General collisionless kinetic approach to studying excitations in arbitrary-spin quantum atomic gases

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Abstract

We develop a general kinetic approach to studying high-frequency collective excitations in arbitrary-spin quantum gases. To this end, we formulate a many-body Hamiltonian that includes the multipolar exchange interaction as well as the coupling of a multipolar moment with an external field. By linearizing the respective collisionless kinetic equation, we find a general dispersion equation that allows us to examine the high-frequency collective modes for arbitrary-spin atoms obeying one or another quantum statistics. We analyze some of its particular solutions describing spin waves and zero sound for Bose and Fermi gases.

Hamiltonian for arbitrary spin-F atoms

$$H = H_0 + H_{\text{int}}, \tag{1}$$

$$H_0 = \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}\alpha} \left[\varepsilon_{\mathbf{p}} \delta_{\alpha\beta} - \sum_{j=0}^{2F} \sum_{m=-j}^{j} (-1)^m h^j_m (T^j_{-m})_{\alpha\beta} \right] a_{\mathbf{p}\beta}, \quad \varepsilon_{\mathbf{p}} = \frac{p^2}{2M}$$

$$H_{\text{int}} = \frac{1}{2V} \sum_{j=0}^{2F} \sum_{m=-j}^{j} (-1)^m \sum_{\mathbf{p}_1 \ldots \mathbf{p}_4} U^{[j]}(\mathbf{p}_1 - \mathbf{p}_4) a^\dagger_{\mathbf{p}_1\alpha} a^\dagger_{\mathbf{p}_2\beta} (T^j_m)_{\alpha\delta} (T^j_{-m})_{\beta\gamma} a_{\mathbf{p}_3\gamma} a_{\mathbf{p}_4\delta}\, \delta_{\mathbf{p}_1+\mathbf{p}_2,\, \mathbf{p}_3+\mathbf{p}_4}.$$

H_0 also includes the coupling of a multipololar moment with an external field. The coupling is specified by two irreducible tensors h^j_m and T^j_m with indices j and m denoting their rank and component, respectively (for a given rank j, both tensors have $2j+1$ components). The quantity h^j_m is constructed from the components of the physical external field and T^j_m represents a spherical tensor operator and describes the multipolar degrees of freedom. $U^{[j]}(\mathbf{p}_1 - \mathbf{p}_4)$ are the Fourier transforms of the energies corresponding to direct ($j = 0$) and multipolar ($i = 1, \ldots, 2F$) interactions.

Kinetic equation and its linearization

In the case of small inhomogeneity and weak interaction, the Wigner density matrix $f_{\alpha\beta}(\mathbf{x}, \mathbf{p})$ satisfies the following kinetic equation:

$$\frac{\partial}{\partial t} f_{\alpha\beta}(\mathbf{x}, \mathbf{p}) + \frac{i}{\hbar} \left[\varepsilon(\mathbf{x}, \mathbf{p}), f(\mathbf{x}, \mathbf{p}) \right]_{\alpha\beta} + \frac{1}{2} \left\{ \frac{\partial \varepsilon(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}}, \frac{\partial f(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}} \right\}_{\alpha\beta} - \frac{1}{2} \left\{ \frac{\partial \varepsilon(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}}, \frac{\partial f(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} \right\}_{\alpha\beta} = 0. \tag{2}$$

The collision integral can be omitted if we are interested in high-frequency collective modes. The Wigner density matrix and the mean-field particle energy $\varepsilon_{\alpha\beta}(\mathbf{x}, \mathbf{p})$ can be decomposed into a complete set of irreducible spherical tensor operators,

$$f_{\alpha\beta}(t, \mathbf{x}, \mathbf{p}) = \sum_{j=0}^{2F} \sum_{m=-j}^{j} (-1)^m f^j_m(t, \mathbf{x}, \mathbf{p}) (T^j_{-m})_{\alpha\beta},$$

$$\varepsilon_{\alpha\beta}(t, \mathbf{x}, \mathbf{p}) = \sum_{j=0}^{2F} \sum_{m=-j}^{j} (-1)^m \varepsilon^j_m(t, \mathbf{x}, \mathbf{p}) (T^j_{-m})_{\alpha\beta}. \tag{3}$$

The coefficients f^j_m determine, in the spherical basis, the physical quantities such as three components of the magnetization vector for $j = 1$, five components of the quadrupolar tensor for $j = 2$, seven components of the octupolar tensor for $j = 3$, etc.
We solve the respective kinetic equation assuming that the tensorial components of the Wigner distribution function $f^j_m(\mathbf{x}, \mathbf{p})$ slightly deviate from a homogeneous stationary state:

$$f^j_m(t, \mathbf{x}, \mathbf{p}) = (f^j_0)_{\mathbf{p}} \delta_{m0} + (\tilde{f}^j_m)_{\mathbf{p}}, \quad \varepsilon^j_m(t, \mathbf{x}, \mathbf{p}) = (\varepsilon^j_m)_{\mathbf{p}} + (\tilde{\varepsilon}^j_m)_{\mathbf{p}},$$

where

$$(\varepsilon^j_m)_{\mathbf{p}} = \sqrt{2F+1}\varepsilon_{\mathbf{p}} \delta_{j0} \delta_{m0} - h^j_0 + \frac{1}{V} \sum_{\mathbf{p}'} J^{[j]}(\mathbf{p} - \mathbf{p}') (f^j_m)_{\mathbf{p}} \delta_{m0},$$

$$(\tilde{\varepsilon}^j_m)_{\mathbf{p}} = \frac{1}{V} \sum_{\mathbf{p}'} J^{[j]}(\mathbf{p} - \mathbf{p}') (\tilde{f}^j_m)_{\mathbf{p}'}. \tag{4}$$

By applying the Fourier-Laplace transform,

$$\{\tilde{f}^j_m\}_{\mathbf{p}} = \int d^3x \int_0^\infty dt\, e^{-i\mathbf{k}\mathbf{x} - at} (\tilde{f}^j_m)_{\mathbf{p}},$$

we obtain the following **linearized** kinetic equation

$$a\{\tilde{f}^j_m\}_{\mathbf{p}} + \frac{i}{\hbar} B^{j;,j_1 j_2}_{m;m_1 m_2} \left[(\varepsilon^{j_1}_{m_1})_{\mathbf{p}} \{\tilde{f}^{j_2}_{m_2}\}_{\mathbf{p}} + \{\tilde{\varepsilon}^{j_1}_{m_1}\}_{\mathbf{p}} (f^{j_2}_0)_{\mathbf{p}} \delta_{m_2 0} \right] + \frac{i\mathbf{k}}{2} C^{j;j_1 j_2}_{m;m_1 m_2} \left[\partial_{\mathbf{p}} (\varepsilon^{j_1}_{m_1})_{\mathbf{p}} \{\tilde{f}^{j_2}_{m_2}\}_{\mathbf{p}} - \{\tilde{\varepsilon}^{j_1}_{m_1}\}_{\mathbf{p}} \partial_{\mathbf{p}} (f^{j_2}_0)_{\mathbf{p}} \delta_{m_2 0} \right] = (g^j_m)_{\mathbf{p}\mathbf{k}}, \tag{5}$$

where $(g^j_m)_{\mathbf{p}\mathbf{k}}$ is the initial condition determined by

$$(g^j_m)_{\mathbf{p}\mathbf{k}} = \int d^3x\, e^{-i\mathbf{k}\mathbf{x}} g^j_m(\mathbf{x}, \mathbf{p}), \quad g^j_m(\mathbf{x}, \mathbf{p}) = (\tilde{f}^j_m)_{\mathbf{p}} \Big|_{t=0}.$$

$$B^{j;j_1 j_2}_{m;m_1 m_2} = (-1)^{m_1 + m_2} \operatorname{Tr} \left(T^j_m \left[T^{j_1}_{-m_1}, T^{j_2}_{-m_2} \right] \right),$$

$$C^{j;j_1 j_2}_{m;m_1 m_2} = (-1)^{m_1 + m_2} \operatorname{Tr} \left(T^j_m \left\{ T^{j_1}_{-m_1}, T^{j_2}_{-m_2} \right\} \right).$$

The dispersion equations and eigenfrequencies in the case of zero magnetic field, $h^j_m = 0$

The dispersion equation reads,

$$1 - \frac{1}{V} \frac{J^{[j]}(0)}{\sqrt{2F+1}} \sum_{\mathbf{p}} \frac{i\mathbf{k}\partial_{\mathbf{p}} (f^0_0)_{\mathbf{p}}}{a + (i\mathbf{k}\mathbf{p}/M)} = 0. \tag{6}$$

For a **Bose gas** at zero temperature, the equilibrium distribution function is

$$f^{[\alpha]}(\mathbf{p}) \to n_0 (2\pi\hbar)^3 \delta(\mathbf{p}) \implies (f^0_0)_{\mathbf{p}} = \sqrt{2F+1} (2\pi\hbar)^3 n_0 \delta(\mathbf{p}).$$

Therefore, dispersion equation (6) gives $2F + 1$ ($j = 0 \ldots 2F$) undamped modes,

$$\omega^2 = \frac{n_0 J^{[j]}(0)}{M} k^2, \quad \omega = ia. \tag{7}$$

This agrees with the Bogoliubov spectrum at small wave vectors (phonon spectrum).
For a **ground-state Fermi gas**, the tensorial component $(f^0_0)_{\mathbf{p}}$ of the Wigner function is written in terms of the Heaviside step function $\Theta(\varepsilon)$,

$$(f^0_0)_{\mathbf{p}} = \sqrt{2F+1} \Theta(\varepsilon_{\text{F}} - \varepsilon_{\mathbf{p}})$$

The dispersion equation (6) gives

$$\frac{\xi}{2} \ln \frac{\xi + 1}{\xi - 1} - 1 = \frac{2\pi^2\hbar^3}{J^{[j]}(0) M p_{\text{F}}}, \quad \xi = \frac{aM}{ikp_{\text{F}}}. \tag{8}$$

As for bosons, we have $2F + 1$ oscillation modes, which are also well known as zero sound.

The case of nonzero magnetic field, $h^j_m \neq 0$ for the modes with $|m| = 2F$

For a **ferromagnetic Bose gas** in an external field, the distribution function is

$$f^{[\alpha]}(\mathbf{p}) = \frac{2\pi^2\hbar^3 n_0}{M\sqrt{2M\varepsilon_{\mathbf{p}}}} \delta_{\alpha 1} \delta(\varepsilon_{\mathbf{p}} - \varepsilon_{\text{B}}) \implies (f^j_0)_{\mathbf{p}} = \frac{2\pi^2\hbar^3 n_0}{M\sqrt{2M\varepsilon_{\mathbf{p}}}} (T^j_0)_{11} \delta(\varepsilon_{\mathbf{p}} - \varepsilon_{\text{B}}),$$

where $\varepsilon_{\text{B}} = \sum_j h^j_0 (T^j_0)_{11}$. Thus, we obtain the normal frequency:

$$\omega \approx \omega_0 + \omega_2 k^2, \quad \omega_0 = \frac{1}{\hbar} \sum_{j_2} B^{j_2} \left(h^{j_2}_0 + n_0 J^{[j]}(0) (T^{j_2}_0)_{11} \Theta(\varepsilon_{\text{B}}) \right), \tag{9}$$

$$\omega_2 = \hbar \frac{\left(4\varepsilon_B + 3n_0 J^{[j]}(0) \sum_{j_2} C^{j_2} (T^{j_2}_0)_{11} \Theta(\varepsilon_{\text{B}}) \right)}{6Mn_0 J^{[j]}(0) \sum_{j_2} B^{j_2} (T^{j_2}_0)_{11} \Theta(\varepsilon_{\text{B}})}.$$

For a **Fermi gas** at zero temperature, we have

$$f^{[\alpha]}(\mathbf{p}) = \Theta(\varepsilon^{[\alpha]}_{\text{F}} - \varepsilon_{\mathbf{p}}) \implies (f^j_0)_{\mathbf{p}} = \Theta(\varepsilon^{[\alpha]}_{\text{F}} - \varepsilon_{\mathbf{p}}) (T^j_0)_{\alpha\alpha},$$

where $\varepsilon^{[\alpha]}_{\text{F}} = \varepsilon_{\text{F}}(h^j_0) + \sum_{j_1=0} h^{j_1}_0 (T^{j_1}_0)_{[\alpha\alpha]}$. From dispersion equation, we have:

$$\omega \approx \omega_0 + \omega_2 k^2, \quad \omega_0 = \frac{1}{\hbar} \sum_{j_2} B^{j_2} \left(h^{j_2}_0 + \frac{nJ^{[j]}(0)}{2F+1} (T^{j_2}_0)_{\alpha\alpha} \left(\frac{\varepsilon^{[\alpha]}_{\text{F}}}{\varepsilon_{\text{F}}(0)} \right)^{3/2} \Theta(\varepsilon^{[\alpha]}_{\text{F}}) \right), \tag{10}$$

$$\omega_2 = \frac{\hbar \sum_{j_1} C^{j_1} (T^{j_1}_0)_{\alpha\alpha} (\varepsilon^{[\alpha]}_{\text{F}})^{3/2} \Theta(\varepsilon^{[\alpha]}_{\text{F}})}{2M \sum_{j_2} B^{j_2} (T^{j_2}_0)_{\beta\beta} (\varepsilon^{[\beta]}_{\text{F}})^{3/2} \Theta(\varepsilon^{[\beta]}_{\text{F}})} + \frac{2\hbar(2F+1)(\varepsilon_{\text{F}}(0))^{3/2} \sum_{j_1} B^{j_1} (T^{j_1}_0)_{\alpha\alpha} (\varepsilon^{[\alpha]}_{\text{F}})^{5/2} \Theta(\varepsilon^{[\alpha]}_{\text{F}})}{5MnJ^{[j]}(0) \left(\sum_{j_2} B^{j_2} (T^{j_2}_0)_{\beta\beta} (\varepsilon^{[\beta]}_{\text{F}})^{3/2} \Theta(\varepsilon^{[\beta]}_{\text{F}}) \right)^2}.$$

Summary

- We have obtained a many-body Hamiltonian of spin-F atoms, which includes the effects of multipolar exchange interaction and the coupling of a multipole moment with an external field. Then we have employed the Hamiltonian to derive the collisionless kinetic equation for quantum gases valid for small inhomogeneities.
- Having linearised the kinetic equation, we have arrived at the dispersion equation. This has allowed us to study high-frequency oscillations near the equilibrium degenerate state of Bose and Fermi gases, both in the presence and the absence of a magnetic field.
- We have showed that there is no need to make any assumptions about the form of the pairwise interaction potential $J^{[j]}(\mathbf{p})$ when calculating the integrals in the dispersion equation since the later contains the quantity $J^{[j]}(0)$, which appears in a natural way.
- Oscillations with $m \neq 0$ are characterized by a quadratic dispersion law (with a gap) and correspond to spin waves. The excitations with $m = 0$ are determined by the components of the tensorial Wigner distribution function of all ranks. The respective modes have a linear dispersion law and represent zero sound or density excitations.
- For a gas of fermionic atoms in the polarized equilibrium state, the dispersion equation for zero sound ($m = 0$) loses its meaning since all the interaction terms are canceled. This fact can be proved at least for $F = 1/2, 3/2, 5/2$.

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