Quasiclassical energy spectra of the Boussinesq breathers in anharmonic crystals

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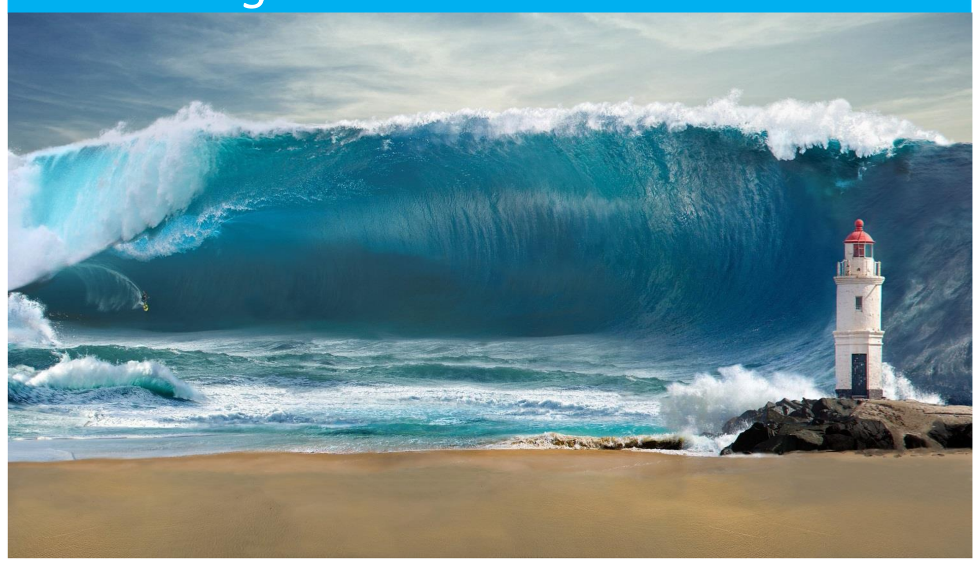
The Boussinesq equation: *Long waves in shallow water*



1. The Boussinesq equation: *Long waves in nonlinear lattices*

$$U(u_n - u_{n-1}) = \frac{1}{2}U''(0)(u_n - u_{n-1})^2 + \frac{1}{6}U'''(0)(u_n - u_{n-1})^3 + \frac{1}{24}U''''(0)(u_n - u_{n-1})^4$$

A.M. Kosevich, *The Crystal Lattice: Phonons, Solitons, Dislocations, Superlattices*, 2nd Ed., WILEY-VCH, Weinheim, Germany (2005).

The Boussinesq equation $\boxed{u_{\tau\tau} - s^2 u_{zz} - B^2 u_{zzzz} + \Lambda^2 u_z u_{zz} = 0}$

$$s^2 = \frac{a^2}{m}U''(0) \qquad B^2 = \frac{a^2}{12}s^2 \qquad \Lambda^2 = -\frac{a^3}{m}U'''(0)$$

The particle-like kink of the Boussinesq equation

$$u_K = \frac{u_0}{1+\exp\left(\frac{z-v\tau}{l}\right)} = \frac{u_0}{2}\left[1 - \tanh\frac{z-v\tau}{2l}\right] \qquad u_0 = 12\frac{v^2 - s^2}{\Lambda^2}l \qquad l = a\cdot\left[12\left(\frac{v^2}{s^2}-1\right)\right]^{-\frac{1}{2}}$$

$$u = aA\varphi \qquad A = \sqrt{3}\frac{s^2}{\Lambda^2} \qquad z_0 = s\tau_0 \qquad \tau_0 = \frac{a}{\sqrt{12}s} \qquad E_0 = 6\sqrt{3}m\frac{s^6}{\Lambda^4}$$

$$\varphi_{tt} - \varphi_{xx} - \varphi_{xxxx} + 6\varphi_x\varphi_{xx} = 0 \qquad E = \int_{-\infty}^{\infty} dx\left\{\left(\frac{1}{2}\varphi_t^2 + \frac{1}{2}\varphi_x^2 - \frac{1}{2}\varphi_{xx}^2 - \varphi_x^3\right)\right\}$$

The shock wave analog in the 1D crystal

$$\varphi_K = K\left[1 - \tanh\frac{K}{2}(x - V_K t)\right] \qquad V_K = \sqrt{1+K^2}$$ The displacement (kink)

$$w_K = \frac{\partial\varphi_K}{\partial x} = -\frac{K^2}{2}\frac{1}{\cosh^2\frac{K}{2}(x - V_K t)}$$ The negative localized deformation

The energy

$$E(K) = \int_{-\infty}^{\infty} dx\left\{\left(\frac{1}{2}(V^2+1)w_K^2 - \frac{1}{2}\left(\frac{\partial w_K}{\partial x}\right)^2 - w_K^3\right)\right\} \qquad E(K) = \frac{2}{3}K^3\left(1 + \frac{4}{5}K^2\right)$$

The field momentum

$$P = -\int_{-\infty}^{\infty} p\,d\varphi \equiv -\int_{-\infty}^{\infty}\frac{\partial L}{\partial\varphi_t}d\varphi = -\int_{-\infty}^{\infty}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x}dx \qquad P(K) = V_K\int_{-\infty}^{\infty} w_K^2 dx = \frac{2}{3}K^3\sqrt{1+K^2} \equiv M_K V_K$$

The Hamiltonian equation $\quad \frac{dE}{dP} = \frac{dE}{dK}\frac{dK}{dP} = \sqrt{1+K^2} = V_K \qquad X_t = V_K = \frac{\partial H}{\partial P}$

2. The Boussinesq breather and its emergence from kink localized mode

The multisoliton solution of the Boussinesq equation

$$w_{tt} - w_{xx} - w_{xxxx} + 3(w^2)_{xx} = 0$$

R. Hirota, *Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices*, J. Math. Phys. **14**, 810 (1973)

$$\varphi = -2\frac{\partial\ln f}{\partial x} \qquad w = -2\frac{\partial^2}{\partial x^2}\ln f \qquad f_K = 1 + \exp K(x - V_K t)$$

The two-soliton solution $\quad f_{2s} = 1 + \exp\eta_1 + \exp\eta_2 + a_{12}\exp(\eta_1+\eta_2) \qquad \eta_i = k_i x - \Omega_i t + \eta_i^0 \qquad V_i = \pm\sqrt{1+k_i^2}$

$i = 1,2$

$$\Omega_i = \pm k_i\sqrt{1+k_i^2} \qquad a_{12} = -\frac{(\Omega_1-\Omega_2)^2 - (k_1-k_2)^2 - (k_1-k_2)^4}{(\Omega_1+\Omega_2)^2 - (k_1+k_2)^2 - (k_1+k_2)^4}$$

Main problem of the solution presentation of M. Tajiri and Y. Murakami

$$(\Omega + i\omega)^2 = (\kappa + ik)^2 + (\kappa + ik)^4 \qquad k_1 = \kappa + ik \qquad k_2 = \kappa - ik$$

We propose a new parameterization for the breather solution :

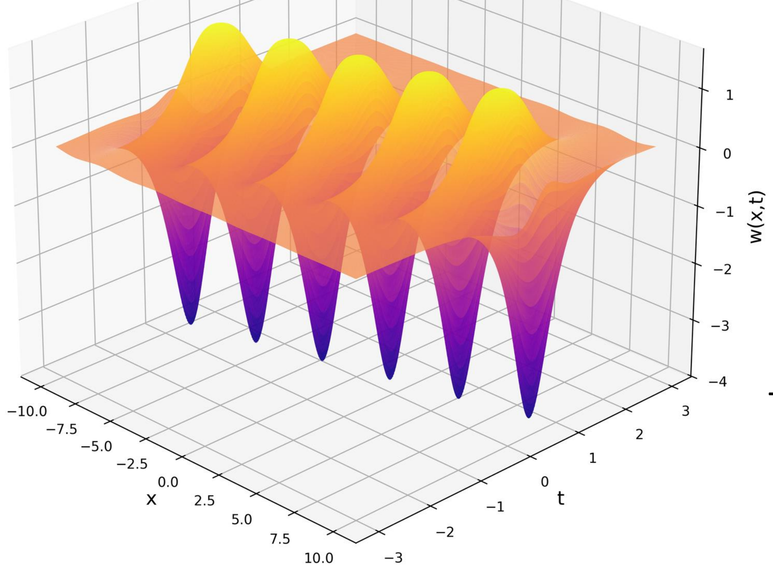
$$\kappa \pm ik \equiv \sinh(\alpha \pm i\beta) \qquad \Omega_1 = \Omega_2^* = \Omega + i\omega = \pm\frac{1}{2}\sinh 2(\alpha + i\beta)$$

$$\kappa = \sinh\alpha\cos\beta \qquad \Omega = \pm\frac{1}{2}\sinh 2\alpha\cos 2\beta$$

$$k = \cosh\alpha\sin\beta \qquad \omega = \pm\frac{1}{2}\cosh 2\alpha\sin 2\beta$$

$$a_{12} = \frac{\sin^2\beta\cdot(4\cosh^2\alpha - 1)}{\sinh^2\alpha\cdot(1 - 4\cos^2\beta)} \qquad w_{br} = -2\frac{\partial^2}{\partial x^2}\ln f_{br} \qquad f_{br} = \cosh(\kappa x - \Omega t) + \lambda\cos(kx - \omega t) \qquad \lambda = \frac{1}{\sqrt{a_{12}}} = \frac{\sinh\alpha\cdot\sqrt{1 - 4\cos^2\beta}}{\sin\beta\sqrt{4\cosh^2\alpha - 1}}$$

The algebraic sum of the pure soliton and the nonlinear breathing mode

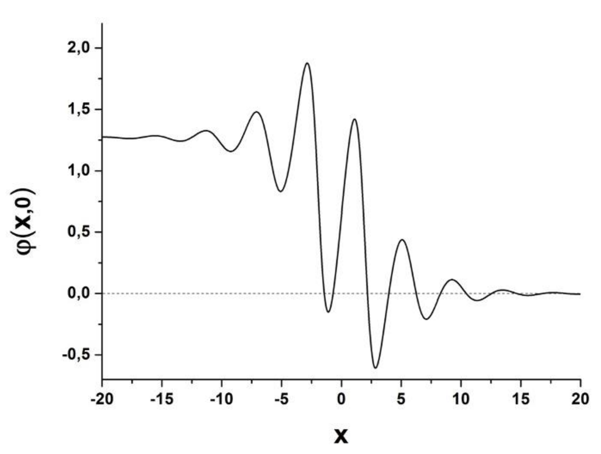


$$w_{br} = -2\frac{\partial^2}{\partial x^2}\ln\cosh(\kappa x - \Omega t) - 2\frac{\partial^2}{\partial x^2}\ln\left\{1 + \frac{\lambda\cos(kx - \omega t)}{\cosh(\kappa x - \Omega t)}\right\}$$

$$V = \frac{\Omega}{\kappa} = \pm\cosh\alpha\cdot\frac{\cos 2\beta}{\cos\beta} \qquad v_r = \frac{\omega}{k} = \pm\cos\beta\cdot\frac{\cosh 2\alpha}{\cosh\alpha}$$

The evolution of the breather solution of the Boussinesq equation

The breather solution as the wobbling kink of the Boussinesq equation



$$\varphi_{br} = -2\frac{\partial}{\partial x}\ln\cosh(\kappa x - \Omega t) - 2\frac{\partial}{\partial x}\ln\left\{1 + \frac{\lambda\cos(kx - \omega t)}{\cosh(\kappa x - \Omega t)}\right\}$$

$$\varphi_{br} = \varphi_K - 2\frac{\partial}{\partial x}\ln\left\{1 + \frac{\lambda\cos(kx - \omega t)}{\cosh(\kappa x - \Omega t)}\right\}$$

The indivisible wobbling kink of the Boussinesq equation

The small-amplitude breather limit

$$\lambda = \frac{1}{\sqrt{a_{12}}} = \frac{\sinh\alpha\cdot\sqrt{1 - 4\cos^2\beta}}{\sin\beta\sqrt{4\cosh^2\alpha - 1}} \qquad \lambda \to 0 \qquad \alpha \to 0 \quad \text{or} \quad \beta \to \beta_0 = \pi/3 \qquad \cos\beta_0 = 1/2$$

$$w_{br} = w_K + \psi \qquad \psi << 1 \qquad \kappa_0 = K/2 \qquad \psi = -2\lambda\frac{\partial^2}{\partial x^2}h(x,t) \qquad w_K = -\frac{2\kappa_0^2}{\cosh^2\kappa_0(x - V_K t)} \qquad h(x,t) = \frac{\cos(kx - \omega t)}{\cosh(\kappa x - \Omega t)}$$

The linearized Boussinesq equation $\quad \psi_{tt} - \psi_{xx} - \psi_{xxxx} + 6(w_K\psi)_{xx} = 0$

J.G. Berryman, *Stability of solitary waves in shallow water*, Phys. Fluids, **19**, 771 (1976)

$$\psi = g_{xx}$$

The *external* linear localized mode

$$g_{tt} - g_{xx} - g_{xxxx} + 6w_K g_{xx} = 0 \qquad V_K = \cosh\alpha \qquad y = \kappa_0(x - V_K t) \qquad g_{tt} - 2V_K g_{ty} + Lg_{yy} = 0$$

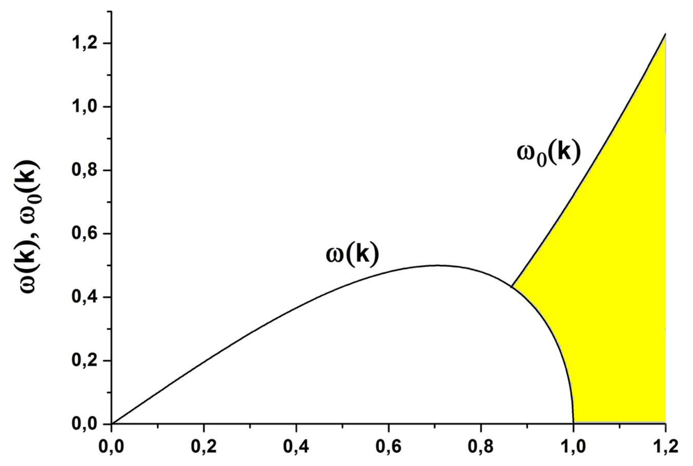
$$L = \kappa_0^2\left(-\frac{\partial^2}{\partial y^2} + 4 - \frac{12}{\cosh^2 y}\right)$$

$$g(y,t) = \frac{\cos(k_m y - \omega_m t)}{\cosh(y)} \qquad k_m = \frac{k_0}{\kappa_0} \qquad \omega_m = \frac{1}{\kappa_0}(\omega_0 - k_0 V_K)$$

$$k_0 = \frac{\sqrt{3}}{2}\cosh\alpha \qquad \Omega_0 = \mp\frac{1}{2}\sinh\alpha\cdot\cosh\alpha \qquad \kappa_0 = \frac{1}{2}\sinh\alpha \qquad \omega_0 = \pm\frac{\sqrt{3}}{4}\cosh 2\alpha$$

The breather existence boundary: $\quad k_0(\kappa_0) = \frac{\sqrt{3}}{2}\sqrt{1 + 4\kappa_0^2}$

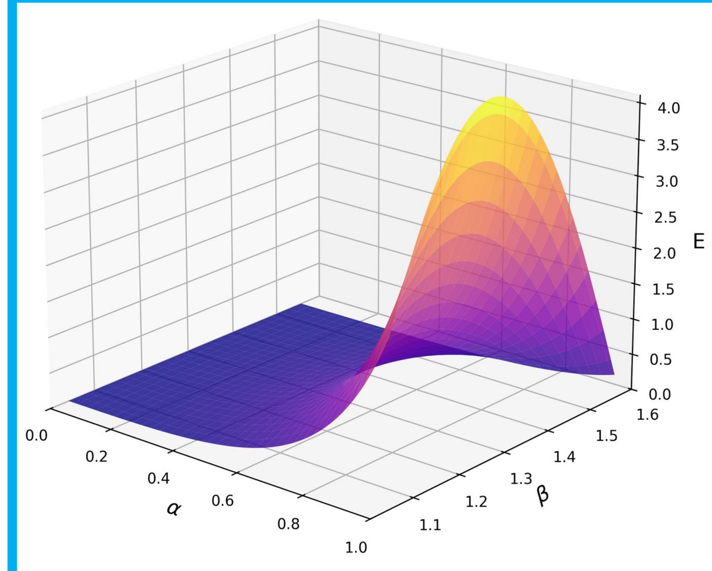
$$\omega_0(k_0) = \frac{2}{\sqrt{3}}\left(k_0^2 - \frac{3}{8}\right)$$ The localized mode frequency



The dispersion law of continuous spectrum waves and the carrier frequency of the kink localized mode.

We call this special type of the wobbling kink the Boussinesq breather.

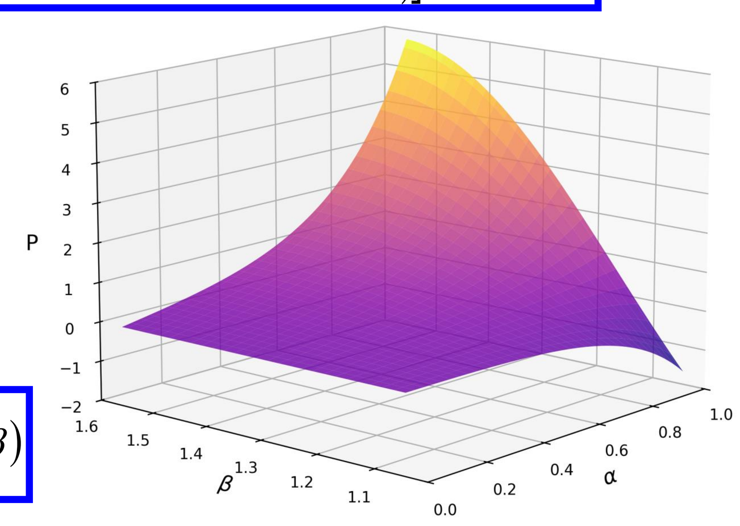
3. Integrals of motion and the quasiclassical spectrum of the Boussinesq breather



The Boussinesq breather energy as a function of the new parameters

$$E_{br} = E(\alpha,\beta) = \frac{4}{15}\sinh\alpha\cos\beta\cdot\left[5\cosh^2\alpha\cdot(4\cosh^2\alpha - 3) - (\cos^2\beta + \sin^2 2\beta)\cdot(16\cosh^4\alpha - 12\cosh^2\alpha + 1)\right]$$

$$P_{br} = P(\alpha,\beta) = \frac{1}{3}\sinh 2\alpha(\cosh 2\alpha\cos 4\beta - \cos 2\beta)$$

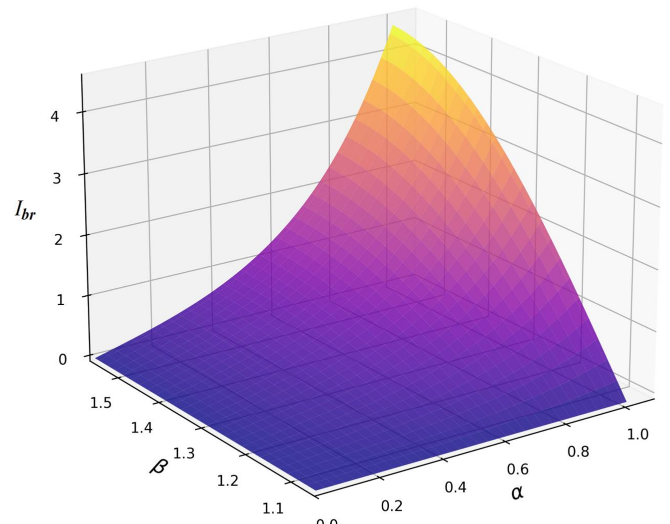


The Boussinesq breather field momentum as a function of the new parameters

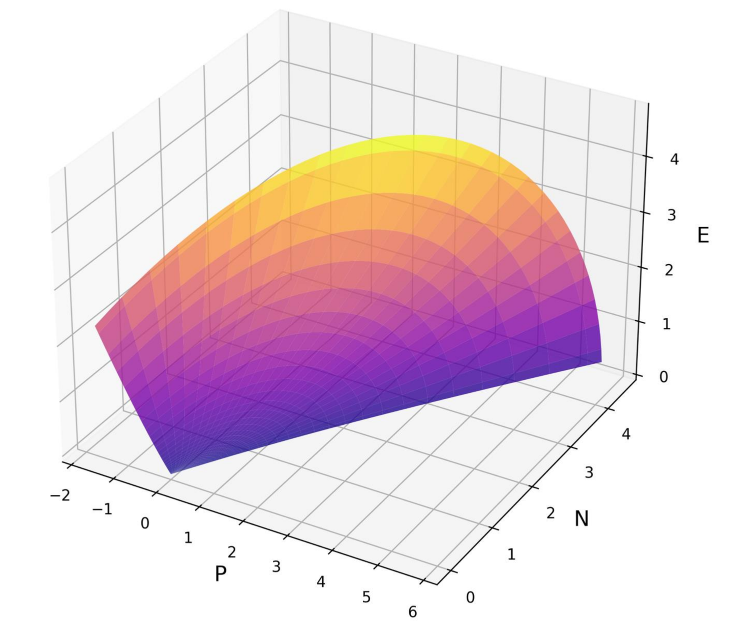
$$I_{br} = I(\alpha,\beta) = \frac{16}{9}\sinh\alpha\sin\beta\cdot(4\cosh^2\alpha - 1)\cdot\left(\sin^2\beta - \frac{3}{4}\right)$$

The Bohr–Sommerfeld quantization rule: $\quad I_d = I_0 I_{br} = \hbar N \qquad N >> 1$

The normalized number of states: $\quad I_{br} = N \equiv \gamma\cdot N \qquad \gamma = \hbar/I_0 \qquad I_0 = E_0\tau_0$



The adiabatic invariant of the Boussinesq breather, i.e. the normalized number of states, as a function of the new parameters



The parametrically defined energy surface, having the quite flat shape, as function of the normalized number of states and the field momentum

The Hamiltonian equations for the Boussinesq breather as a particle-like excitation

Calculating the derivatives and checking the identities: $\quad \frac{\partial E}{\partial\alpha} = \tilde{\omega}\frac{\partial N}{\partial\alpha} + V\frac{\partial P}{\partial\alpha} \qquad \frac{\partial E}{\partial\beta} = \tilde{\omega}\frac{\partial N}{\partial\beta} + V\frac{\partial P}{\partial\beta}$

$$\delta E = \tilde{\omega}\delta N + V\delta P \longleftarrow \frac{\partial E}{\partial\alpha}\delta\alpha + \frac{\partial E}{\partial\beta}\delta\beta = \tilde{\omega}\left(\frac{\partial N}{\partial\alpha}\delta\alpha + \frac{\partial N}{\partial\beta}\delta\beta\right) + V\left(\frac{\partial P}{\partial\alpha}\delta\alpha + \frac{\partial P}{\partial\beta}\delta\beta\right)$$

The Hamiltonian equations for the Boussinesq breather $\quad \frac{dX_{br}}{dt} = V = \frac{\partial H_{br}}{\partial P} \qquad \frac{d\theta_{br}}{dt} = \tilde{\omega} = \omega - kV = \frac{\partial H_{br}}{\partial N}$

4. Conclusion

1.The analytical investigation of the dynamical properties of the breather solution found by M. Tajiri and Y. Murakami for the Boussinesq equation has been carried out in detail. This became possible due to proposing the new parameterization of the solution, which allowed us to show the composite structure of the solution, to find exactly its existence boundary, and to calculate explicitly all the dynamical characteristics of the complex excitation.

2.The complex solution represents the algebraic sum of the kink and the breather expressions. The variability of the solution forms, previously revealed in the numerical simulation, is simply explained by the difference in the amplitudes of these components. In modern terms of the soliton theory this complex solution would be related to the category of the wobbling kinks. We single out the solution into a special category and call it the Boussinesq breather.

3.We show that the breather at the nearest vicinity of its existence boundary emerges from the *linear localized mode* of the single soliton and we demonstrate this analytically for the first time. This fact encourages the search for similar exact solutions in other integrable equations, first of all, with higher dispersive terms.

4.We exactly obtain the first integrals, the energy and the field momentum, for the Boussinesq breather and explicitly calculated the adiabatic invariant for the complex excitation. We carried out the quasiclassical quantization of the nonlinear oscillating solution, obtaining its energy spectrum, i.e., the energy dependence on the field momentum and the number of states, and established the Hamiltonian equations for this particle-like excitation.

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