

# UNIVERSALITY OF THE LOCAL EIGENVALUE STATISTICS FOR A CLASS OF UNITARY INVARIANT RANDOM MATRIX ENSEMBLES

L.Pastur<sup>1</sup> and M.Shcherbina<sup>1</sup>

## Abstract

The paper is devoted to the rigorous proof of the universality conjecture of the random matrix theory, according to which the limiting eigenvalue statistics of  $n \times n$  random matrices within spectral intervals of the order  $O(n^{-1})$  is determined by the type of matrices (real symmetric, Hermitian or quaternion real) and by the density of states. We prove this conjecture for a certain class of the Hermitian matrix ensembles that arose in the quantum field theory and have the unitary invariant distribution defined by a certain function (the potential in the quantum field theory) satisfying some regularity conditions.

**Key words:** random matrices, local asymptotic regime, universality conjecture, orthogonal polynomial technique.

## 1 Introduction. Problem and results.

The random matrix theory (RMT) has been extensively developed and used in a number of areas of theoretical and mathematical physics. In particular the theory provides quite satisfactory description of fluctuations in spectra of complex quantum systems such as heavy nuclei, small metallic particles, classically chaotic quantum models, etc. One of the important ingredients of this description is the universality conjecture of the RMT according to which the local eigenvalue statistics on  $n \times n$  random matrices (probabilistic properties of their spectra within intervals of the order of  $1/n$ ) do not depend on a particular ensemble in the limit  $n = \infty$  and is completely determined by the invariance group of the ensemble probability distribution. There are three invariance groups (orthogonal, unitary and symplectic) and three respective classes of the random matrix ensembles that model quantum systems possessing respective invariance under the time reflection and the space rotations. The explicit form of the local eigenvalue statistics in the limit  $n = \infty$  for each of these classes was found in sixties by Wigner, Mehta, Dyson and others who introduced and studied the explicitly solvable Gaussian and circular ensembles (see Ref.1 and references therein).

In this paper we consider the technically simplest case of the unitary invariant ensembles. Moreover we will study the class defined by the density

$$p_n(M)dM = Z_n^{-1} \exp\{-n\text{Tr}V(M)\}dM \tag{1.1}$$

where  $M$  is a  $n \times n$  Hermitian matrix,

$$dM = \prod_{j=1}^n dM_{jj} \prod_{j < k} d\Im M_{jk} d\Re M_{jk}$$

is the "Lebesgue" measure for Hermitian matrices,  $Z_n$  is the normalization factor and  $V(\lambda)$  is a real valued function (see the Theorem below for explicit conditions).

The case  $V(\lambda) = \frac{\lambda^2}{2}$  corresponds to the Gaussian unitary ensemble (GUE) which was introduced by Wigner in fifties. Ensembles with an arbitrary  $V(\lambda)$  were introduced in sixties,<sup>(2-4)</sup>

---

<sup>1</sup>Mathematical Division of the Institute for Low Temperature Physics of the National Academy of Sciences of Ukraine, 310164, Kharkov, Ukraine

when some particular cases were studied. The new wave of interest to this class of unitary invariant ensembles was caused by quantum field theory, where they arise in large- $n$  limit of quantum chromodynamics, 2-dimensional quantum gravity and bosonic string theory (see the review papers 5, 6). Analogous ensembles are used in condensed matter theory and statistical mechanics of random surfaces.<sup>(7,8)</sup>

Denote by  $p_n(\lambda_1, \dots, \lambda_n)$  the joint probability density of all eigenvalues which we assume to be symmetric without loss of generality. Let

$$p_l^{(n)}(\lambda_1, \dots, \lambda_l) = \int p_n(\lambda_1, \dots, \lambda_l, \lambda_{l+1}, \dots, \lambda_n) d\lambda_{l+1} \dots d\lambda_n \quad (1.2)$$

be its  $l$ -th marginal distribution density. The simplest case of  $p_1^{(n)}(\lambda_1)$  is of particular interest. Indeed, denote by  $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$  eigenvalues of a random Hermitian matrix  $M$  and set

$$N_n(\Delta) = \frac{1}{n} \sum_{\lambda_i^{(n)} \in \Delta} 1, \quad \Delta = (a, b). \quad (1.3)$$

This is their normalized counting function (empirical eigenvalue distribution). Then

$$E\{N_n(\Delta)\} = \int_{\Delta} p_1^{(n)}(\lambda) d\lambda \equiv \int_{\Delta} \rho_n(\lambda) d\lambda \quad (1.4)$$

where  $E\{\dots\}$  denotes the expectation with respect to density (1.1).

In the recent paper<sup>(9)</sup> it was proved that if  $V(\lambda)$  is bounded below for all  $\lambda \in \mathbf{R}$  and satisfies the conditions

$$|V(\lambda)| \geq (2 + \epsilon) \log |\lambda|, \quad |\lambda| \geq L_0 \quad (1.5a)$$

for some  $L_0$  and

$$|V(\lambda_1) - V(\lambda_2)| \leq C(A) |\lambda_1 - \lambda_2|^\gamma, \quad |\lambda_{1,2}| \leq A \quad (1.5b)$$

for any  $0 < A < \infty$  and some  $\gamma > 0$ , then  $\rho_n(\lambda)$  converges to the limiting density  $\rho(\lambda)$  (density of states) in the Hilbert space defined by the norm

$$\left( - \int \log |\lambda - \mu| \rho(\lambda) \rho(\mu) d\lambda d\mu \right)^{1/2} \quad (1.6)$$

and  $\rho(\lambda)$  can be found from the certain variational procedure analogous to that known in the mean field theory of statistical mechanics. Moreover, there exist positive numbers  $L, L_1, (L > L_1), \delta_1$  and  $a$  such that

$$V(\lambda) - \max_{|\mu| \leq L_1} V(\mu) > \delta_1, \quad \text{and} \quad \rho_n(\lambda) \leq \exp\{-na[V(\lambda) - \max_{|\mu| \leq L_1} V(\mu)]\}, \quad |\lambda| > L \quad (1.7)$$

and for any differentiable on  $(-L, L)$  function  $\phi(\mu)$  which grows not faster than  $e^{bV(\mu)}$ ,  $b > 0$  as  $|\mu| \rightarrow \infty$

$$\left| \int \phi(\mu) \rho_n(\mu) d\mu - \int \phi(\mu) \rho(\mu) d\mu \right| \leq C \|\phi\|_2^{1/2} \|\phi\|_2^{1/2} n^{-1/2} \log^{1/2} n \quad (1.8)$$

where symbol  $\|\dots\|_2$  denotes the  $L_2$ -norm on  $(-L, L)$ .

Here and below the symbols  $C$  and  $C_i$  denote  $n$ -independent positive constants that may be different in different formulae.

Now we formulate the universality conjecture following Dyson.<sup>(4)</sup>

**Universality conjecture.** For any  $n$ -independent integer  $l$ ,  $\lambda_0$  such that  $\rho(\lambda_0) \neq 0$  and arbitrary fixed  $(x_1, \dots, x_l) \in \mathbf{R}^l$

$$\lim_{n \rightarrow \infty} [n\rho_n(\lambda_0)]^{-l} p_l^{(n)}(\lambda_0 + \frac{x_1}{n\rho_n(\lambda_0)}, \dots, \lambda_0 + \frac{x_l}{n\rho_n(\lambda_0)}) = \det \|S(x_1 - x_k)\|_{j,k=1}^l \quad (1.9)$$

where

$$S(x) = \frac{\sin \pi x}{\pi x}. \quad (1.10)$$

In other words, the limit in the r.h.s. of (1.9) is the same for all  $V(\lambda)$ 's in (1.1) (modulo some weak conditions) and all  $\lambda_0$  that belong to the "bulk" of the spectrum where  $\rho(\lambda_0) \neq 0$ . Thus the limit (1.9) for arbitrary  $V$  has to coincide with the same limit for the archetype Gaussian case  $V(\lambda) = \lambda^2/2$ , whose form is given by the r.h.s. of (1.9) and is known since early sixties (see Ref.1 for respective results and discussions).

In this paper we prove the following

**Theorem.** Assume that the function  $V(\lambda)$  satisfies the conditions

(i)

$$V(\lambda) \geq (2 + \epsilon) \log |\lambda|, \quad |\lambda| \geq L_0 \quad (1.11a)$$

for some  $L_0 < \infty$  (cf.(1.5));

(ii)

$$\sup |V'''(\lambda)| \leq C(L) < \infty, \quad |\lambda| \leq L \quad (1.11b)$$

where  $L$  is defined in (1.7);

(iii)  $V'(\lambda)$  exists for all  $\lambda$  and

$$\int (V')^2(\lambda) e^{-kV(\lambda)} d\lambda \quad (1.11c)$$

for some  $k > 0$ .

Then the universality conjecture (1.9) is true uniformly in  $(x_1, \dots, x_l)$  varying on compact sets of  $\mathbf{R}^l$ .

**Remarks.**

1. In fact the Theorem is valid without any assumptions on the growth of  $V(\lambda)$  provided that conditions (i) and (ii) are valid. However our proofs are simpler if  $V(\lambda)$  satisfies (iii). To prove the Theorem without this condition we have to restrict from the very beginning all integrals to the finite interval  $(-L, L)$  outside of which the estimate (1.7) is valid. The latter allows us to control the remainders and to prove that they vanish exponentially as  $n \rightarrow \infty$ . This approach was used in the work<sup>(9)</sup> where a kind of the variational arguments were applied to study the density of states  $\rho(\lambda)$ . In our case we can do the same restriction, but since, unlike the work<sup>(9)</sup> here we use extensively the orthogonal polynomials technique that relies strongly on integral relations, respective estimates are more tedious and require more space. That is why we impose the technical conditions (iii).

2. Denote by  $P_l^{(n)}(\lambda)$ ,  $l = 0, 1, \dots$  orthogonal polynomials on  $\mathbf{R}$  associated with the weight

$$w_n(\lambda) = e^{-nV(\lambda)}, \quad (1.12)$$

$$\int P_l^{(n)}(\lambda) P_m^{(n)}(\lambda) e^{-nV(\lambda)} d\lambda = \delta_{l,m} \quad (1.13)$$

and by

$$\psi_l^{(n)}(\lambda) = \exp\{-nV(\lambda)/2\} P_l^{(n)}(\lambda), \quad l = 0, 1, \dots \quad (1.14)$$

respective orthonormal system

$$\int \psi_l^{(n)}(\lambda) \psi_m^{(n)}(\lambda) d\lambda = \delta_{l,m}. \quad (1.15)$$

Then the joint probability density of all eigenvalues of ensemble (1.1) is<sup>(1)</sup>

$$p_n(\lambda_1, \dots, \lambda_n) = \hat{Z}_n^{-1} \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k)^2 \exp\{-n \sum_{j=1}^n V(\lambda_j)\} = \quad (1.16)$$

$$(n!)^{-1} \left( \det \|\psi_{j-1}(\lambda_k)\|_{j,k=1}^n \right)^2, \quad (1.17)$$

and marginal densities (1.4) are

$$p_l^{(n)}(\lambda_1, \dots, \lambda_l) = \frac{(n-l)!}{n!} \det \|k_n(\lambda_j, \lambda_k)\|_{j,k=1}^l \quad (1.18)$$

where

$$k_n(\lambda, \mu) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda) \psi_l^{(n)}(\mu) \quad (1.19)$$

is known as the reproducing kernel of system (1.14). In particular

$$\rho_n(\lambda) \equiv p_1^{(n)}(\lambda) = K_n(\lambda, \lambda) \quad (1.20)$$

where

$$K_n(\lambda, \mu) = n^{-1} k_n(\lambda, \mu). \quad (1.21)$$

In view of (1.18) the proof of the universality conjecture (1.9) for the random matrix ensemble (1.1) reduces to the proof of the limiting relation

$$\lim_{n \rightarrow \infty} [\rho_n(\lambda_0)]^{-1} K_n\left(\lambda_0 + \frac{x}{n\rho_n(\lambda_0)}, \lambda_0 + \frac{y}{n\rho_n(\lambda_0)}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)} \quad (1.22)$$

that being rewritten as

$$\lim_{n \rightarrow \infty} [k_n(\lambda_0, \lambda_0)]^{-1} k_n\left(\lambda_0 + \frac{x}{k_n(\lambda_0, \lambda_0)}, \lambda_0 + \frac{y}{k_n(\lambda_0, \lambda_0)}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)} \quad (1.23)$$

can be regarded as a conjecture of purely analytic nature concerning the orthogonal polynomial (1.13). Since for a complete systems of orthonormal functions we have the relation

$$\sum_{j=0}^{\infty} \psi_j^{(n)}(\lambda) \psi_j^{(n)}(\mu) = \delta(\lambda - \mu) \quad (1.24)$$

the result (1.23) can be viewed as the one saying that the fine ("magnified") structure of the  $\delta$ -function in (1.24) is universal and is given by the r.h.s. of (1.23). The result (1.23) can be readily proven if the precise enough asymptotic formula for respective orthogonal polynomials is known. Let us consider the simplest ("toy") case of an  $n$ -independent weight supported on a finite interval, say interval  $[-1, 1]$ . By using classical asymptotic formulae<sup>(10)</sup> we find that in this case  $\rho(\lambda) = (\pi\sqrt{1-\lambda^2})^{-1}$ ,  $|\lambda| \leq 1$  and relation (1.9) is valid for any  $|\lambda| < 1$ . Less trivial case corresponds to the weight (1.12) in which  $V(\lambda) = |\lambda|^\alpha/\alpha$  with a positive  $\alpha$ . In this case  $P_l^{(n)}(\lambda) = n^{1/2\alpha} \pi_l(n^{1/\alpha} \lambda)$  where  $\{\pi_l(x)\}_{l=0}^{\infty}$  are orthogonal polynomials associated with the  $n$ -independent weight  $w(x) = \exp\{-|x|^\alpha/\alpha\}$ . The case  $\alpha = 2$  corresponds to the Gaussian unitary ensemble and the Hermite polynomials as  $\pi_l(x)$ . This case was studied in great details<sup>(1)</sup> on the basis of the Plancherel-Rotah asymptotic formula<sup>(10)</sup> describing the semiclassical regime of quantum oscillator. For the general case  $\alpha > 1$  asymptotic formulae were recently obtained in papers<sup>(11,12)</sup>. By using these formulae the limiting density  $\rho(\lambda)$  can be found and relation (1.9)

can be checked for  $\lambda = 0$  <sup>(13)</sup>. Unfortunately asymptotic formulae <sup>(11,12)</sup> are not precise enough to prove (1.9) for  $\lambda \neq 0$ . This can be done only for  $\alpha = 4, 6$  where more precise asymptotic formulae are known.

3. We mention several physical works, related to the subject of the paper. In the paper <sup>(14)</sup> the scheme of proof (1.22) for  $\lambda = 0$  is proposed. It is based on the formalism developed in studying so called double scaling limit of the quantum field theory. In the paper <sup>(15)</sup> the new asymptotic formula for the orthogonal polynomials  $P_l^{(n)}(\lambda)$ ,  $l = n + o(1)$  is proposed in the case, when the support of the density of states  $\rho(\lambda)$  is an interval. By using this formula authors derived (1.22) and, moreover, found the new asymptotic regime for the smoothed correlation function of eigenvalues for  $1 \gg \lambda \gg n^{-1}$  ("mesoscopic" scale). These results was improved and developed in the subsequent paper <sup>(16)</sup>. In the paper <sup>(17)</sup> the universality conjecture was considered by studying the generating functional of densities (1.18) that was computed by applying the Laplace method to its Grassman integral representation.

4. We would like to stress, that our approach is "local", i.e. it is not sensitive to the form of the support of  $\rho(\lambda)$  provided that  $\rho(\lambda) > 0$ . On the other hand it is known that if  $V(\lambda)$  is a polynomial of degree  $2m$ , then the support of  $\rho(\lambda)$  may consist of several (at most  $m$ ) intervals. The physical papers <sup>(15,16)</sup> are based on the asymptotic formulas for the orthogonal polynomials with the weight  $e^{-nV(x)}$  that are valid for such  $V(\lambda)$ , which produce the one-interval support of  $\rho(\lambda)$ . This is the case if, for instance,  $V(\lambda)$  is a convex function (not necessary a polynomial) <sup>(9)</sup>.

We will prove the Theorem by using the orthogonal polynomials technique that is rather powerful and widely used in in the random matrix theory and its numerous applications. However, since the asymptotic formulae for the general case treated in the Theorem are not known we combine the orthogonal polynomial technique with certain identities that were introduced in the random matrix theory in the seminal paper <sup>(5)</sup>.

Our paper is organized as follows. In Section 2 we give the proof of the Theorem following the main line of the arguments. The important ingredient of our arguments is the pointwise convergence of  $\rho_n(\lambda)$  to  $\rho(\lambda)$  on the set  $\{\lambda : \rho(\lambda) > 0\}$ .

**Proposition.** *Under the conditions of the Theorem we have for all  $\lambda$  and  $n$  such that  $\rho(\lambda) > n^{-1/9}$*

$$|\rho_n(\lambda) - \rho(\lambda)| \leq C \left(1 + \frac{1}{\rho(\lambda)}\right) n^{-1/4} \quad (1.25)$$

for some positive  $n$ -independent constant  $C$ .

The Proposition is also proved in Section 2. Auxiliary facts which we need to establish the Theorem and the Proposition are proved in Section 3. We discuss some consequences of our results in Section 4.

## 2 Proofs of the Proposition and the Theorem

**Proof of the Proposition.** Consider the Stieltjes transform of the normalized counting measure (1.3)

$$f_n(z) \equiv \int \frac{N_n(d\lambda)}{\lambda - z} = \frac{1}{n} \sum_{l=1}^n \frac{1}{\lambda_l - z}, \quad (2.1)$$

and denote

$$g_n(z) \equiv E\{f_n(z)\} = \int \frac{\rho_n(\lambda)d\lambda}{\lambda - z}. \quad (2.2)$$

According to the spectral theorem

$$f_n(z) = \frac{1}{n} \text{Tr}G(z)$$

where  $G(z) = (M - z)^{-1}$  is the resolvent of a Hermitian matrix  $M$ . By using Lemma 1 for  $F(M) = G_{ik}(z)$  (a matrix element of the resolvent) and  $B = B^{(ik)} = \{B_{jm}^{(ik)}\}_{j,m=1}^n$ ,  $B_{jm}^{(ik)} = \zeta \delta_{ij} \delta_{km} + \bar{\zeta} \delta_{im} \delta_{kj}$ , where  $\zeta \in \mathbf{C}$  is a free parameter, it is easy to derive the identity<sup>(5)</sup>

$$E\{\zeta G_{ii} G_{kk} + \bar{\zeta} G_{ik}^2 + n G_{ik} (\zeta (V'(M))_{ki} + \bar{\zeta} (V'(M))_{ik})\} = 0$$

Since  $\zeta$  is arbitrary we conclude that

$$E\{G_{ii} G_{kk} + n G_{ik} (V'(M))_{ki}\} = 0.$$

Now if we sum over  $i, k = 1, \dots, n$  this inequality and divide the result by  $n^2$  we get

$$E\{f_n^2\} + E\{n^{-1} \text{Tr} V'(M) G(z)\} = 0. \quad (2.3)$$

By applying Lemma 3 to  $f(\mu) = (\mu - z)^{-1}$ ,  $z = \lambda + i\eta$ ,  $\eta > 0$ , we find that

$$E\{f_n^2\} = E^2\{f_n\} + O(n^{-2} \eta^{-4}). \quad (2.4)$$

This bound, (2.1) and (2.2) yield the relation

$$g_n^2(z) + V'(\lambda) g_n(z) + Q_n(z) = O(n^{-2} \eta^{-4}) \quad (2.5)$$

where

$$Q_n(z) = \int \frac{V'(\mu) - V'(\lambda)}{\mu - z} \rho_n(\mu) d\mu.$$

is well defined due to (1.7) and our conditions on  $V(\lambda)$  (see the Theorem and Remark 1). To proceed further we use the result (1.8) combining it with condition (1.11b). We obtain

$$Q_n(\lambda + i\eta) = Q(\lambda) + O(\eta^{-1/4} n^{-1/2} \log^{1/2} n) + O(\eta) \quad (2.6)$$

where

$$Q(\lambda) = \int \frac{V'(\mu) - V'(\lambda)}{\mu - \lambda} \rho(\mu) d\mu. \quad (2.7)$$

Combining (2.5) and (2.6) we find that

$$g_n(\lambda + in^{-2/3}) = -\frac{V'(\lambda)}{2} + \sqrt{\left(\frac{V'(\lambda)}{2}\right)^2 - Q(\lambda) + O(n^{-1/3} \log^{1/2} n)} \quad (2.8)$$

Thus on the basis of Lemma 4 we get

$$\pi^{-1} \Im g_n(\lambda + in^{-1/3}) = \rho(\lambda) + O(n^{-1/3} \log^{1/2} n) \frac{1}{\rho(\lambda)}. \quad (2.9)$$

On the other hand it follows from Lemmas 5 and 6 that

$$|\pi^{-1} \Im g_n(\lambda + in^{-1/3}) - \rho_n(\lambda)| \leq C \left(1 + \frac{1}{\rho(\lambda)}\right) n^{-1/4}.$$

This bound and (2.9) imply (1.25).

**Proof of the Theorem.** According to (1.18) the proof of the Theorem reduces to the proof of the limiting relation (1.22) to the reproducing kernel (1.21) of orthonormal systems (1.14). We use the representation

$$K_n(\lambda, \mu) = Z_n^{-1} \int \prod_{j=2}^n d\lambda_j (\lambda - \lambda_j) (\mu - \lambda_j) \times$$

$$\prod_{2 \leq j < k \leq n} (\lambda_j - \lambda_k)^2 \exp \left\{ -\frac{n}{2} V(\lambda) - \frac{n}{2} V(\mu) - \sum_{j=2}^n V(\lambda_j) \right\} \quad (2.10)$$

which can be derived from the well-known in the RMT<sup>(1)</sup> identities

$$\prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k) = \left( \prod_{l=1}^{n-1} \gamma_l^{(n)} \right)^{-1} \det \| P_{j-1}^{(n)}(\lambda_k) \|_{j,k=1}^n, \quad Z_n = n! \prod_{l=0}^{n-1} (\gamma_l^{(n)})^{-2}$$

where  $\gamma_l^{(n)}$  is the coefficient in front of  $\lambda^l$  in the polynomial  $P_l^{(n)}$ . If we substitute these identities into the r.h.s. of (2.10), set in the one of determinant  $\lambda_1 = \lambda$ , in other  $\lambda_1 = \mu$  and then integrate the result with respect to  $\lambda_2, \dots, \lambda_n$ , using the orthogonality of polynomials  $P_l^{(n)}$  we obtain the l.h.s. of (2.10).

We will consider the function  $K_n(\lambda_0, \lambda_0 + s/n)$ . General case of function  $K_n(\lambda_0 + s/n, \lambda_0 + t/n)$  can be reduced to  $K_n(\lambda_0, \lambda_0 + (s-t)/n)$  by using Lemma 7. Let us choose

$$\delta = \frac{\log n}{n}$$

and rewrite (2.10) in the form

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = \exp \left\{ \frac{n}{2} V(\lambda_0) - \frac{n}{2} V(\lambda_0 + \frac{s}{n}) \right\} \left\langle \prod_{j=2}^n \left[ \frac{s}{n} \cdot \frac{\chi_\delta(\lambda_0 - \lambda_j)}{\lambda_0 - \lambda_j} + e^{u(\lambda_j)} \right] \right\rangle. \quad (2.11)$$

Here and below the symbol  $\langle \dots \rangle$  denotes the operation  $E\{\delta(\lambda_0 - \lambda_1) \dots\}$ ,  $\chi_\delta(\lambda)$  is the indicator of the interval  $|\lambda| \leq \delta$ , and

$$u(\lambda) = (1 - \chi_\delta(\lambda_0 - \lambda)) \log \left( 1 + \frac{s}{n(\lambda_0 - \lambda)} \right).$$

Rewrite (2.11) as

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[ 1 + \sum_{l=1}^{n-1} C_n^l \left(\frac{s}{n}\right)^l \left\langle \prod_{j=2}^{l+1} \frac{\chi_\delta(\lambda_0 - \lambda_j)}{\lambda_0 - \lambda_j} e^{U_n(\lambda_0)} \right\rangle Z_n^{-1}(\lambda_0) \right], \quad (2.12)$$

where  $C_n^l = \frac{n!}{l!(n-l)!}$  and

$$T_n(\lambda_0) = \exp \left\{ -\frac{n}{2} V(\lambda_0) + \frac{n}{2} V(\lambda_0 + \frac{s}{n}) \right\} Z_n(\lambda_0), \quad (2.13)$$

$$Z_n(\lambda_0) = \left\langle e^{U_n(\lambda_0)} \right\rangle, \quad (2.14)$$

$$U_n(\lambda_0) = \sum_{j=2}^n u(\lambda_j). \quad (2.15)$$

Introduce the probability density (cf. (1.16))

$$p_{tn}(\lambda_2, \dots, \lambda_n) = Z_{tn}^{-1}(\lambda_0) \prod_{j=2}^n (\lambda_0 - \lambda_j)^2 \prod_{2 \leq j < k \leq n} (\lambda_j - \lambda_k)^2 \exp \left\{ -n \sum_{j=2}^n V(\lambda_j) + t \sum_{j=2}^n u(\lambda_j) \right\}, \quad (2.16)$$

where  $Z_{tn}^{-1}(\lambda_0)$  is the normalization factor, and respective marginal densities

$$p_{tl}^{(n)}(\lambda_2, \dots, \lambda_{l+1}) = \int p_{tn}(\lambda_2, \dots, \lambda_n) d\lambda_{l+2} \dots d\lambda_n. \quad (2.17)$$

In particular for  $t = 0$

$$p_{0l}^{(n)}(\lambda_2, \dots, \lambda_{l+1}) = \rho_n^{-1}(\lambda_0) p_{l+1}^{(n)}(\lambda_0, \lambda_2, \dots, \lambda_{l+1}). \quad (2.18)$$

This allows us to rewrite (2.12) as follows

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[ 1 + \sum_{l=1}^{n-1} C_{n-1}^l \left(\frac{s}{n}\right)^l \int \prod_{j=2}^{l+1} \frac{\chi_\delta(\lambda_0 - \lambda_j)}{\lambda_0 - \lambda_j} p_{1n}(\lambda_2, \dots, \lambda_{l+1}) d\lambda_2 \dots d\lambda_{l+1} \right]. \quad (2.19)$$

Introduce

$$R_{tn}(\lambda, \mu) = \frac{1}{n-1} \sum_{k=0}^{n-2} \psi_{tk}(\lambda) \psi_{tk}(\mu) \quad (2.20)$$

where

$$\psi_{tk}(\lambda) = (\lambda - \lambda_0) \exp\left\{-\frac{n}{2}V(\lambda) + \frac{t}{2}u(\lambda)\right\} P_{tk}(\lambda)$$

and  $\{P_{tk}(\lambda)\}_{k=0}^{\infty}$  are polynomials that are orthogonal with respect to the weight  $(\lambda - \lambda_0)^2 \exp\{-nV(\lambda) + tu(\lambda)\}$ :

$$\int P_{tl}(\lambda) P_{tm}(\lambda) (\lambda - \lambda_0)^2 \exp\{-nV(\lambda) + tu(\lambda)\} d\lambda = \delta_{lm}.$$

Then (cf.(1.18))

$$p_{tl}^{(n)}(\lambda_1, \dots, \lambda_{l+1}) = \frac{(n-1)^l}{(n-1)\dots(n-l-1)} \det \|R_{tn}(\lambda_{j+1}, \lambda_{k+1})\|_{j,k=1}^l$$

and after the change of variables  $x_j = n(\lambda_{j+1} - \lambda_0)$  (2.19) can be rewritten in the form

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[ 1 + \sum_{l=1}^{n-1} \frac{s^l (n-1)^l}{n^l l!} \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \|R_{1n}(\lambda_0 + \frac{x_j}{n}, \lambda_0 + \frac{x_k}{n})\|_{j,k=1}^l \right]. \quad (2.21)$$

We will prove that

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[ 1 + \sum_{l=1}^{n-1} \frac{s^l}{l!} \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \|R_{0n}(\lambda_0 + \frac{x_j}{n}, \lambda_0 + \frac{x_k}{n})\|_{j,k=1}^l + O\left(\frac{1}{n\delta}\right) \right]. \quad (2.22)$$

To this end we use Lemma 9. Therefore we have to check conditions (3.41)-(3.45) of the Lemma for  $A = R_{0n}$  and  $B = R_{1n}$ . Inequality (3.41) follows from (2.18) and Lemma 8, inequality (3.43) follows from (2.18) and Lemma 7, and inequality (3.42) follows from representation (2.20). To check (3.44) and (3.45) consider the derivative  $R'_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n})$  of (2.20) with respect to  $t$ . By using arguments similar to those in the proof of Lemma 5 we obtain

$$R'_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) = \frac{1}{2} (u(\lambda_0 + \frac{x}{n}) + u(\lambda_0 + \frac{y}{n})) R_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) -$$



$$(n-2) \int R_{tn}(\lambda_0 + \frac{x}{n}, \mu) R_{tn}(\lambda_0 + \frac{y}{n}, \mu) u(\mu) d\mu. \quad (2.23)$$

If  $|x| < n\delta$  and  $|y| < n\delta$ , then the first term in the r.h.s. of (2.23) is zero. The second term can be estimated by using the Schwarz inequality and the analogue of (3.8) for (2.20)

$$\begin{aligned} & \left| (n-2) \int R_{tn}(\lambda_0 + \frac{x}{n}, \mu) R_{tn}(\lambda_0 + \frac{y}{n}, \mu) u(\mu) d\mu \right| \leq \\ & \left| (n-2) \int R_{tn}(\lambda_0 + \frac{x}{n}, \mu)^2 |u(\mu)| d\mu \right|^{1/2} \left| (n-2) \int R_{tn}(\lambda_0 + \frac{y}{n}, \mu)^2 |u(\mu)| d\mu \right|^{1/2} \leq \\ & \max_{\lambda} |u(\lambda)| \cdot \left[ R_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) \cdot R_{tn}(\lambda_0 + \frac{y}{n}, \lambda_0 + \frac{y}{n}) \right]^{1/2}. \end{aligned} \quad (2.24)$$

Hence

$$\left| R'_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) \right| \leq \frac{C}{n\delta} \left[ R_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) R_{tn}(\lambda_0 + \frac{y}{n}, \lambda_0 + \frac{y}{n}) \right]^{1/2}. \quad (2.25)$$

Besides

$$\begin{aligned} & \max_t R_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) = R_{t^*n}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) = \\ & R_{0n}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) + \int_0^{t^*} d\tau R'_{\tau n}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) \leq \\ & R_{0n}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) + \frac{C}{n\delta} R_{t^*n}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}). \end{aligned} \quad (2.26)$$

Thus it follows from (2.26) that for all  $x$  and  $t$  (cf.(3.44))

$$R_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) \leq C R_{0n}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}). \quad (2.27)$$

Combining (2.25) and (2.27) we obtain (cf.(3.44))

$$|R_{1n}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) - R_{0n}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n})| \leq \frac{C}{n\delta} R_{0n}^{1/2}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) R_{0n}^{1/2}(\lambda_0 + \frac{y}{n}, \lambda_0 + \frac{y}{n}). \quad (2.28)$$

Inequality (2.28), identity (2.18) and Lemma 8 guarantee condition (3.44) of Lemma 9. Condition (3.45) can be proved by similar arguments. Thus we can apply Lemma 9 to the expression in the r.h.s. of (2.21) and obtain (2.22).

By using the analogue of representation (2.10) for  $R_{0n}(\lambda, \mu)$  we get

$$R_{0n}(\lambda, \mu) = K_n(\lambda, \mu) - \frac{K_n(\lambda_0, \lambda) K_n(\lambda_0, \mu)}{K_n(\lambda_0, \lambda_0)}. \quad (2.29)$$

We will use this representation to prove that we can replace the function  $R_{0n}(\lambda, \mu)$  in the r.h.s. of (2.22) by

$$R^*(x_j, x_k) = K_n(\lambda_0, \lambda_0 + \frac{x_k - x_j}{n}) - \frac{K_n(\lambda_0, \lambda_0 - x_j/n) K_n(\lambda_0, \lambda_0 + x_k/n)}{K_n(\lambda_0, \lambda_0)}. \quad (2.30)$$

We use again Lemma 9 for  $A = R_{0n}$ ,  $B = R^*$ . As it was explained above, conditions (3.41)-(3.43) for this  $A$  are true and thus we have to check (3.44) and (3.45). Since according to (2.29) and (2.30)

$$|R^*(x, y) - R_{0n}(x, y)| \leq \left| K_n(\lambda_0, \lambda_0 + \frac{y-x}{n}) - K_n(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) \right| +$$

$$\left| \frac{K_n(\lambda_0, \lambda_0 + y/n)}{K_n(\lambda_0, \lambda_0)} \right| |K_n(\lambda_0, \lambda_0 + x/n) - K_n(\lambda_0, \lambda_0 - x/n)|$$

it suffices to check that uniformly in  $|y| \leq n\delta$  and  $n \rightarrow \infty$

$$|K_n(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) - K_n(\lambda_0, \lambda_0 + \frac{y-x}{n})|^2 \leq \frac{Cx^2}{n^{1/4}}, \quad |x| \leq 1, \quad (2.31)$$

and

$$\int_{-n\delta}^{n\delta} |K_n(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) - K_n(\lambda_0, \lambda_0 + \frac{x-y}{n})|^2 dx \leq \frac{C(n\delta)^3}{n^{1/4}}. \quad (2.32)$$

Estimate (2.31) follows from Lemma 7, because  $|x|, |y| \leq n\delta = \log n$ . Estimate (2.32) can be obtained if we integrate (2.31) with respect to  $x$ . Thus we have proved that

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[ 1 + \sum_{l=1}^{n-1} \frac{s^l}{l!} \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \|R^*(x_j, x_k)\|_{j,k=1}^l + O\left(\frac{1}{n\delta}\right) \right]. \quad (2.33)$$

The next step is to prove that we can replace the integral over the interval  $(-n\delta, n\delta)$  in the r.h.s. of (2.33) by the integral over the whole axis  $\mathbf{R}$ . To this end let us notice first that since  $R^*(x, x) = R^*(-x, -x)$

$$\int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \|R^*(x_j, x_k)\|_{j,k=1}^l = \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \|R^*(x_j, x_k)(1 - \delta_{jk})\|_{j,k=1}^l. \quad (2.34)$$

Besides

$$\begin{aligned} \Delta_l &\equiv \left| \int \prod_{j=1}^l \frac{dx_j}{x_j} \det \|R^*(x_j, x_k)(1 - \delta_{jk})\|_{j,k=1}^l - \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \|R^*(x_j, x_k)(1 - \delta_{jk})\|_{j,k=1}^l \right| \leq \\ &\sum_{p=1}^l C_l^p \int \prod_{j=1}^p \frac{(1 - \chi_{n\delta}(x_j)) dx_j}{|x_j|} \prod_{j=p+1}^l \frac{\chi_{n\delta}(x_j) dx_j}{|x_j|} \left| \det \|R^*(x_j, x_k)(1 - \delta_{jk})\|_{j,k=1}^l \right| \leq \\ &\sum_{p=1}^l C_l^p \sum_{m=0}^{l-p} C_{l-p}^m \int \prod_{j=1}^p \frac{(1 - \chi_{n\delta}(x_j)) dx_j}{|x_j|} \times \\ &\prod_{i=p+1}^{p+m} \frac{(1 - \chi_1(x_i)) dx_i}{|x_i|} \prod_{k=p+m+1}^l \frac{\chi_1(x_k) dx_k}{|x_k|} \left| \det \|R^*(x_j, x_k)(1 - \delta_{jk})\|_{j,k=1}^l \right| \leq \\ &\sum_{p=1}^l C_l^p \sum_{m=0}^{l-p} C_{l-p}^m \left\{ \int \prod_{j=1}^p \frac{(1 - \chi_{n\delta}(x_j)) dx_j}{|x_j|^{5/4}} \prod_{i=p+1}^{p+m} \frac{(1 - \chi_1(x_i)) dx_i}{|x_i|^{5/4}} \prod_{k=p+m+1}^l dx_k \chi_1(x_k) \right\}^{1/2} \times \\ &\left\{ \int \prod_{j=1}^p \frac{(1 - \chi_{n\delta}(x_j)) dx_j}{|x_j|^{3/4}} \prod_{i=p+1}^{p+m} \frac{(1 - \chi_1(x_i)) dx_i}{|x_i|^{3/4}} \times \right. \\ &\left. \prod_{k=p+m+1}^l \frac{\chi_1(x_k) dx_k}{|x_k|^2} \left| \det \|R^*(x_j, x_k)(1 - \delta_{jk})\|_{j,k=1}^l \right|^2 \right\}^{1/2}. \end{aligned}$$

The first factor in the r.h.s. of the last inequality can be estimated by  $(n\delta)^{-p/4} C^{l-p}$ . To estimate the second one we repeat almost literally the arguments of Lemma 9. We obtain

$$\Delta_l \leq l^{(l+2)/2} C^l (n\delta)^{-1/4}.$$

Therefore

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[ 1 + \sum_{l=1}^{n-1} \frac{s^l}{l!} \int \prod_{j=1}^l \frac{dx_j}{x_j} \det \|R^*(x_j, x_k)\|_{j,k=1}^l + O\left(\frac{1}{(n\delta)^{1/4}}\right) \right]. \quad (2.35)$$

Now by using the formula

$$\det \|a_{jk}\|_{j,k=0}^l = a_{00} \det \|a_{jk} - \frac{a_{j0}a_{0k}}{a_{00}}\|_{j,k=1}^l$$

we obtain from (2.35)

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[ 1 + \sum_{l=1}^{n-1} \frac{s^l}{l! \rho_n(\lambda_0)} \int \prod_{j=1}^l \frac{dx_j}{x_j} \det \|S_n(x_j - x_k)\|_{j,k=0}^l + O\left(\frac{1}{(n\delta)^{1/4}}\right) \right]. \quad (2.36)$$

where  $x_0 = 0$  and  $S_n(x) = K_n(\lambda_0, \lambda_0 + \frac{x}{n})$ . The integral in the r.h.s. of (2.36) can be computed by using the Fourier integral technique. This is done in Lemma 11 of Section 3. According to that Lemma

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[ \frac{\sin \pi \rho_n(\lambda_0) s}{\pi \rho_n(\lambda_0) s} + O\left(\frac{1}{(n\delta)^{1/4}}\right) \right]. \quad (2.37)$$

Comparing this expression with (1.22) we see that to finish the proof of the Theorem we have to establish the relation

$$\lim_{n \rightarrow \infty} T_n(\lambda_0) = \rho(\lambda_0).$$

This relation follows from the Proposition and Lemma 10 of Section 3. Theorem is proved.

### 3 . Auxiliary results

In this section we prove a number of facts that we use in the proofs of the Theorem and the Proposition (Sec.2).

**Lemma 1** *Let  $F(t), t \in \mathbf{R}$  be a continuously differentiable and polynomially bounded function, and  $B$  be an arbitrary Hermitian matrix. Then*

$$E\{F_B^l(M)\} - nE\{F(M) \text{Tr}V'(M)B\} = 0 \quad (3.1)$$

where  $F_B^l(M) = \lim_{\epsilon \rightarrow \infty} \epsilon^{-1} [F(M + \epsilon B) - F(M)]$ .

**Proof.** We obtain the Lemma by differentiating with respect to  $t$  the identity

$$\int \exp\{-n\text{Tr}V(M + tB)\} F(M + tB) dM = \int \exp\{-n\text{Tr}V(M)\} F(M) dM$$

that follows from the invariance of the measure  $dM$  with respect to shift  $M \rightarrow M + B$  by an arbitrary Hermitian matrix  $B$ .

**Remark.** This lemma was in fact proved by Bessis et al.<sup>(5)</sup>

**Lemma 2** *Let  $k_n(\lambda, \mu)$  be defined by (1.19). Then*

$$\int (\lambda - \mu)^2 K_n^2(\lambda, \mu) d\lambda d\mu \leq \frac{C}{n^2}, \quad (3.2)$$

and for  $\alpha = 1, 2$

$$|\int (\lambda - \mu)^\alpha K_n^2(\lambda, \mu) d\mu| \leq \frac{C}{n^2} \{(\psi_{n-1}^{(n)}(\lambda))^2 + (\psi_n^{(n)}(\lambda))^2\}. \quad (3.3)$$

**Proof.** It follows from the orthogonality relations (1.15) that for  $j = 0, 1, 2, \dots$

$$r_j P_{j+1}(\lambda) + r_{j-1} P_{j-1}(\lambda) = \lambda P_j(\lambda) \quad (r_{-1} = 0) \quad (3.4)$$

where

$$r_j = \int \lambda P_j(\lambda) P_{j+1}(\lambda) e^{-nV(\lambda)} d\lambda \quad (3.5)$$

and we omit superscript  $n$  to simplify notations. Denote by  $J = \{J_{jk}\}_{j,k=1}^{\infty}$  the Jacobi matrix defined by (3.4):

$$J_{jk} = r_j \delta_{j+1,k} + r_{j-1} \delta_{j-1,k}. \quad (3.6)$$

Then for any nonnegative integer  $p$

$$(J^p)_{jk} = \int \lambda^p \psi_j^{(n)}(\lambda) \psi_k^{(n)}(\lambda) d\lambda. \quad (3.7)$$

By using the identity

$$\int K_n^2(\lambda, \mu) d\mu = \frac{1}{n} K_n(\lambda, \lambda), \quad (3.8)$$

and (3.7) for  $p = 1, 2$  we find that the l.h.s. of (3.2) is

$$2 \left( \sum_{j=0}^{n-1} (J^2)_{jj} - \sum_{j,k=0}^{n-1} J_{jk}^2 \right). \quad (3.9)$$

This relation and (3.6) yield

$$n^2 \int (\lambda - \mu)^2 K_n^2(\lambda, \mu) d\lambda d\mu = 2r_{n-1}^2. \quad (3.10)$$

Using (1.7) and (1.19)-(1.21) we obtain that for some  $n$ -independent  $a, A, L > 0$

$$[\psi_l(\lambda)]^2 \leq n \rho_n(\lambda) \leq n \exp\{-an[V(\lambda) - \max_{|\mu| \leq L_1} V(\mu)]\}, \quad |\lambda| \geq L. \quad (3.11)$$

and then (3.5) implies the bound

$$|r_l| \leq C \quad (3.12)$$

for some  $C$ . This bound and (3.10) imply (3.2). Similar arguments and equation (3.4) yield

$$n^2 \int (\lambda - \mu) K_n^2(\lambda, \mu) d\mu = \psi_{n-1}(\lambda) \psi_n(\lambda) r_{n-1}.$$

Now (3.3) follows from this identity and (3.12). The case  $\alpha = 2$  in the l.h.s. of (3.3) can be proved analogously. Lemma 2 is proved.

**Lemma 3** *Let  $f(\mu)$ ,  $\mu \in \mathbf{R}$ , be a bounded and Holder continuous function:*

$$|f(\lambda) - f(\mu)| \leq C |\lambda - \mu|^\alpha \quad (3.13)$$

for some  $C > 0$  and  $0 < \alpha \leq 1$  and

$$f_n = \frac{1}{n} \sum_{j=1}^n f(\lambda_j^{(n)})$$

where  $\{\lambda_j^{(n)}\}_{j=1}^n$  are eigenvalues of a random matrix. Then

$$D\{f_n\} \equiv E\{|f_n - E\{f_n\}|^2\} \leq C_1 n^{-1-\alpha}. \quad (3.14)$$

**Proof.** By using (1.18) and (1.19) we can write (3.14) as follows

$$D\{f_n\} = \frac{1}{2} \int |f(\lambda) - f(\mu)|^2 K_n^2(\lambda, \mu) d\lambda d\mu.$$

This representation, (3.13), the Holder inequality and the relation

$$\int K_n^2(\lambda, \mu) d\lambda d\mu = n^{-1} \quad (3.15)$$

yield the bound

$$D\{f_n\} \leq \frac{C^2}{2} \left[ \int |\lambda - \mu|^2 K_n^2(\lambda, \mu) d\lambda d\mu \right]^\alpha \left[ n^{-1} \right]^{1-\alpha}$$

which implies (3.14) in view of Lemma 2. Lemma 3 is proved.

**Lemma 4** *Assume that  $\lambda$  is a point of the spectral axis at which  $\rho(\lambda) > 0$ . Then*

$$\rho(\lambda) = \frac{1}{\pi} \sqrt{Q(\lambda) - (V'(\lambda))^2/4} \quad (3.16)$$

where  $Q(\lambda)$  is defined by (2.7).

**Proof.** According to (1.8)  $\rho_n(\lambda)$  converges weakly to  $\rho(\lambda)$ . This result allows us to perform the limiting transition in (2.5) and to obtain for nonreal  $z$ 's the relation

$$g^2(z) + V'(z)g(z) + Q(z) = 0 \quad (3.17)$$

where  $g(z)$  is the Stieltjes transform of the limiting density  $\rho(\lambda)$ . Definition (2.7) and condition (1.11b) of the Theorem imply that  $Q(\lambda + i0)$  is a real valued, bounded function with a bounded derivative. Then by general principles

$$\rho(\lambda) = \frac{1}{\pi} \Im g(\lambda + i0) \quad (3.18)$$

is also bounded. Computing the real and the imaginary parts of (3.17) rewritten as

$$g = -\frac{Q}{V' + g} \quad (3.19)$$

we find (3.16). Lemma 4 is proved.

**Lemma 5** *Under the conditions of the Theorem*

$$\sup_{n, \lambda} \rho_n(\lambda) \leq C \quad (3.20)$$

and

$$\left| \frac{d\rho_n(\lambda)}{d\lambda} \right| \leq C_1(\psi_{n-1}^2(\lambda) + \psi_n^2(\lambda)) + C_2. \quad (3.21)$$

**Proof.** We start from a simple identity  $\frac{d\rho_n(\lambda)}{d\lambda} = \frac{d\rho_n(\lambda + t)}{dt} \Big|_{t=0}$ . Performing the change of variables in the integral (1.2)  $\lambda_i - t = \mu_i$ ,  $i = 2, \dots, n$  we rewrite  $\rho_n(\lambda + t)$  as follows:

$$\rho_n(\lambda + t) = Z_n^{-1} \int \exp \left\{ -nV(\lambda) - n \sum_{j=2}^n V(t + \mu_j) \right\} \prod_{j=2}^n (\lambda - \mu_j)^2 \prod_{i>j \geq 2}^n (\mu_i - \mu_j)^2 d\mu_j$$

Thus after differentiating with respect to  $t$  we get for  $t = 0$

$$\begin{aligned} \frac{d\rho_n(\lambda)}{d\lambda} &= -nV'(\lambda)\rho_n(\lambda) - n(n-1) \int V'(\lambda_2)p_n(\lambda, \lambda_2, \dots, \lambda_n)d\lambda_2 \dots d\lambda_n = \\ &= -nV'(\lambda)K_n(\lambda, \lambda) - n^2 \int V'(\lambda_2)[K_n(\lambda, \lambda)K_n(\lambda_2, \lambda_2) - K_n^2(\lambda, \lambda_2)]d\lambda_2 \end{aligned} \quad (3.22)$$

The identity (3.1) for  $F(M) = 1$  and  $B = 1$  yields

$$E\{\text{Tr}V'(M)\} = n \int V'(\lambda)K_n(\lambda, \lambda)d\lambda = 0.$$

Hence by (3.22)

$$\rho'_n(\lambda) = n^2 \int [V'(\mu) - V'(\lambda)]K_n^2(\lambda, \mu)d\mu. \quad (3.23)$$

Now we split this integral in two parts corresponding to the intervals  $|\mu| > L$  and  $|\mu| \leq L$  where  $L$  is defined by (1.7). The former integral is bounded because of the inequality  $K_n^2(\lambda, \mu) \leq K_n(\lambda, \lambda)K_n(\mu, \mu)$  and bound (1.7) for  $K_n(\lambda, \lambda) = \rho_n(\lambda)$ . In the latter integral we write

$$V'(\mu) - V'(\lambda) = (\mu - \lambda)V''(\lambda) + \frac{(\mu - \lambda)^2}{2}V'''(\xi)$$

for some  $\xi$  depending on  $\lambda$  and  $\mu$  and use Lemma 2 and condition (1.11b) of the Theorem. Combining the bounds for these two integrals we obtain (3.21). To obtain (3.20) we have to use (3.23) and (1.13). Lemma 5 is proved.

**Lemma 6** Take  $\epsilon = O(n^{-1/4})$ . Then for any  $\lambda$  and  $n$  such that  $\rho(\lambda) > n^{-1/9}$  we have

$$\int_{|\mu - \lambda| \leq \epsilon} (\psi_{n-1}^2(\mu) + \psi_n^2(\mu))d\mu \leq C_1 \left(1 + \frac{1}{\rho(\lambda)}\right) n^{-1/4}, \quad (3.24)$$

$$\frac{1}{n}(\psi_{n-1}^2(\mu) + \psi_n^2(\mu)) \leq C_2 \left(1 + \frac{1}{\rho(\lambda)}\right) n^{-1/8}, \quad |\mu - \lambda| \leq \epsilon. \quad (3.25)$$

**Proof.** Let us introduce the density

$$p_n^-(\lambda_1, \dots, \lambda_{n-1}) = \frac{1}{Z_n^-} \exp \left\{ -n \sum_{j=1}^{n-1} V(\lambda_j) \right\} \prod_{1 \leq j < k \leq n-1} (\lambda_j - \lambda_k)^2. \quad (3.26)$$

The difference of this density from density (1.2) written for  $n-1$  variables  $\lambda_1, \dots, \lambda_{n-1}$  is that in the former we have the factor  $n$  in the exponent while in the latter we would have  $n-1$ . Set

$$\rho_n^-(\lambda) = \frac{n-1}{n} \int p_n^-(\lambda, \lambda_2, \dots, \lambda_{n-1})d\lambda_2 \dots d\lambda_{n-1} = \frac{1}{n} \sum_{j=0}^{n-2} [\psi_j(\lambda)]^2. \quad (3.27)$$

Then

$$\psi_{n-1}(\lambda) = n[\rho_n(\lambda) - \rho_n^-(\lambda)]. \quad (3.28)$$

Furthermore, by using the analogue of identity (3.1) for the density  $p_n^-$  and the arguments similar to those proving (2.5) we obtain the relation

$$[g_n^-(z)]^2 + \int \frac{V'(\mu)\rho_n^-(\mu)}{\mu - z}d\mu = O\left(\frac{1}{n^2\eta^4}\right) \quad (3.29)$$

for the Stieltjes transform  $g_n^-(z)$  of  $\rho_n^-(\mu)$  and  $z = \lambda + i\eta$ ,  $\eta > 0$ . Denote

$$\Delta_n(z) \equiv n(g_n(z) - g_n^-(z)) \equiv \int \frac{\psi_{n-1}^2(\mu)}{\mu - z} d\mu, \quad (3.30)$$

subtract (3.29) from (2.5) and multiply the result by  $n$ . We obtain

$$\Delta_n(z)(g_n(z) + g_n^-(z)) + \int \frac{V'(\mu)\psi_{n-1}^2(\mu)}{\mu - z} d\mu = O\left(\frac{1}{n\eta^4}\right).$$

For  $z = \lambda + in^{-1/4}$  this relation takes the form

$$\Delta_n(z)(g_n(z) + g_n^-(z) - V'(\lambda)) = \int \frac{(V'(\lambda) - V'(\mu))\psi_{n-1}^2(\mu)}{\mu - z} d\mu + O(1).$$

Then relation (2.8), (3.16) and (3.18) imply that

$$\Im \Delta_n(\lambda + in^{-1/4}) \leq C_3 \left(1 + \frac{1}{\rho(\lambda)}\right).$$

Using definition (3.30) we obtain for  $z = \lambda + i\epsilon$  with  $\epsilon = n^{-1/4}$

$$\int_{|\mu - \lambda| \leq \epsilon} \psi_{n-1}^2(\mu) d\mu \leq 2\epsilon^2 \int \frac{\psi_{n-1}^2(\mu)}{(\mu - \lambda)^2 + \epsilon^2} d\mu \leq C_4 \left(1 + \frac{1}{\rho(\lambda)}\right) \epsilon. \quad (3.31)$$

Now we derive (3.25) from (3.31). Set

$$\Psi_{n-1} = \psi_{n-1}^2(\mu^*) = \max_{|\mu - \lambda| \leq \epsilon} \{\psi_{n-1}^2(\mu)\}, \quad \mu_1 = \sup\{\mu : \mu \in (\lambda - \epsilon, \mu^*), \psi_{n-1}^2(\mu) \leq \frac{\Psi_{n-1}}{2}\}.$$

Since  $|\mu_1 - \mu^*| \leq \epsilon$  we have from (3.31)

$$\frac{\Psi_{n-1}}{2}(\mu^* - \mu_1) \leq \int_{\mu_1}^{\mu^*} \psi_{n-1}^2(\mu) d\mu \leq C_4 \left(1 + \frac{1}{\rho(\lambda)}\right) n^{-1/4}. \quad (3.32)$$

On the other hand

$$\left(\frac{\Psi_{n-1}^{1/2} - (\frac{1}{2}\Psi_{n-1})^{1/2}}{\mu^* - \mu_1}\right)^2 \cdot (\mu^* - \mu_1) \leq \int_{\mu_1}^{\mu^*} (\psi'_{n-1})^2(\mu) d\mu \leq \int (\psi'_{n-1})^2(\mu) d\mu$$

and since

$$\int (\psi'_{n-1})^2(\mu) d\mu = \int \frac{n^2}{4} (V'(\mu))^2 (\psi'_{n-1})^2(\mu) d\mu \leq C_5 n^2$$

we obtain

$$\frac{\Psi_{n-1}}{\mu^* - \mu_1} \leq C_5 n^2 \quad (3.33)$$

Now if we multiply (3.32) by (3.33) we get (3.25) for  $\psi_{n-1}$ .

To prove the analogous bounds for  $\psi_n^2$  we have to repeat above arguments for the density (cf.(3.26))

$$p_n^+(\lambda_1, \dots, \lambda_{n+1}) = \frac{1}{Z_n^+} \exp \left\{ -n \sum_{j=1}^{n+1} V(\lambda_j) \right\} \prod_{1 \leq j < k \leq n+1} (\lambda_j - \lambda_k)^2$$

and for

$$\rho_n^+(\lambda) = \frac{n+1}{n} \int p_n^+(\lambda, \lambda_2, \dots, \lambda_{n+1}) d\lambda_2 \dots d\lambda_{n+1} = \frac{1}{n} \sum_{j=0}^n [\psi_j(\lambda)]^2,$$

so that  $\psi_n^2(\lambda) = n(\rho_n^+(\lambda) - \rho_n(\lambda))$  (cf.(3.28)). Lemma 6 is proved.

**Lemma 7** *If  $\rho(\lambda_0) \neq 0$ , then*

$$|K_n(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) - K_n(\lambda_0, \lambda_0 + \frac{y-x}{n})| \leq C|x| \left( \frac{1}{n^{1/8}} + \frac{|x-y|^2}{n^2} + e^{-na\delta_1/2} \right). \quad (3.34)$$

**Proof.** Repeating almost literally the derivation of (3.22) we get

$$\begin{aligned} & \frac{d}{dt} K_n(\lambda_0 + \frac{x-tx}{n}, \lambda_0 + \frac{y-tx}{n}) = \\ & xn \int K_n(\lambda_0 + \frac{x-tx}{n}, \lambda) K_n(\lambda_0 + \frac{y-tx}{n}, \lambda) \left( V'(\lambda) - \frac{1}{2}V'(\lambda_0 + \frac{x-tx}{n}) - \frac{1}{2}V'(\lambda_0 + \frac{y-tx}{n}) \right) d\lambda. \end{aligned}$$

To estimate the r.h.s. of this relation we split this integral in two parts corresponding to the intervals  $|\lambda| > L$  and  $|\lambda| \leq L$  where  $L$  is defined by (1.7). The former integral is bounded by  $C \exp\{-na\delta_1/2\}$  because of the inequality  $K_n^2(\lambda, \mu) \leq K_n(\lambda, \lambda)K_n(\mu, \mu)$  and bound (1.7) for  $K_n(\lambda, \lambda) = \rho_n(\lambda)$ . In the latter integral we write

$$\begin{aligned} & V'(\lambda) - \frac{1}{2}V'(\lambda_0 + \frac{x-tx}{n}) - \frac{1}{2}V'(\lambda_0 + \frac{y-tx}{n}) = \frac{1}{2}(\lambda - \lambda_x)V''(\lambda_x) + \frac{1}{2}(\lambda - \lambda_y)V''(\lambda_y) + \\ & O\left((\lambda - \lambda_x)^2 + (\lambda - \lambda_y)^2\right) = \frac{1}{2}(\lambda - \lambda_x)V''(\lambda_x) + \frac{1}{2}(\lambda - \lambda_y)V''(\lambda_y) + O\left((\lambda - \lambda_x)(\lambda - \lambda_y) + \frac{|x-y|^2}{n^2}\right) \end{aligned}$$

where

$$\lambda_x = \lambda_0 + \frac{x-tx}{n}, \quad \lambda_y = \lambda_0 + \frac{y-tx}{n}.$$

According to (3.7)

$$n \int K_n(\lambda_x, \lambda) K_n(\lambda_y, \lambda) (\lambda - \lambda_{x,y}) d\lambda = \frac{r_{n-1}}{n} \psi_n^{(n)}(\lambda_x) \psi_n^{(n)}(\lambda_y)$$

Besides, by the Schwartz inequality

$$\begin{aligned} & n \left| \int K_n(\lambda_x, \lambda) K_n(\lambda_y, \lambda) (\lambda - \lambda_x)(\lambda - \lambda_y) d\lambda \right| \leq \\ & n \left[ \int K_n(\lambda_x, \lambda) (\lambda - \lambda_x)^2 d\lambda \int K_n(\lambda_y, \lambda) (\lambda - \lambda_y)^2 d\lambda \right]^{1/2}. \end{aligned}$$

Now the arguments similar to those used in the proof of Lemma 2 and Lemma 5 yield the estimate

$$\begin{aligned} & \left| \frac{d}{dt} K(\lambda_0 + \frac{x-tx}{n}, \lambda_0 + \frac{y-tx}{n}) \right| \leq \\ & \frac{C}{n} |x| \left( \psi_n^2(\lambda_x) + \psi_{n-1}^2(\lambda_x) + \psi_n^2(\lambda_y) + \psi_{n-1}^2(\lambda_y) + \frac{|x-y|^2}{n} + e^{-na\delta_1/2} \right). \end{aligned}$$

Combining this estimate with (3.25) we obtain (3.34).

**Lemma 8** *Let  $p_2^{(n)}(\lambda_1, \lambda_2)$  be specified by (1.18) for  $l = 2$ . Then uniformly in  $n$*

$$\int_{-1}^1 \frac{p_2^{(n)}(\lambda_0 + \frac{x}{n}, \lambda_0)}{x^2} dx \leq C. \quad (3.35)$$



**Proof.** Consider

$$W = \left\langle \prod_{i=2}^n \left| 1 - \frac{1}{n^2(\lambda_i - \lambda_0)^2} \right| \right\rangle$$

where symbol  $\langle \dots \rangle$  was defined in (2.11). By the Schwarz inequality  $W^2$  is bounded from above by the product of integrals

$$Z_n^{-1} \int \prod_{2 \leq j < k \leq n} (\lambda_i - \lambda_j)^2 \prod_{2 \leq j \leq n} (\lambda_0 + \sigma - \lambda_j)^2 \exp\{-nV(\lambda_0) - n \sum_{j=2}^n V(\lambda_j)\}$$

for  $\sigma = \pm 1/n$ . Besides,  $n(V(\lambda_0) - V(\lambda_0 + \sigma))$  is bounded in  $n$  due to (1.11b). This allows us to write the bound

$$W \leq C \cdot K_n^{1/2}(\lambda_0 + \frac{1}{n}, \lambda_0 + \frac{1}{n}) K_n^{1/2}(\lambda_0 - \frac{1}{n}, \lambda_0 - \frac{1}{n}) \leq C_1. \quad (3.36)$$

On the other hand  $W$  can be written as

$$\begin{aligned} W &= \left\langle \prod_{i=2}^n (\phi_1(\lambda_i) + \phi_2(\lambda_i)) \right\rangle = \\ &= \left\langle \prod_{i=2}^n \phi_2(\lambda_i) \right\rangle + \sum_{k=1}^{n-1} C_{n-1}^k \left\langle \prod_{i=2}^{k+1} \phi_1(\lambda_i) \prod_{i=k+2}^n \phi_2(\lambda_i) \right\rangle \end{aligned}$$

where

$$\phi_1(\lambda) = \begin{cases} \frac{(1 - n^2(\lambda - \lambda_0)^2)^2}{n^2(\lambda - \lambda_0)^2}, & n|\lambda - \lambda_0| < 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\phi_2(\lambda) = \begin{cases} 1 - n^2(\lambda - \lambda_0)^2, & n|\lambda - \lambda_0| < 1, \\ \frac{n^2(\lambda - \lambda_0)^2 - 1}{n^2(\lambda - \lambda_0)^2}, & \text{otherwise.} \end{cases}$$

Since  $0 \leq \phi_2(\lambda) \leq 1$ ,  $\phi_1(\lambda) \geq 0$  and  $\langle 1 \rangle = \rho_n(\lambda_0)$ , we get from this representation

$$W \geq -\rho_n(\lambda_0) + (n-1) \int d\lambda \phi_1(\lambda) \left\langle \delta(\lambda_2 - \lambda) \exp \left\{ \sum_{i=3}^n \log \phi_2(\lambda_i) \right\} \right\rangle. \quad (3.37)$$

By using the Jensen inequality and (1.2) we have

$$\begin{aligned} & \frac{\langle \delta(\lambda_2 - \lambda) \exp \{ \sum_{i=3}^n \log \phi_2(\lambda_i) \} \rangle}{\langle \delta(\lambda_2 - \lambda) \rangle} \geq \\ & \exp \left\{ \left\langle \delta(\lambda_2 - \lambda) \sum_{i=3}^n \log \phi_2(\lambda_i) [p_2^{(n)}(\lambda_0, \lambda)]^{-1} \right\rangle \right\} = \\ & \exp \left\{ (n-2) \int \log \phi_2(\lambda') p_3^{(n)}(\lambda_0, \lambda, \lambda') d\lambda' [p_2^{(n)}(\lambda_0, \lambda)]^{-1} \right\}. \end{aligned} \quad (3.38)$$

According to (1.18)

$$p_3^{(n)}(\lambda_0, \lambda, \lambda') = \frac{n^2}{(n-1)(n-2)} \left( \frac{n-1}{n} \rho_n(\lambda') p_2^{(n)}(\lambda_0, \lambda) + 2K_n(\lambda_0, \lambda) K_n(\lambda_0, \lambda') K_n(\lambda, \lambda') - \right.$$

$$\rho_n(\lambda_0)K_n^2(\lambda, \lambda') - \rho_n(\lambda)K_n^2(\lambda_0, \lambda'). \quad (3.39)$$

Besides, since  $\log \phi_2(\lambda') \leq 0$  and

$$\begin{aligned} 2K_n(\lambda_0, \lambda)K_n(\lambda_0, \lambda)K_n(\lambda', \lambda) &\leq 2K_n^{1/2}(\lambda_0, \lambda_0)K_n^{1/2}(\lambda, \lambda')|K_n(\lambda_0, \lambda')||K_n(\lambda', \lambda)| \leq \\ &\rho_n(\lambda_0)K_n^2(\lambda', \lambda) + \rho_n(\lambda)K_n^2(\lambda_0, \lambda') \end{aligned}$$

we have

$$\int d\lambda' \log \phi_2(\lambda') \left( 2K_n(\lambda_0, \lambda)K_n(\lambda_0, \lambda')K_n(\lambda', \lambda) - \rho_n(\lambda_0)K_n^2(\lambda, \lambda') - \rho_n(\lambda)K_n^2(\lambda_0, \lambda') \right) \geq 0.$$

Hence taking into account that  $\rho_n(\lambda)$  is bounded from above uniformly in  $n$  we get

$$\begin{aligned} W &\geq -\rho_n(\lambda_0) + (n-1) \int d\lambda \phi_1(\lambda) p_2^{(n)}(\lambda_0, \lambda) \exp \left\{ (n-1) \int \rho_n(\lambda') \log \phi_2(\lambda') d\lambda' \right\} \geq -\rho_n(\lambda_0) + \\ &\int_{-1}^1 \frac{(1-x^2)^2}{x^2} p_2^{(n)}(\lambda_0, \lambda_0 + \frac{x}{n}) dx \cdot \exp \left\{ -C \left( \int_0^1 |\log(1-y^2)| dy + \int_1^\infty \log(1-y^{-2}) dy \right) \right\}. \quad (3.40) \end{aligned}$$

From (3.40) and (3.36) it is easy to derive (3.35).

**Lemma 9** *Let the functions  $A(x, y)$ ,  $B(x, y)$  be defined for  $|x|, |y| \leq n\delta$ ,  $\delta = n^{-1} \log n$  and satisfy the conditions:*

$$A(x, y) \leq a(x) \leq C_0, \quad \int_{-1}^1 \frac{a^2(x)}{x^2} dx \leq C_1^2, \quad (3.41)$$

$$\int |A(x, y)|^2 dx \leq C_2^2, \quad (3.42)$$

$$|A(x, x) - A(-x, -x)| \leq \epsilon_1 x, \quad (3.43)$$

$$|A(x, y) - B(x, y)| \leq \epsilon_2 b(x), \quad \int_{-1}^1 \frac{b^2(x)}{x^2} dx \leq C_3^2, \quad |b(x)| \leq C_4, \quad (3.44)$$

$$\int_{-n\delta}^{n\delta} |A(x, y) - B(x, y)|^2 dx \leq C_5^2 \epsilon_3^2. \quad (3.45)$$

Then

$$\int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \| |A(x_j, x_k)| \|_{j,k=1}^l \leq (lC)^{l/2}, \quad (3.46)$$

and

$$\left| \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} (\det \| |A(x_j, x_k)| \|_{j,k=1}^l - \det \| |B(x_j, x_k)| \|_{j,k=1}^l) \right| \leq \epsilon l (lC)^{l/2} \quad (3.47)$$

where  $\epsilon = n\delta(\epsilon_1 + \epsilon_2) + \epsilon_3$  and  $C$  depend only on  $C_i$ ,  $i = 1, \dots, 5$

**Proof.** It suffices to prove estimates (3.46) and (3.47) for

$$A_0(x_j, x_k) = A(x_j, x_k)(1 - \delta_{jk}),$$

and

$$B_0(x_j, x_k) = B(x_j, x_k)(1 - \delta_{jk})$$

Indeed, due to conditions (3.43) and (3.44) the following inequalities are true

$$\left| \int_{-n\delta}^{n\delta} \frac{A(x, x)}{x} dx \right| \leq 2n\delta\epsilon_1$$

and

$$\left| \int_{-n\delta}^{n\delta} \frac{B(x, x)}{x} dx \right| \leq 2n\delta\epsilon_1 + 2\epsilon_2(C_3 + C_4 \log n\delta)$$

and we can easily obtain (3.46) and (3.47) for general  $A$  and  $B$  from the respective bounds for  $A_0$  and  $B_0$ .

Consider

$$F(t) = \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \|A_t(x_j, x_k)\|_{j,k=1}^l$$

where  $A_t(x_j, x_k) = A_0(x_j, x_k) + t(B_0 - A_0)(x_j, x_k)$ . To obtain (3.46) we have to estimate  $|F(1) - F(0)|$ . Therefore it suffices to estimate  $\frac{dF}{dt}$ . Differentiating  $F(t)$ , making respective permutations of columns and rows and the same renumbering of variables we obtain

$$\frac{dF}{dt} = l \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{x_j} \det \|D_t(x_j, x_k)\|_{j,k=1}^l$$

where

$$D_t(x_1, x_k) = (A_0 - B_0)(x_1, x_k), \quad D_t(x_j, x_k) = A_t(x_j, x_k), \quad j \geq 2.$$

Thus

$$\begin{aligned} l^{-1} \left| \frac{dF}{dt} \right| &\leq \int_{-1}^1 \prod_{j=1}^l \frac{dx_j}{|x_j|} \left| \det \|D_t(x_j, x_k)\|_{j,k=1}^l \right| + \int_{-n\delta}^{n\delta} \prod_{j=1}^l \frac{dx_j}{|x_j|} (1 - \chi_1(x_j)) \left| \det \|D_t(x_j, x_k)\|_{j,k=1}^l \right| + \\ &\sum_{m=1}^l C_l^m \left[ \int_{-n\delta}^{n\delta} \prod_{j=1}^m \frac{dx_j}{|x_j|} \chi_1(x_j) \prod_{j=m+1}^l (1 - \chi_1(x_j)) \frac{dx_j}{|x_j|} \left| \det \|D_t(x_j, x_k)\|_{j,k=1}^l \right| + \right. \\ &\left. \int_{-n\delta}^{n\delta} \prod_{j=1}^m \frac{dx_j}{|x_j|} (1 - \chi_1(x_j)) \prod_{j=m+1}^l \chi_1(x_j) \frac{dx_j}{|x_j|} \left| \det \|D_t(x_j, x_k)\|_{j,k=1}^l \right| \right] \end{aligned} \quad (3.48)$$

where  $\chi_1(x)$  is the indicator of the interval  $(-1, 1)$ . Let us estimate the last term

$$F_t^{(m)} = \int_{-n\delta}^{n\delta} \prod_{j=1}^m \frac{dx_j}{|x_j|} (1 - \chi_1(x_j)) \prod_{j=m+1}^l \chi_1(x_j) \frac{dx_j}{|x_j|} \left| \det \|D_t(x_j, x_k)\|_{j,k=1}^l \right|.$$

Other terms in the r.h.s. of (3.48) can be estimated similarly.

$$\begin{aligned} |F_t^{(m)}| &\leq \left\{ \int_{-n\delta}^{n\delta} \prod_{j=1}^m \frac{dx_j}{|x_j|^{3/4}} (1 - \chi_1(x_j)) \prod_{j=m+1}^l \chi_1(x_j) \frac{dx_j}{|x_j|^2} \left| \det \|D_t(x_j, x_k)\|_{j,k=1}^l \right|^2 \right\}^{1/2} \times \\ &\left\{ \int_{-n\delta}^{n\delta} \prod_{j=1}^m \frac{dx_j}{|x_j|^{5/4}} (1 - \chi_1(x_j)) \prod_{j=m+1}^l \chi_1(x_j) dx_j \right\}^{1/2} \leq \\ &\left\{ \int_{-n\delta}^{n\delta} \prod_{j=1}^m \frac{dx_j}{|x_j|^{3/4}} (1 - \chi_1(x_j)) \prod_{j=m+1}^l \chi_1(x_j) \frac{dx_j}{|x_j|^2} \prod_{j=1}^l \sum_{k=1}^l D_t^2(x_j, x_k) \right\}^{1/2} \cdot 2^{m+l/2}. \end{aligned} \quad (3.49)$$

Here we have used the Schwartz inequality and then the Hadamard estimate for determinants<sup>(16)</sup>. Now the r.h.s. of (3.49) can be rewritten as the sum of the integrals

$$I_{j_1 \dots j_l} = \int_{-n\delta}^{n\delta} \prod_{j=1}^m \frac{dx_j}{|x_j|^{3/4}} (1 - \chi_1(x_j)) \prod_{j=m+1}^l \chi_1(x_j) \frac{dx_j}{|x_j|^2} D_t^2(x_1, x_{j_1}) \dots D_t^2(x_l, x_{j_l}) \leq$$

$$\int_{-n\delta}^{n\delta} \prod_{j=1}^m \frac{dx_j}{|x_j|^{3/4}} (1 - \chi_1(x_j)) \prod_{j=m+1}^l \chi_1(x_j) \frac{dx_j}{|x_j|^2} D_t^2(x_1, x_{j_1}) \dots D_t^2(x_m, x_{j_m}) a_t^2(x_{m+1}) \dots a_t^2(x_l) \quad (3.50)$$

with  $a_t(x) = a(x) + t\epsilon_3 b(x)$ .

To estimate the last integral we start integrating with respect to the "free" variables, i.e. the variables that do not enter the set  $(x_{j_1}, \dots, x_{j_m})$ . We use bound (3.42) for the integral with respect to  $x_1$  and bounds (3.44) and (3.45) for integrals with respect to  $x_2, \dots, x_m$ . If there is no free variables, then we use the inequality  $D_t^2(x_m, x_{j_m}) \leq (C_0 + t\epsilon_2 C_4)^2$  which makes the variable  $x_{j_m}$  free. Repeating this procedure we end up either with the estimate

$$I_{j_1 \dots j_l} \leq C^{l-1} \Delta_{1,j} \quad (3.51)$$

or with the estimate

$$I_{j_1 \dots j_l} \leq \epsilon_3^2 C^{l-1} \Delta_{i,j} \quad (3.52)$$

where  $C = \max\{2(C_1 + \epsilon_2 C_3), C_0 + \epsilon_2 C_4, 2(C_2 + \epsilon_3 C_5)\}$  and

$$\Delta_{ij} = \int_{-n\delta}^{n\delta} D_t^2(x_i, x_j) (1 - \chi_1(x_i)) (1 - \chi_1(x_j)) \frac{dx_i}{|x_i|^{3/4}} \frac{dx_j}{|x_j|^{3/4}}$$

with some  $i, j \leq m$ . Regarding  $D_t^2(x_i, x_j)$  as the kernel of an integral operator acting in  $L_2(-n\delta, n\delta)$  and using the bound  $\sup_{x_i} \int_{-n\delta}^{n\delta} D_t^2(x_i, x_j) dx_j$  for the norm of this operator and bounds (3.42) and (3.45) we obtain that

$$I_{j_1 \dots j_l} \leq \epsilon_3^2 C^l.$$

Repeating similar argument to estimate all other terms in (3.48) we obtain (3.46) and (3.47)

**Lemma 10** *Let  $T_n$  be defined by (2.13). Then*

$$|T_n(\lambda_0) - \rho_n(\lambda_0)| \leq \frac{C}{\log n}. \quad (3.53)$$

**Proof.** We will prove the following bounds:

$$|E\{U_n(\lambda_0)\} - \frac{s}{2} V'(\lambda_0)| \leq C s n^{-1/4} \log n \quad (3.54)$$

$$\langle \exp\{2U_n(\lambda_0)\} \rangle \leq C \quad (3.55)$$

$$\langle |U_n(\lambda_0) - E\{U_n(\lambda_0)\}|^2 \rangle \leq \frac{C}{n\delta} \quad (3.56)$$

where  $U_n(\lambda_0)$  is specified by (2.15). Assuming that these bounds are true it is easy to prove (3.53) by using the Schwarz inequality and elementary inequality  $|e^x - 1| \leq |x|(e^x + 1)$ .

To prove (3.54) take  $\delta_1 = n^{-1/4}$  and rewrite  $E\{U_n(\lambda_0)\}$  as follows

$$E\{U_n(\lambda_0)\} = (n-1) \int u(\lambda) \rho_n(\lambda) d\lambda = \int ((n-1)u(\lambda) - \phi(\lambda)) (\rho_n(\lambda) - \rho(\lambda)) d\lambda +$$

$$\int \phi(\lambda) (\rho_n(\lambda) - \rho(\lambda)) d\lambda + (n-1) \int u(\lambda) \rho(\lambda) d\lambda \quad (3.57)$$

where  $\phi(\lambda)$  is a differentiable function of the form:

$$\phi(\lambda) = \begin{cases} (n-1) \frac{\lambda_0 + \delta_1 - \lambda}{2\delta_1} \ln \left(1 - \frac{s}{n\delta_1}\right) + (n-1) \frac{\lambda - \lambda_0 + \delta_1}{2\delta_1} \ln \left(1 + \frac{s}{n\delta_1}\right), & |\lambda - \lambda_0| < \delta_1, \\ (n-1)u(\lambda), & \text{otherwise} \end{cases}$$

Using the Proposition and Lemma 4 one can estimate the first integral  $I_1$  in the r.h.s. of (3.57) as follows

$$I_1 \leq Cn^{-1/4} \int_{|\lambda - \lambda_0| < \delta_1} ((n-1)|u(\lambda)| + |\phi(\lambda)|) d\lambda \leq Csn^{-1/4} \log n \quad (3.58)$$

To estimate the second integral we use inequality (1.8) according to which

$$\left| \int \phi(\lambda) (\rho_n(\lambda) - \rho(\lambda)) d\lambda \right| \leq Cn^{-1/2} \log^{1/2} n \|\phi'\|_2^{1/2} \|\phi\|_2^{1/2} = C\delta_1^{-1} n^{-1/2} \log^{1/2} n \quad (3.59)$$

And the last integral  $I_3$  in the r.h.s. of (3.57) can be calculated by using the result<sup>(9)</sup>, according to which for any  $\lambda$ :  $\rho(\lambda) \neq 0$

$$\int \log |\lambda - \lambda'| \rho(\lambda') d\lambda' = \frac{1}{2} V(\lambda) + \text{const} \quad (3.60)$$

Thus we have

$$I_3 = \frac{n-1}{2} (V(\lambda_0 + s/n) - V(\lambda_0)) + O((n\delta_1)^{-1}) \quad (3.61)$$

Relations (3.57)-(3.61) prove (3.54).

To prove (3.55) consider  $f(t) \equiv \log \langle \exp\{tU_n(\lambda_0)\} \rangle$ . Since  $f(t)$  is a convex function

$$f(2) \leq f(0) + 2f'(2) = \log \rho_n(\lambda_0) + 2f'(2) \quad (3.62)$$

In view of (2.16)

$$\begin{aligned} f'(2) &= (n-1) \int u(\lambda) R_{2n}(\lambda, \lambda) d\lambda = \\ &= (n-1) \int u(\lambda) R_{0n}(\lambda, \lambda) d\lambda + (n-1) \int_0^2 dt \int u(\lambda) R'_{tn}(\lambda, \lambda) d\lambda \end{aligned} \quad (3.63)$$

where  $R_{tn}$  and  $R'_{tn}$  are specified by (2.20) and (2.23). According to (2.23) the second integral  $I_2$  in (3.63) can be rewritten as

$$\begin{aligned} I_2 &= (n-1) \int_0^2 dt \left[ \int u^2(\lambda) R_{tn}(\lambda, \lambda) d\lambda + (n-1) \int u(\lambda) u(\lambda') R_{tn}^2(\lambda, \lambda') d\lambda d\lambda' \right] \leq \\ &= C \int_{n\delta}^{\infty} \log^2 \left(1 + \frac{s}{x}\right) dx + (n-1)^2 \int_0^2 dt \int u(\lambda) u(\lambda') R_{tn}^2(\lambda, \lambda') d\lambda d\lambda' \end{aligned} \quad (3.64)$$

where we have used (2.29) for  $\lambda = \mu$  and (1.20) according to which  $R_{0n}(\lambda, \lambda) \leq K_n(\lambda, \lambda) = \rho_n(\lambda)$ , the boundedness of  $\rho_n(\lambda)$  and (2.27). Regarding  $R_{tn}^2(\lambda, \lambda')$  as a kernel of integral operator  $\overline{R}$  in  $L_2(\mathbf{R})$  one can estimate its norm as  $\|\overline{R}\| \leq \max_{\lambda} \left\{ \int R_{tn}^2(\lambda, \lambda') d\lambda' \right\} = n^{-1}$ . Thus

$$(n-1)^2 \int u(\lambda) u(\lambda') R_{tn}^2(\lambda, \lambda') d\lambda d\lambda' \leq n \int u^2(\lambda) d\lambda \leq C \int_{n\delta}^{\infty} \log^2(1+x^{-1}) dx \leq \frac{C_1}{n\delta} \quad (3.65)$$

To estimate the first integral  $I_1$  in the r.h.s. of (3.63) we use again (2.29). Then

$$I_1 = (n-1) \int u(\lambda) R_{0n}(\lambda, \lambda) d\lambda = (n-1) \int u(\lambda) K_n(\lambda, \lambda) d\lambda - \frac{n-1}{\rho_n(\lambda_0)} \int u(\lambda) K_n^2(\lambda_0, \lambda) d\lambda =$$

$$\frac{s}{2}V'(\lambda_0) + O(n^{-1/4} \log n) + O\left(n \frac{\max |u(\lambda)|}{\rho_n(\lambda_0)} \int K_n^2(\lambda_0, \lambda) d\lambda\right) = \frac{s}{2}V'(\lambda_0) + O((n\delta)^{-1}) \quad (3.66)$$

Here we have used (3.54) to calculate  $(n-1) \int u(\lambda) K_n(\lambda, \lambda) d\lambda$ . Relations (3.62)-(3.66) prove (3.55).

To prove (3.66) let us note that in view of (3.54) and (3.66)

$$E\{U_n(\lambda_0)\} = \left\langle \sum_{j=2}^n u(\lambda_j) \right\rangle_0 + O((n\delta)^{-1})$$

where  $\langle \dots \rangle_0$  denotes the expectation with respect to density (2.18). This expectation is related to the operation  $\langle \dots \rangle = E\{\delta(\lambda_0 - \lambda_1) \dots\}$  as  $\langle \dots \rangle = \rho_n(\lambda_0) \langle \dots \rangle_0$ . Thus to estimate the r.h.s. of (3.56) it is enough to estimate

$$\left\langle \left( \sum_{j=2}^n u(\lambda_j) - \left\langle \sum_{j=2}^n u(\lambda_j) \right\rangle_0 \right)^2 \right\rangle_0 \leq$$

$$(n-1) \int u^2(\lambda) R_{0n}(\lambda, \lambda) d\lambda - (n-1)(n-2) \int u(\lambda) u(\lambda') R_{0n}^2(\lambda, \lambda') d\lambda d\lambda'$$

Combining this inequality with estimate (3.64) we get (3.56). Lemma 10 is proved.

**Lemma 11** *Let  $X(x)$ ,  $x \in \mathbf{R}$  be a smooth enough and fast decaying function. Then*

$$\int \prod_{j=1}^l \frac{dx_j}{x_j} \det \|X(x_j - x_k)\|_{j,k=0}^l = \frac{(i\pi)^l X^{l+1}(0)}{l+1} \cdot \frac{1 - (-1)^{l+1}}{2} \quad (3.67)$$

where  $x_0 = 0$ .

**Proof.** By using the Fourier integral representation of  $X(x)$  we can write

$$\det \|X(x_j - x_k)\|_{j,k=0}^l = \int \prod_{j=0}^l dp_j \hat{X}(p_j) \det \|\exp\{ix_j(p_j - p_k)\}\|_{j,k=0}^l$$

where  $\hat{X}(p)$  is the Fourier transform of  $X(x)$ . This representation and the identity

$$\int \frac{e^{ipx}}{x} dx = i\pi \text{sign} p \equiv i\pi \theta(p)$$

allow us to rewrite the integral in the l.h.s. of (3.64) as follows

$$(i\pi)^l \int \prod_{j=0}^l dp_j \hat{X}(p_j) \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \theta(p_1 - p_0) & 0 & \cdot & \cdot & \theta(p_1 - p_l) \\ \theta(p_2 - p_0) & \theta(p_2 - p_1) & 0 & \cdot & \theta(p_2 - p_l) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \theta(p_l - p_0) & \cdot & \cdot & \cdot & 0 \end{vmatrix}$$

Let us compute the determinant in the domain  $p_1, \dots, p_m < p_0$ ,  $p_{m+1}, \dots, p_l \geq p_0$ . Without loss of generality we can assume that  $p_1 < p_2 \dots < p_m < p_0 < p_{m+1} < \dots < p_l$ . Then the determinant

will have the form

$$\begin{vmatrix} +1 & +1 & +1 & +1 & \cdot & \cdot & +1 & +1 \\ -1 & 0 & -1 & -1 & \cdot & \cdot & -1 & -1 \\ -1 & +1 & 0 & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & +1 & +1 & \cdot & \cdot & \cdot & -1 & -1 \\ +1 & +1 & +1 & +1 & \cdot & \cdot & -1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & +1 & +1 & \cdot & \cdot & +1 & +1 & 0 \end{vmatrix}.$$

Subtracting the first row from  $l$ -th,  $(l-1)$ -th, ...,  $(l-m)$ -th ones and then the first column from the second, ...,  $m$ -th ones we find that determinant is equal to  $(-1)^{l-m}$ . Therefore the l.h.s. of (3.67) is equal to

$$(i\pi)^l \sum_{m=0}^l C_l^m \int dp_0 \hat{X}(p_0) \left( \int_{-\infty}^{p_0} dp \hat{X}(p) \right)^m \left( \int_{p_0}^{\infty} \hat{X}(p) dp \right)^{l-m} = \int dp_0 \hat{X}(p_0) \left( 2 \int_{-\infty}^{p_0} dp \hat{X}(p) - X(0) \right)^l = \frac{(i\pi)^l X^{l+1}(0)}{l+1} \cdot \frac{1 - (-1)^{l+1}}{2}.$$

Lemma 11 is proved.

## 4 . Discussion

Let us regard the set  $\{\lambda_j^{(n)}\}_{j=1}^n$  of eigenvalues of random matrices as a point process, i.e. as the random counting measure

$$\nu_n(\Delta) \equiv nN_n(\Delta) = \sum_{\lambda_i^{(n)} \in \Delta} 1. \quad (4.1)$$

Keeping in mind that we are studying the asymptotic behavior of the eigenvalue statistics for large  $n$  we can define this point process either by system (1.4) of its marginal distributions or by its generating functional

$$\Phi_n[\phi] = E \left\{ \exp \left[ \int \phi(\lambda) \nu_n(d\lambda) \right] \right\} \quad (4.2)$$

defined on a suitable space of test functions  $\phi(\lambda)$ ,  $\lambda \in \mathbf{R}$ . We use the simplest case of bounded piece-wise continuous functions with a compact support. Then, by using (1.17) we find that

$$\Phi_n[\phi] = \det(1 - k_n[\phi]) \quad (4.3)$$

where  $k_n[\phi]$  is the integral operator defined on the support  $\sigma_\phi$  of  $\phi$  by the kernel

$$k_n(\lambda, \mu)(1 - e^{\phi(\mu)}). \quad (4.4)$$

According to the Theorem the "scaling" limit (1.9) of all marginal densities (1.4) is given by (1.9) for all unitary invariant ensembles defined by (1.1), (1.7) and (1.11). To find the same limit for the generating functional we have to replace the test function  $\phi(\lambda)$  by  $\phi_n(x) = \phi(x/n\rho_n(\lambda_0))$ . Then

$$\Phi[\phi] \equiv \lim_{n \rightarrow \infty} \Phi_n[\phi_n] = \det(1 - Q_\phi) \quad (4.5)$$

where  $Q_\phi$  is the integral operator defined on  $\sigma_\phi$  by the formula

$$(Q_\phi f)(x) = \int_{\sigma_\phi} S(x-y)(1 - e^{\phi(y)})f(y)dy, \quad x \in \sigma_\phi. \quad (4.6)$$

and  $S(x)$  is defined in (1.10). These formulae contain in fact the same information as (1.9) saying that in our case the point process

$$\nu_{\lambda_0}^{(n)}(t) = \nu_n(\lambda_0, \lambda_0 + \frac{t}{n\rho_n(\lambda_0)}) \quad (4.7)$$

converges weakly as  $n \rightarrow \infty$  to the random process defined by (4.5) and (4.6) or by (1.9).

Consider now the probability

$$R_n \left( \{\Delta_j\}_{j=1}^l \right) = \Pr \{ \nu_n(\Delta_j) = 0, \quad j = 1, \dots, l \} \quad (4.8)$$

that an ordered set of disjoint intervals  $\Delta_i = (a_i, b_i)$  does not contain eigenvalues. Then the arguments similar to those proving (4.3) imply that

$$R_n \left( \{\Delta_j\}_{j=1}^l \right) = \det(1 - K_{n\Delta}) \quad (4.9)$$

where  $\Delta = \cup_{j=1}^l \Delta_j$  and  $K_{n\Delta}$  is the integral operator defined on  $\Delta$  by the kernel

$$\sum_{j=1}^l \chi_{\Delta_j}(\lambda) k_n(\lambda, \mu) \chi_{\Delta_j}(\mu).$$

Setting

$$a_i = \lambda_0 + \frac{\alpha_i}{n\rho_n(\lambda_0)}, \quad b_i = \lambda_0 + \frac{\beta_i}{n\rho_n(\lambda_0)} \quad (4.10)$$

$$\delta_i = (\alpha_i, \beta_i), \quad \delta = \cup_{j=1}^l \delta_j$$

and using the Theorem we obtain that

$$\lim_{n \rightarrow \infty} R_n \left( \{\Delta_j\}_{j=1}^l \right) = r(\delta) \quad (4.11)$$

where  $r(\delta)$  is the Fredholm determinant of the kernel

$$\sum_{j=1}^l \chi_{\delta_j}(x) S(x-y) \chi_{\delta_j}(y). \quad (4.12)$$

We can also introduce more general kernel

$$\sum_{j=1}^l \tau_j \chi_{\delta_j}(x) S(x-y) \chi_{\delta_j}(y). \quad (4.13)$$

for an arbitrary collection of real  $\tau_j$ 's. Then if for an arbitrary collection  $k = (k_1, \dots, k_l)$  of positive integers we consider the probability

$$R_n \left( \{\Delta_j\}_{j=1}^l, \{k_j\}_{j=1}^l \right) = \Pr \{ \nu_n(\Delta_j) = k_j \},$$

its limit  $r(\delta, k)$  is

$$r(\delta, k) = \frac{(-1)^k}{k_1! \dots k_l!} \cdot \frac{\partial^{k_1 + \dots + k_l}}{\partial \tau_1^{k_1} \dots \partial \tau_l^{k_l}} r(\delta, \tau) \Big|_{\tau_j=1} \quad (4.14)$$

where  $r(\delta, \tau)$  is the Fredholm determinant of the kernel (4.13).



The case  $l = 1$  of (4.8) and (4.11) determines<sup>(1)</sup> the limiting probability distribution of distances between nearest neighbor eigenvalues (spacings) lying in the  $O(n^{-1})$  neighborhood of  $\lambda_0$ . Thus in the limit (4.10) the spacing probability distribution is the same for all ensembles satisfying the conditions of the Theorem. For the Gaussian case formula (4.14) was obtained in Ref.18 where some other kernels are also considered and various links of determinant (4.13) with integrable systems and related topics are discussed.

We can also consider another asymptotic regime, making "windows" in  $O(n^{-1})$ -neighborhood of different spectral points, i.e. considering joint probability distribution of the counting functions  $\nu_{\lambda_1}^{(n)}(t_1), \dots, \nu_{\lambda_k}^{(n)}(t_k)$  for distinct  $n$ -independent  $\lambda_1, \dots, \lambda_k$ . Take for simplicity  $k = 2$ . Then we have to consider generating functional (4.2) on functions

$$\phi(\mu) = \phi_1(n\rho_n(\lambda_1)(\mu - \lambda_1)) + \phi_2(n\rho_n(\lambda_2)(\mu - \lambda_2)).$$

Inserting this  $\phi(\mu)$  in (4.2) and using the result<sup>(9)</sup> according to which  $p_2^{(n)}(\lambda_1, \lambda_2) \rightarrow \rho(\lambda_1)\rho(\lambda_2)$  as  $n \rightarrow \infty$  for distinct  $n$ -independent  $\lambda_1, \lambda_2$  and the Theorem we obtain

$$\lim_{n \rightarrow \infty} \Phi_n[\phi] = \Phi[\phi_1]\Phi[\phi_2]$$

where  $\Phi[\phi]$  is defined by (4.5) and (4.6).

We conclude that in the "scaling" limit the local statistics eigenvalues lying in  $O(n^{-1})$ -neighborhoods of distinct spectral points are independent.

**Acknowledgments.** The research described in this publication was made possible in part by Grant N U2S00 from the International Science Foundation and by Grant 1/3/132 from the State Committee for Science and Technology of Ukraine. We are thankful the CNRS for the hospitality in the Laboratory of Mathematical Physics and Geometry of the University Paris VII, where this work was completed.

## References

- [1] M.L.Mehta, Random Matrices (Academic Press, New York, 1991) .
- [2] D.Fox, P.Kahn, Phys.Rev. 134:B1151 (1964).
- [3] H.Leff, J.Math.Phys. 5:761 (1964).
- [4] F.J.Dyson , J.Math.Phys. 13:90 (1972).
- [5] D.Bessis , C.Itzykson , and J.Zuber , Adv.Appl.Math. 1: 109 (1980).
- [6] K.Demetrefi, Internat.J.Mod.Phys. A8 :1185 (1993).
- [7] J.-L.Pichard, In: Quantum Coherence in Mesoscopic Systems (B.Kramer (Ed.), Plenum, New York, 1991).
- [8] R.Fernandez , J.Frohlich, and A.Sokal, Random Walks, Critical Phenomena and Triviality in the Quantum Field Theory (Springer Verlag, Heidelberg, 1992).
- [9] A.Boutet de Monvel, L.Pastur and M.Shcherbina, J.Stat.Phys. 79:585 (1995).
- [10] G.Szego. Orthogonal Polynomials. AMS, New York, 1959.
- [11] D.Lubinsky, Strong Asymptotics for Extremal Polynomials Associated with the Erdos-type Weights ( Longmans , Harlow, 1989).
- [12] E.Rakhmanov, Lectures Notes in Mathem. 1550:71 (1993).
- [13] L.Pastur, Lett.Math.Phys. 25:259 (1992).
- [14] G.Moore, Progr.Th.Phys.Suppl. 102:255 (1990).
- [15] E.Brezin, A.Zee Nucl.Phys. B402 : 613 (1993).
- [16] B.Eynard, Large Random Matrices: Eigenvalue Distribution. Preprint SPhT/93-999, Saclay, 1993.
- [17] G.Hackenbroich, H.A.Weidenmuller, Universality of Random Matrix Results for non Gaussian Ensembles (The Heidelberg University Preprint, 1994).
- [18] C.Tracy, H.Widom, Comm.Math.Phys 163:35 (1994).