}(z) - S^{(p)}_{k,k}(z)| \le \frac{\varepsilon_n + C_1 n^{-1}}{(\delta^{p+1}(z)n^p} + \frac{e^{-C_2\delta(z)M}}{|\Im z|\delta(z)}.$$

Now, using (2.9) and the fact, that both functions $R_{k,k}$ and $S^{(p)}_{k,k}(z)$ are analiticalin $\mathbf{C}\setminus\sigma_\varepsilon$, we obtain the first inequality in the first line of (2.41).

To prove the second inequality in this line, we use again (3.5). Taking the second power, using the bounds

$$|\dot{R}^{(n,M_1)}_{kj}(z)|, |\mathcal{R}_{kj}(z)| \le \frac{1}{|\Im z|},$$

and also (3.6), we get

$$\begin{gathered} |\sum_{j=1}^{\infty}(\mathcal{R}_{k,j}(z))^2 - \sum_{j=1}^{\infty}(\dot{R}^{(n,M_1)}_{k,j}(z))^2| \le \\ \frac{4}{|\delta(z)|} e^{-C_2\delta(z)M}(\sum_{j=1}^{\infty}|\mathcal{R}_{n-M_1,j}(z)|^2 + \sum_{j=1}^{\infty}|\mathcal{R}_{n+M_1,j}(z))^2|) \le \frac{8}{|\Im z|^2\delta(z)} e^{-C_2\delta(z)M}. \end{gathered} \tag{3.13}$$

Here to estimate the sums of the type $\sum_j |\mathcal{R}_{n-M_1,j}(z)|^2$ we have used the simple inequalities

$$\sum_{j=1}^{\infty} |\mathcal{R}_{n-M_1,j}(z)|^2 = \sum_{j}^{\infty} \mathcal{R}_{n-M_1,j}(z)\mathcal{R}_{j,n-M_1}(\overline{z}) \le (\mathcal{R}(z)\cdot\mathcal{R}(\overline{z}))_{n-M_1,n-M_1} \le \frac{1}{|\Im z|^2}$$

Similarly,

$$|\sum_{m=k+1}^{\infty} (\tilde{R}_{k,m}(z))^2 - \sum_{m=k+1}^{\infty} (\dot{R}_{km}^{(n,M_1)}(z))^2| \le 2\frac{1}{|\Im z|^2\delta(z)} e^{-C_2\delta(z)M}. \tag{3.14}$$

And then, by the same way as in (3.10)-(3.12) we get the second inequality in the first line of (2.41).

The proof of the inequality on the second line of (2.41) is the same.

Let us remark here, that in fact we prove the first and the second lines of (2.41) for $n-(M_1-M) \le k \le n+(M_1-M)$

To prove the third line of (2.41) we need to make one more step. Let us prove that for $n-(M_1-2M) \le k \le n+(M_1-2M)$

$$|\sum_{j=1}^{n-(M_1-M)} [-R'_{j,j}(z) - \sum_{m=1}^{k} R_{j,m}^2(z)]| \le \frac{1}{|\Im z|^2\delta(z)} e^{-C_2\delta(z)M/2}. \tag{3.15}$$

To this end we consider one more "blok" symmetric Jacobi matrix $\dot{J}^{(n,(M_1-2M))}$ which can be obtained from J if we put $J_{n-(M_1-2M)-1,n-(M_1-2M)} = 0$. Using identity (3.4) for J and $\dot{J}^{(n,(M_1-2M))}$ and (3.6) for $\dot{J}^{(n,(M_1-2M))}$, we get similarly to (3.13)

$$\begin{gathered}|\sum_{j=1}^{n-(M_1-M)} \sum_{m=k+1}^{\infty} (\mathcal{R}_{j,m}(z))^2 - \sum_{j=1}^{n-(M_1-M)} \sum_{m=k+1}^{\infty} (\dot{R}_{j,m}^{(n,(M_1-2M))}(z))^2| \le \\ \frac{2n}{|\Im z|^3} e^{-C_2\delta(z)M} \le \frac{1}{|\Im z|^3} e^{-C_2\delta(z)M/2}.\end{gathered} \tag{3.16}$$

Then, using the estimate (3.6) for $\dot{R}_{j,m}^{(n,(M_1-2M))}(z)$ with $j \le n-(M_1-M)$ and $m \ge k+1 > n-(M_1-2M)$ we get

$$|\sum_{j=1}^{n-(M_1-M)} \sum_{m=k+1}^{\infty} (\dot{R}_{j,m}^{(n,(M_1-2M))}(z))^2| \le \frac{2n}{|\Im z|^2} e^{-C_2\delta(z)M} \le \frac{1}{|\Im z|^2} e^{-C_2\delta(z)M/2}.$$

This inequality combined with (3.16) proves that

$$|\sum_{j=1}^{n-(M_1-M)} [(\mathcal{R}\cdot\mathcal{R})_{j,j}(z) - \sum_{m=1}^{k} \mathcal{R}_{j,m}^2(z)| = |\sum_{j=1}^{n-(M_1-M)} \sum_{m=k+1}^{\infty} \mathcal{R}_{j,m}^2(z)| \le \frac{1}{|\Im z|^3} e^{-C_2\delta(z)M/2}.$$

Now, using (2.9), we can substitute $(\mathcal{R} * \mathcal{R})_{j,j}(z)$ by $(-R'_{j,j}(z))$ and $\mathcal{R}_{j,m}(z)$ by $R_{j,m}(z)$ to get (3.15). Applying the the first and the second line of (2.41) for $n-(M_1-M) \le k \le n+(M_1-M)$ we get the third line of (2.41).

Proof of Proposition 2

To find $D_k^{(i)}$ let us find at first

$$\begin{gathered}(-1)^{k-n}(R^{(0)}(\zeta)J^{(\pm)}R^{(0)}(\zeta))_{k,k+1} = \sum_{j=-\infty}^{\infty} (R_{k,j}^{(0)}(\zeta)(-1)^{k-j}R_{j,k}^{(0)} + R_{k,j+1}^{(0)}(\zeta)(-1)^{k-j}R_{j+1,k}^{(0)} = \\ \sum_{j=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} dxdy \frac{e^{i(k-j)(x-y-\pi)}(1+e^{-i(x+y)})}{(\zeta - a\cos x)(\zeta - a\cos x)} = \\ \frac{1}{2\pi}\int_0^{2\pi} dx \frac{1-cos2x}{(\zeta^2 - a^2\cos^2 x)} = 2(1-\frac{z^2}{a^2})\frac{1}{2\pi}\int_0^{2\pi} dx \frac{1}{(\zeta^2 - a^2\cos^2 x)} + \frac{1}{\pi a^2} = \frac{2}{\zeta}(1-\frac{\zeta^2}{a^2})X^{-1}(\zeta).\end{gathered}$$

Then, using the simple formula $R^{(0)}_{k,k+1}(\zeta)=a^{-1}\zeta R^{(0)}_{k,k}(\zeta)-1=a^{-1}\zeta X^{-1}(\zeta)-1$, we find

$$(-1)^{k-n}D^{(i)}_k=\frac{1}{2\pi i}\int_L V'(\zeta)(\frac{\zeta}{a}+\frac{a}{\zeta}(1-\frac{\zeta^2}{a^2}))X^{-1}(\zeta)d\zeta=\frac{a}{2\pi i}\int_L\frac{V'(\zeta)}{\zeta}d\zeta=aP(0)\neq 0.$$

Here we have used representation (2.22).

Similar calculations show us that $A^{(i)}_k=k-n$, so it follows from equation (2.51) that $c=0$ and we get (1.25).

In the case (ii) let us start from the explanation why are equations (2.53) and similar equations which appears at the each step of the approximation process linearly dependent. To this end we come back to the equations (2.50) and add the k-th equation to the $(k+1)$-th one. We get

$$\frac{1}{2\pi i}\int_L V'(\zeta)(J_{k,k-1}R_{k-1,k}(\zeta)+J_{k,k+1}R_{k+1,k}(\zeta))d\zeta=\frac{2k-1}{n}$$

But since $J_{k,k-1}R_{k-1,k}(\zeta)+J_{k,k+1}R_{k+1,k}(\zeta)=\zeta R_{k,k}(\zeta)-1$, we get

$$\frac{1}{2\pi i}\int_L V'(\zeta)\zeta R_{k,k}(\zeta)d\zeta=\frac{2k-1}{n}\tag{3.17}$$

But this equation can be derived from equation (2.15), if we take in it only the second order term with respect to z^{-2}. So it is easy to understand, that if we solve equation (2.15) up to the $(p+1)$-th order in n^{-1}, then we solve (3.17) up to the same order. Therefore the sum of equations (2.53) is the corollary of equation (2.48) (or in the case of higher orders of equation (2.62)), which we have used in order to derive (2.53).

To find $D^{(ii)}_{2k-1,2k}$ and $D^{(ii)}_{2k+1,2k}$ we again find first $(R^{(0)}(\zeta)J^{(\pm)}R^{(0)}(\zeta))_{2k\pm1,2k}$. Let us note, that

$$(R^{(0)}(\zeta)J^{(\pm)}R^{(0)}(\zeta))_{2k\pm1,2k}=(\frac{1}{a_1}\frac{\partial}{\partial a_1}-\frac{1}{b_1}\frac{\partial}{\partial b_1})R^{(0)}(\zeta)_{2k\pm1,2k}\tag{3.18}$$

On the other hand,

$$\begin{aligned}R^{(0)}(\zeta))_{2k,2k}&=\frac{\zeta}{X(\zeta)}=\frac{\zeta}{\sqrt{\zeta^4-2(a_1^2+b_1^2)\zeta^2+(a_1^2-b_1^2)^2}}\\ R^{(0)}(\zeta))_{2k-1,2k}&=\text{const}+\frac{\zeta^2+a_1^2-b_1^2}{2a_1\zeta}R^{(0)}(\zeta))_{2k,2k}=\text{const}+\frac{\zeta^2+a_1^2-b_1^2}{2a_1X(\zeta)}\\ R^{(0)}(\zeta))_{2k+1,2k}&=\text{const}+\frac{\zeta^2-a_1^2+b_1^2}{2b_1\zeta}R^{(0)}(\zeta))_{2k,2k}=\text{const}+\frac{\zeta^2-a_1^2+b_1^2}{2b_1X(\zeta)},\end{aligned}\tag{3.19}$$

where symbols const means that this term is independent of ζ and so after integration with respect to ζ gives us 0. Now, using formulae (3.18) and (3.19) we get

$$\begin{aligned}(R^{(0)}(\zeta)J^{(\pm)}R^{(0)}(\zeta))_{2k-1,2k}=\text{const}-\frac{\zeta^2+a_1^2-b_1^2}{2a_1^3X(\zeta)}+\frac{2}{a_1X(\zeta)}+\\ 2\frac{(\zeta^2+a_1^2-b_1^2)(a_1^2-b_1^2)}{a_1^3(X(\zeta))^3}.\end{aligned}\tag{3.20}$$

Then, using definition (2.54), we get

$$\begin{aligned}D^{(ii)}_{2k-1,2k}=\frac{1}{2\pi i}\int_L d\zeta V'(\zeta)[\frac{1}{a_1}R^{(0)}(\zeta))_{2k-1,2k}+a_1(R^{(0)}(\zeta)J^{(\pm)}R^{(0)}(\zeta))_{2k-1,2k}]=\\ \frac{2}{2\pi i}\int_L d\zeta V'(\zeta)\frac{1}{X(\zeta)}+\frac{2}{2\pi i}\int_L d\zeta V'(\zeta)\frac{(\zeta^2+a_1^2-b_1^2)(a_1^2-b_1^2)}{a_1^2(X(\zeta))^3}=\\ \frac{2}{2\pi i}\int_L d\zeta\frac{V'(\zeta)}{X(\zeta)}+\frac{8}{2\pi i}\int_L d\zeta V'(\zeta)\frac{(\zeta^2+ab)ab}{(a+b)^2(\zeta^2-a^2)(\zeta^2-b^2)X(\zeta)}\end{aligned}$$

Here we have used expressions for a_1 and b_1 given after formula (1.26). But it is easy to see that

$$\frac{1}{2\pi i}\int_L d\zeta \frac{V'(\zeta)}{X(\zeta)} = \frac{1}{\pi}\int_\sigma d\lambda \frac{V'(\lambda)}{X(\lambda)} = 0$$

Therefore

$$D^{(ii)}_{2k-1,2k} = \frac{8}{2\pi i}\int_L d\zeta V'(\zeta)\frac{(\zeta^2+ab)ab}{(a+b)^2(\zeta^2-a^2)(\zeta^2-b^2)X(\zeta)} \tag{3.21}$$

Similarly

$$D^{(ii)}_{2k+1,2k} = -\frac{8}{2\pi i}\int_L d\zeta V'(\zeta)\frac{(\zeta^2-ab)ab}{(a-b)^2(\zeta^2-a^2)(\zeta^2-b^2)X(\zeta)} \tag{3.22}$$

It is easy to see, that $D^{(ii)}_{2k-1,2k}$ and $D^{(ii)}_{2k+1,2k}$ can be zeros at the same time if and only if

$$\frac{1}{2\pi i}\int_L d\zeta \frac{V'(\zeta)}{(\zeta^2-a^2)X(\zeta)} = \frac{1}{2\pi i}\int_L d\zeta \frac{V'(\zeta)}{(\zeta^2-b^2)X(\zeta)} = 0. \tag{3.23}$$

But on the basis of representation (2.22) one can get easily, that

$$\frac{1}{2\pi i}\int_L d\zeta \frac{V'(\zeta)}{(\zeta^2-a^2)X(\zeta)} = \frac{P(a)}{a}, \quad \frac{1}{2\pi i}\int_L d\zeta \frac{V'(\zeta)}{(\zeta^2-b^2)X(\zeta)} = \frac{P(b)}{b}$$

and since under conditions of the theorem $P(a) \neq 0$ and $P(b) \neq 0$, we proved so that $D^{(ii)}_{2k-1,2k}$ and $D^{(ii)}_{2k+1,2k}$ cannot be zeros at the same time.

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