AIP | Low Temperature Physics

Dynamical features of bound states of topological solitons in highly dispersive low-dimensional systems

M. M. Bogdan and O. V. Charkina

Citation: Low Temperature Physics **33**, 1073 (2007); doi: 10.1063/1.2807243
View online: http://dx.doi.org/10.1063/1.2807243
View Table of Contents: http://scitation.aip.org/content/aip/journal/ltp/33/12?ver=pdfcov
Published by the AIP Publishing

SHORT NOTES

Dynamical features of bound states of topological solitons in highly dispersive low-dimensional systems

M. M. Bogdan[a)] and O. V. Charkina

B. Verkin Institute for Low Temperature Physics and Engineering, National Academy of Sciences of Ukraine, pr. Lenina 47, Kharkov 61103, Ukraine

(Submitted July 19, 2007; revised August 6, 2007)
Fiz. Nizk. Temp. **33**, 1406–1410 (December 2007)

The nonstationary dynamics and interaction of topological solitons (dislocations, domain walls, fluxons) in one-dimensional systems with high dispersion are investigated. Processes of soliton complex formation are studied analytically and numerically in relation to the strength of the dispersion, soliton velocity, and distance between solitons. It is demonstrated that stable bound soliton states with complex internal structure can propagate in a dissipative medium owing to their stabilization by external forces. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2807243]

The dynamical properties of topological defects and structural inhomogeneities in low-dimensional crystals can be described adequately in terms of soliton theory.[1–3] For this purpose it is customary to use quasi-one-dimensional lattice models; in particular, for describing the dynamics of dislocations (crowdions) the Frenkel–Kontorova model is used,[4,5] and the properties of magnetic domain walls are investigated in the framework of anisotropic models of Heisenberg chains.[6] As a consequence of the discreteness of such systems there is strong spatial dispersion of waves. Continuum models can also have strong dispersion, e.g., in the case when nonlocal interactions are taken into account. This is valid, in particular, for macroscopic quantum systems, specifically for processes of propagation of magnetic flux quanta (fluxons) in long Josephson junctions.[7]

Dislocations, domain walls, and fluxons are the simplest examples of effectively one-dimensional topological solitons—kinks. In the long-wavelength limit their properties are described in terms of the sine-Gordon (SG) and double sine-Gordon (DSG) equations.[2] When the strong dispersion due to nonlocal interactions or discreteness is taken into account, it becomes necessary to introduce integral terms or higher-order spatial derivatives in the SG and DSG equations.[7–12] Here it would seem that the propagation of solitons in a dispersive medium must necessarily be accompanied by strong radiation. However, first for discrete systems[13] and then for a number of continuum models with strong dispersion[7–12] a universal phenomenon was revealed: the possibility of practically radiationless fast motion of bound soliton complexes. This unique property makes multisoliton excitations extremely attractive from the standpoint of applications. In this paper we investigate the dynamical features of soliton complexes formed by strongly interacting one-dimensional kinks in a highly dispersive medium.[11] Physically such two-soliton states correspond, e.g., to a moving defect consisting of two neighboring dislocation half-planes, or to a narrow 360° magnetic domain wall, which arises even in the absence of magnetic field, or to a bound pair of fluxons. Theoretically the internal structure of these solitons can be studied in detail in models that lead to piecewise linear equations with strong dispersion.[10–12] At the same time, the nonstationary dynamics of complexes, the conditions of their formation and stability, and the influence of dissipative and external forces on them remain largely open questions. In this paper we consider this circle of problems in the framework of *regularized* SG and DSG equations with an additional fourth spatio-temporal derivative.

Ordinarily in the long-wavelength approximation for discrete systems, e.g., for the Frenkel–Kontorova model, which is described by the discrete sine-Gordon equation[13]

$$\frac{\partial^2 u_n}{\partial \tau^2} + 2u_n - u_{n-1} - u_{n+1} + \frac{1}{d^2}\sin u_n = 0, \tag{1}$$

the higher dispersion is taken into account by keeping the fourth spatial derivative in the expansion of the second difference $u_{n-1}+u_{n+1}-2u_n \approx u_{xx}+\gamma u_{xxxx}$. Here the variable u_n is the displacement of atom number n, d is the discreteness parameter, $x=n/d$, and the parameter $\gamma=1/12d^2$. The result obtained from the equation with the fourth spatial derivative has an important shortcoming: an artificially arising instability of states with $u=0,2\pi,4\pi\ldots$ with respect to excitation of short waves. To avoid such instability in the equations of hydrodynamics, Boussinesq first proposed to use a mixed spatio-temporal derivative instead of the fourth spatial derivative. Such a replacement was justified in the lattice theory by Rosenau.[14] The same idea was applied to the SG and DSG equations with higher dispersion in Refs. 9–11. At present this approach is actively being used for analytical description of discreteness effects.[15–17]

The DSG equation with mixed fourth derivative has the form[9–11]

$$u_{tt} - u_{xx} - \beta u_{xxtt} + \sin u + 2H\sin(u/2) = 0. \tag{2}$$

The constant β in this equation is the dispersion parameter. At $H=0$ and $\beta=0$, Eq. (2) becomes the usual SG equation, which has a host of other applications besides the crowdion model. In particular, the variable $u(x,t)$ in the SG equation describes the phase difference of the wave functions in superconductors in the model of a long Josephson junction.[2] For magnetic applications the constant H in Eq. (2) has the physical meaning of the magnetic field, and the variable $\varphi = u/2$ corresponds, for example, to the azimuthal angle of the magnetization vector in an easy-plane ferromagnet.[6]

The dispersion relation of linear waves for Eq. (2) has the form

$$\omega(k) = \sqrt{(1 + H + k^2)/(1 + \beta k^2)}. \quad (3)$$

Spectrum (3) has the peculiarity of being bounded in frequency not only from below but also from above. This property makes it similar to the spectrum of the initial discrete model (1). Moreover, at $H=0$ it simply coincides with the spectrum of the SG model with a nonlocal interaction.[7] It follows from the dispersion relation (3) that the regularized equation (2) does not suffer from the instability of states with $u=0, 2\pi, 4\pi\ldots$ with respect to short-wavelength excitations. Thus in the framework of the regularized equation (2) it becomes possible to investigate in detail the dynamical features of soliton complexes both analytically and numerically.

Equation (2) has important exact solutions, making it possible to state certain rigorous results. The solution $u_{2\pi}(x) = 4\arctan(\exp(x))$ for a static soliton (kink) of the usual SG equation remains an exact solution of Eq. (2) for $H=0$ and arbitrary β. Analogously the static solution of the DSG equation, the so-called *wobbler*:[2] $u_w(x) = 4\arctan(\exp(qx+R)) + 4\arctan(\exp(qx-R))$, where $q = \sqrt{1+H}$ and $\sinh R = 1/\sqrt{H}$, is the exact solution of equation (2) for any β. It describes, e.g., a 360° domain wall formed from two identical 180° domain walls in a ferromagnet.

The total spectrum of linear excitations of a static kink of equation (2) at $H=0$ can be found in explicit form.[18,19] It turns out that, owing to dispersion, the kink now has internal modes, and the number of them increases with increasing parameter β, whereas the region of the continuous spectrum becomes narrower and narrower. It is natural to suppose that under conditions of strong dispersion the presence of internal modes becomes the dominant factor in both the dynamics of solitary kinks and kink coupling processes. Indeed, for Eq. (2) a solution describing a moving complex consisting of two strongly coupled π kinks has been found:[9–11]

$$u_{4\pi}(x,t) = 8\arctan\left\{\exp\left(\frac{x - V_0 t}{l_0}\right)\right\}. \quad (4)$$

The velocity V_0 of such a complex, its effective width l_0, and its energy E_0 are specified functions of the parameters β and H:

$$V_0(\beta,H) = \sqrt{1 + \frac{\beta}{3}\left(1 + \frac{3}{2}H\right)^2} - \sqrt{\frac{\beta}{3}}\left(1 + \frac{3}{2}H\right),$$

$$l_0 = (3\beta V_0^2)^{1/4}, \quad E_0 = 32\left(l_0^{-1} - \frac{l_0}{9}\right). \quad (5)$$

For $H=0$ expression (4) goes over to the exact solution for the regularized SG equation. Equation (2) for any $H \geqslant 0$ has other two-soliton solutions that correspond to "excited" states of the soliton complex and have a characteristic internal structure.[7,10,11] These solutions are found numerically, and they correspond to a discrete set of velocity values.

The spectrum of oscillations of a moving kink at $H=0$ is determined by a complicated linearized equation that can be analyzed in the case of low velocities V and small values of the parameter β. In a dispersive medium the motion of the kink leads to coupled oscillations of its velocity and effective width.[18] For small β and V the dynamical properties of equation (2) should be close to those for a Lorentz-invariant SG system.

In this study we have carried out a numerical simulation of the dynamics of solitary kinks and soliton complexes. For the solution of the equations in the fourth spatio-temporal derivative we used a difference scheme analogous to that proposed in Ref. 7, which has high stability, since it entails the method of tridiagonal inversion. In the calculations the time step was chosen equal to $\Delta t = 0.0001$ and the coordinate step, as a rule, equal to $\Delta x = 0.02$; here the size of the system was usually chosen equal to $L = N \cdot \Delta x$, where $N=3000$. In cases when boundary effects need to be eliminated or minimized, we chose values $N=6000$ and 10000. The initial conditions were chosen in accordance with the expressions for the kink, wobbler, and soliton complex (4), while the boundary conditions were chosen in the form of fixed boundaries with values $u(0,t)=0$ and, accordingly, $u(L,t)=2\pi$ for a kink, and $u(L,t)=4\pi$ for a complex. The initial velocities V_{in} of the kinks and complexes lay in the interval between 0.1 and 0.9, and typical values of the dispersion parameter were $\beta = 1/12$, 1/4, and 1.

As a result, for the case $H=0$ it was found that a solitary kink at small values of the parameters β and V_{in} moves in a practically steady manner, generating weak radiation with a wave vector determined from the equation $\omega(k_0) = Vk_0$. With increasing dispersion parameter the forward radiation vanishes, i.e., at fixed β there exist critical velocities V_{in} above which only radiation backwards is possible. If the parameter β and velocity V_{in} are not small, then the kink dynamics even in the initial stage becomes highly nonstationary and dissipative, as in discrete systems.[13] In actuality, because of the presence of an internal mode in the spectrum of excitations of the kink, an important channel of energy loss by the soliton is the process of formation of a wobbling kink. It begins with the excitation of an internal mode, which rapidly transforms into a self-localized oscillation, the so-called *breather*,[3] which is a dynamical bound state of two kinks with opposite topological charges. For the usual SG equation such an oscillating kink can be found explicitly.[20] Here the breather, localized at the kink, has the symmetry of an internal mode. Because of the dispersion, its velocity rapidly becomes less than the velocity of the kink, and the breather is found on the wake of the topological soliton.

We made further analytical and numerical studies of kink interaction processes and the conditions for the formation of bound soliton complexes in relation to the initial velocity of the kinks, the distance between them, and the value of the dispersion parameter. It turned out that for describing the process of coupling of the kinks and their interaction with a low-amplitude breather mode, $f_b(\xi,t) = a\sin(\Omega t - k(\xi - \xi_0))/\cosh(\varepsilon(\xi - \xi_0))$ it is sufficient to use the following ansatz:

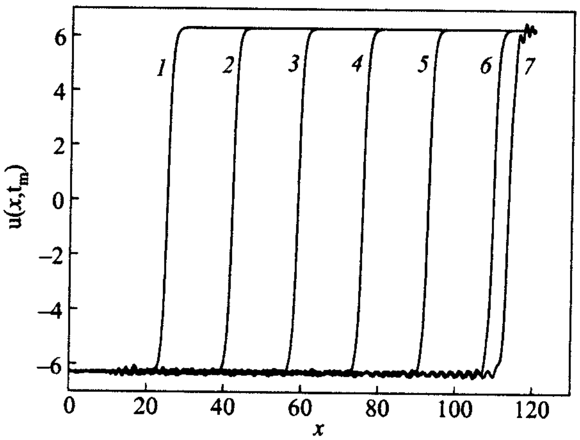

FIG. 1. Propagation of a stable soliton complex. Each curve labeled with a number m corresponds to a time $t_m=20\cdot m$. The profile of the solution with number $m=7$ corresponds to a complex that has reflected off the boundary.

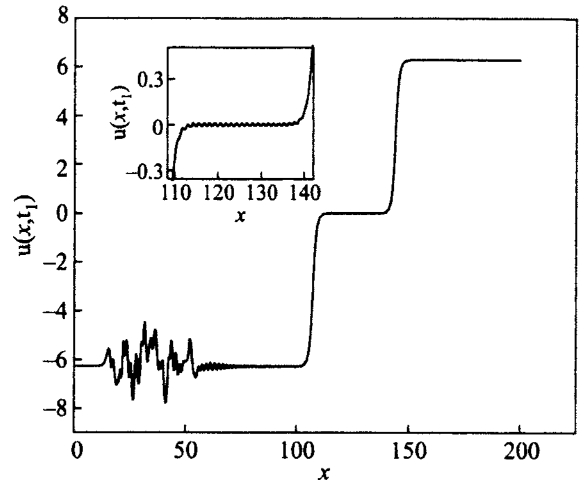

FIG. 2. Decay of a soliton complex for $\beta=1$, $V_{in}=0.9$ and $t_1=500$. The first kink moves with a constant velocity $V_1=0.152$. Behind the second kink are breather modes. The inset shows the spatial modulation of the field between kinks on an expanded scale.

$$u_{wb}(x,t)=4\arctan(\exp(\xi+R))+4\arctan(\exp(\xi-R))+f_b(\xi,t)(1-\tanh\xi), \quad (6)$$

in which $\xi=(x-Vt)/l$, and the parameters V, l, R, a, ε, Ω, k, and ξ_0 are assumed to be time-dependent functions. For $R=a=0$, $l=l_0$, and $V=V_0$, expression (6) reduces to the exact solution (4). First, we were convinced from the numerical calculations that the soliton complex (4), moving steadily with velocity V_0, is stable against small perturbations. If we set $H=0$ and use the solution (4) as the initial profile but choose a small value for the velocity, then the soliton complex dissociates in an explosive manner. However, the repulsive forces fall off quite rapidly with increasing initial velocity of the complex. When V_{in} approaches the velocity of steady motion V_0 from below, the two interacting kinks pass through a stage of formation of "excited" states of the soliton complex.[7,10,11] However, such states are metastable, although they have a rather large lifetime. We then studied the character of the interaction of rapidly moving kinks in relation to the distance between their centers. It was found that at small $R\leqslant 1$ an attraction occurs, as a result of which a stable soliton complex forms. This case is shown in Fig. 1 for $\beta=1/12$ and $V_{in}=0.9$. It is seen that the soliton complex is not destroyed even after reflection from the boundary. With increasing distance between kinks, attraction gives way to repulsion, and, as a result, the complex decomposes into two kinks. At moderate initial velocities $V_{in}-V_0\leqslant V_0/3$ and $\beta\leqslant 1$ the soliton complex survives, throwing off excess energy in the form of breather modes. In this case, however, there exists yet another critical velocity, above which the complex dissociates into two kinks. Such a decay of a "high-energy" complex, with the formation of several breather modes, occurs for $\beta=1$ and $V_{in}=0.9$, for example (Fig. 2).

Finally, we have investigated the influence of driving forces and dissipation on the dynamics of soliton complexes. For this we took Eq. (2) for $H=0$ and added the terms $j_0-\lambda u_t$ to the right-hand side, where the first term corresponds to the bias current in a Josephson junction, for example, and λ is the dissipation coefficient. The result of a numerical modeling are presented in Fig. 3 for $\lambda=0.1$ and six sequential values of j_0 from -0.1 to -0.35. It turns out that the external influences under conditions of dissipation permit stabilization not only of the soliton complex but also of its "excited" states. For waves of stationary profile, the derivatives u_t and u_x are proportional to each other, and both have the form of closely spaced double peaks. These derivatives are directly related to experimentally measurable quantities, in particular, the voltage $U\sim u_t$ and magnetic field $h\sim u_x$ in the case of a long Josephson junction, and in a crystal with dislocations the derivative u_x determines the elastic deformation of the medium. We note that the possibility in principle of observing multi-soliton excitations in long Josephson junctions was demonstrated quite some time ago.[21]

Thus the results obtained can be used for explanation and description of new effects in the dynamics of topological solitons in highly dispersive media—in particular, dislocations in nonideal lattices, fluxons in Josephson junction systems, and magnetic domain walls in anisotropic magnets.

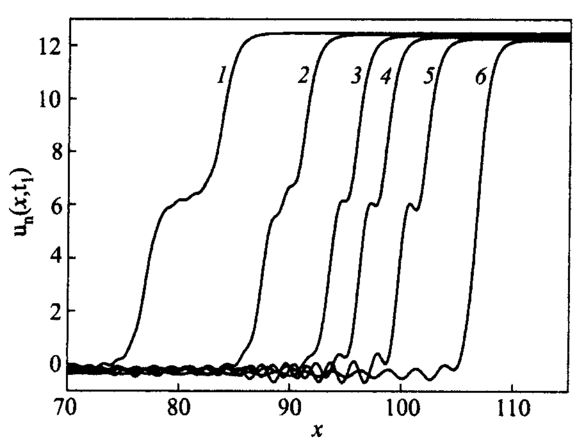

FIG. 3. Propagation of stable soliton complexes with an internal structure under the influence of external forces and in the presence of dissipation. The coefficient $\lambda=0.1$ and the curves with numbers $n=1,\ldots,6$ corresponds to $j_0=-0.05(1+n)$ and to the same instant of time $t_1=100$.

[a)]E-mail: bogdan@ilt.kharkov.ua

[1]S. V. Manakov, S. P. Novikov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: the Inverse Scattering Method*, Consultants Bureau, New York (1984), Moscow, Nauka (1980).
[2]R. K. Dodd, J. C. Eilbeck, J. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations*, Academic Press, New York (1982), Mir, Moscow (1988).
[3]A. M. Kosevich, and A. S. Kovalev, *Introduction to Nonlinear Physical Mechanics* [in Russian], Naukova Dumka, Kiev (1989).
[4]A. M. Kosevich, *Theory of Crystal Lattice*, WILEY-VCH, Berlin, New York (1999) Vishcha Shkola, Kharkov (1988).
[5]V. D. Natsik, S. N. Smirnov, and E. I. Nazarenko, Fiz. Nizk. Temp. **27**, 316 (2001) [Low Temp. Phys. **27**, 233 (2001)].
[6]A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, Nelineĭnye volny namagnichennosti. Dinamicheskie i topologicheskie solitony, Kiev, Naukova Dumka (1983).
[7]G. L. Alfimov, V. M. Eleonskii, N. E. Kulagin, and N. N. Mitskevich, Chaos **3**, 405 (1993).
[8]M. M. Bogdan and A. M. Kosevich, in *Nonlinear Coherent Structures in Physics and Biology*, Vol. 329 of NATO ASI Series B: Physics, Plenum Press, New York (1994), p. 373.
[9]M. M. Bogdan and A. M. Kosevich, Proc. Est. Acad. Sci., Phys., Math. **46**, 14 (1997).
[10]M. M. Bogdan, A. M. Kosevich, and G. A. Maugin, Condens. Matter Phys. **2**, 255 (1999).
[11]M. M. Bogdan, A. M. Kosevich, and G. A. Maugin, Wave Motion **34**, 1 (2001).
[12]A. S. Malyshevskiĭ, V. P. Silin, and S. A. Uryupin, Zh. Éksp. Teor. Fiz. **117**, 771 (2000) [JETP **90**, 671 (2000)].
[13]M. Peyrard and M. D. Kruskal, Physica D **14**, 88 (1984).
[14]P. Rosenau, Phys. Rev. B **36**, 5868 (1987).
[15]P. G. Kevrekidis, I. G. Kevrekidis, A. R. Bishop, and E. S. Titi, Phys. Rev. E **65**, 046613-1 (2002).
[16]B. F. Feng, Y. Doi, and T. Kawahara, J. Phys. Soc. Jpn. **73**, 2100 (2004).
[17]P. Rosenau and S. Schochet, Chaos **15**, 015111 (2005).
[18]O. V. Charkina and M. M. Bogdan, Uzhgorod Univ. Sci. Herald. Series Physics **17**, 30 (2005).
[19]O. V. Charkina and M. M. Bogdan, Symmetry, Integrability and Geometry: Methods and Applications **2**, 047-12 (2006).
[20]H. Segur, J. Math. Phys. **24**, 1439 (1983).
[21]D. Dueholm, O. A. Levring, J. Mygind, N. F. Pedersen, O. H. Soerensen, and M. Cirillo, Phys. Rev. Lett. **46**, 1299 (1981).

Translated by Steve Torstveit