Many-body hamiltonian on the basis of spherical tensor operators for studying collective phenomena in quantum high-spin systems

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Abstract

We employ the representation properties of the rotation group through the apparatus of spherical tensor operators to construct a many-body Hamiltonian of pairwise interacting spin-F atoms. This Hamiltonian, valid for finite-range potentials, includes the effects of multipolar exchange interaction as well as the coupling of any multipole moment (dipole, quadrupole, octupole, etc.) with an external field. It can be applied to study the collective phenomena in quantum Bose and Fermi gases of high-spin atoms, whose interaction is not specified by s-wave scattering length associated with delta-like contact pseudopotential. In particular, we examine all magnetic states emerging in a gas of spin-1 atoms with condensate. Besides, following the reduced description method of quantum systems, we obtain the respective kinetic equation in the collisionless approximation. Next, we employ it to spin-3/2 atoms to explore the high-frequency collective excitations known as zero sound.

Hamiltonian for arbitrary spin-F atoms

$$H = H_0 + H_{\text{int}}, \tag{1}$$

$$H_0 = \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}\alpha}\left[\varepsilon_{\mathbf{p}}\delta_{\alpha\beta} - \sum_{i=0}^{2F}\sum_{m=-i}^{i}(-1)^m h^i_m (T^i_{-m})_{\alpha\beta}\right]a_{\mathbf{p}\beta}, \quad \varepsilon_{\mathbf{p}} = \frac{p^2}{2m}$$

$$H_{\text{int}} = \frac{1}{2V}\sum_{i=0}^{2F}\sum_{m=-i}^{i}(-1)^m \sum_{\mathbf{p}_1\ldots\mathbf{p}_4} U^{[i]}(\mathbf{p}_1-\mathbf{p}_4)a^\dagger_{\mathbf{p}_1\alpha}a^\dagger_{\mathbf{p}_2\beta}(T^i_m)_{\alpha\delta}(T^i_{-m})_{\beta\gamma}a_{\mathbf{p}_3\gamma}a_{\mathbf{p}_4\delta}\,\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}.$$

H_0 also includes the coupling of a multipole moment with an external field. The coupling is specified by two irreducible tensors h^i_m and T^i_m, where h^i_m is constructed from the components of the physical external field and T^i_m represents a spherical tensor operator and describes the multipole degrees of freedom. $U^{[i]}(\mathbf{p}_1-\mathbf{p}_4)$ are the Fourier transforms of the energies corresponding to direct ($i=0$) and multipolar ($i=1,\ldots,2F$) interactions.

Zero sound in spin-3/2 atomic gas

In the case of small inhomogeneity and weak interaction, the Wigner density matrix $f_{\alpha\beta}(\mathbf{x},\mathbf{p})$ satisfies the following kinetic equation:

$$\frac{\partial}{\partial t}f_{\alpha\beta}(\mathbf{x},\mathbf{p}) + \frac{i}{\hbar}\left[\varepsilon(\mathbf{x},\mathbf{p}), f(\mathbf{x},\mathbf{p})\right]_{\alpha\beta} + \frac{1}{2}\left\{\frac{\partial\varepsilon(\mathbf{x},\mathbf{p})}{\partial\mathbf{p}}, \frac{\partial f(\mathbf{x},\mathbf{p})}{\partial\mathbf{x}}\right\}_{\alpha\beta} - \frac{1}{2}\left\{\frac{\partial\varepsilon(\mathbf{x},\mathbf{p})}{\partial\mathbf{x}}, \frac{\partial f(\mathbf{x},\mathbf{p})}{\partial\mathbf{p}}\right\}_{\alpha\beta} = 0. \tag{2}$$

The particle energy $\varepsilon_{\alpha\beta}(\mathbf{x},\mathbf{p})$, being dependent both on interaction amplitudes and Wigner density matrix includes the mean-field effects,

$$\varepsilon_{\alpha\beta}(\mathbf{x},\mathbf{p}) = \varepsilon_{\mathbf{p}}\delta_{\alpha\beta} - h(T^1_0)_{\alpha\beta} + \frac{1}{\mathcal{V}}\sum_{i=0}^{3}\sum_{m=-i}^{i}(-1)^m\sum_{\mathbf{p}'}U^{[i]}(0)(T^i_m)_{\alpha\beta}(T^i_{-m})_{\delta\gamma}f_{\gamma\delta}(\mathbf{x},\mathbf{p}')$$
$$-\frac{1}{\mathcal{V}}\sum_{i=0}^{3}\sum_{m=-i}^{i}(-1)^m\sum_{\mathbf{p}'}U^{[i]}(\mathbf{p}-\mathbf{p}')(T^i_m)_{\alpha\gamma}f_{\gamma\delta}(\mathbf{x},\mathbf{p}')(T^i_{-m})_{\delta\beta}.$$

The specific feature of zero-sound is that it propagates even at zero temperature. In this temperature limit, the distribution function, $f_{\alpha\beta}(\mathbf{p}) = f^{[\alpha]}_{\mathbf{p}}\delta_{\alpha\beta}$, takes the form

$$f^{[\alpha]}_{\mathbf{p}} = \Theta(\varepsilon^{[\alpha]}_{\text{F}} - \varepsilon_{\mathbf{p}}), \quad \varepsilon^{[\alpha]}_{\text{F}}(h) = \varepsilon_{\text{F}}(h) + h\left(\frac{5}{2}-\alpha\right).$$

Linearising the kinetic equation, Eq. (2), we come to the dimensionless dispersion equation:

$$Y(w,h,\kappa) \approx 0, \tag{3}$$

$$Y(w,h,\kappa) = 1 - \frac{m^{3/2}\sqrt{2\varepsilon_{\text{F}}(0)}}{(2\pi\hbar)^3}\sum_{i=0}^{3}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\frac{\epsilon^{[\alpha]}_{\text{F}}\cdot\Theta(\epsilon^{[\alpha]}_{\text{F}})\cdot\cos\theta}{w - \left[\sqrt{\epsilon^{[\alpha]}_{\text{F}}} - a^{[\alpha]}_{\text{F}}\right]\cdot\cos\theta}$$
$$\times\left(U^{(i)}(0) - U^{(i)}(2\epsilon^{[\alpha]}_{\text{F}}[1-\cos\chi(\theta,\phi,\kappa)])\right)((T^i_0)_{\alpha\alpha})^2.$$

where $w = \dfrac{\omega}{k}\sqrt{\dfrac{m}{2\varepsilon_{\text{F}}(0)}}$ is the dimensionless speed of zero sound. Three angles are defined as follows: $\kappa = \angle(\mathbf{p},\mathbf{k})$, $\theta = \angle(\mathbf{p}',\mathbf{k})$ and $\chi = \angle(\mathbf{p},\mathbf{p}')$.

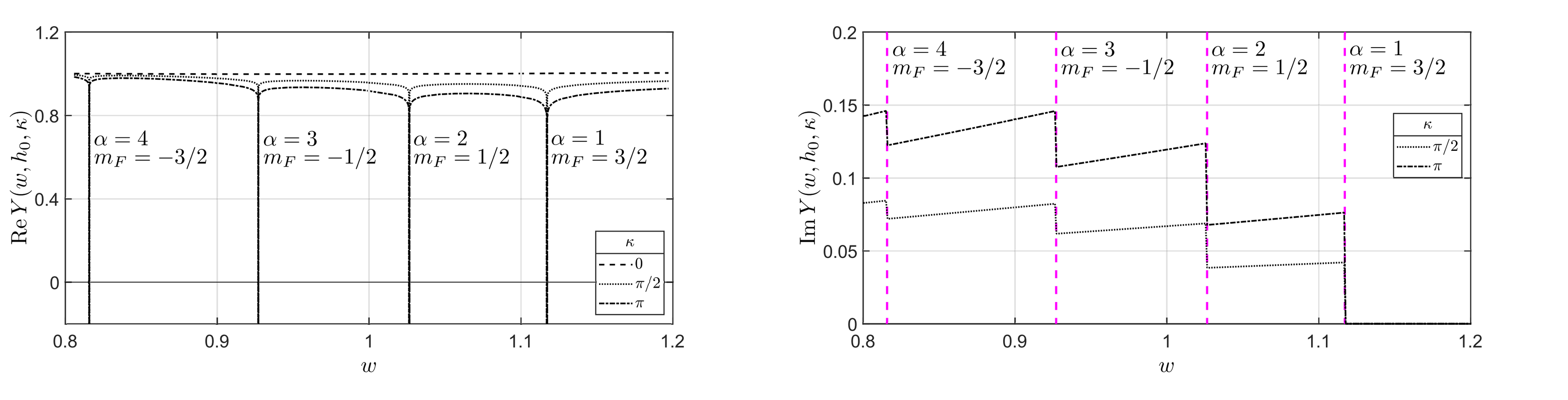

Fig. 1: (left panel) The solutions of Eq. (3) are realized at the intersection of the function $Y(w,h_0,\kappa)$ with a horizontal solid line. (right panel) The dependence of the decrement γ, $\operatorname{Im}Y(w,h,\kappa)\propto\gamma$, on w at fixed magnetic field $h_0 = 0.2\varepsilon_{\text{F}}(0)$. The magenta dashed lines indicate the positions of poles. All calculations are performed for the following values of the physical parameters: $n = 10^{15}\,\text{cm}^{-3}$, $a^{(i)} = 100a_0$, $r^{(0)}_0 = 760a_0$, $r^{(1)}_0 = 780a_0$, $r^{(2)}_0 = 800a_0$, $r^{(3)}_0 = 820a_0$, and $h_0 = 0.2\varepsilon_{\text{F}}(0)$, where $a_0 \approx 53$ pm is the Bohr radius.

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Spin-1 BEC

The Hamiltonian (1), can be also applied to describe the magnetic states of a weakly interacting Bose gas of spin-1 atoms with condensate. To this end, we employ the well-known Bogoliubov model, which treats the creation and annihilation operators with zero momentum as c-numbers ($a_{0\alpha} \to \sqrt{N_0}\zeta_\alpha$, $a^\dagger_{0\alpha} \to \sqrt{N_0}\zeta^*_\alpha$, where ζ_α is the order parameter or the normalized condensate state vector).
The state vector ζ_α is found by minimizing the grand thermodynamic potential.

State	State vector, ζ	Thermodynamic potential, $\mathcal{W}/n_0$
$\text{F}_\pm$ ferromagnetic	$\begin{pmatrix}1\\0\\0\end{pmatrix}$ or $\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\dfrac{v+c}{2} - \chi \mp h - \tilde{\mu}$
Q quadrupolar	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\dfrac{v}{2} - \tilde{\mu}$
P paramagnetic	$\begin{pmatrix}\dfrac{e^{i\phi_+}}{\sqrt{2}}\sqrt{1+\dfrac{h}{c}}\\0\\-\dfrac{e^{i\phi_-}}{\sqrt{2}}\sqrt{1-\dfrac{h}{c}}\end{pmatrix}$	$\dfrac{v}{2} - \dfrac{h^2}{2c} - \chi - \tilde{\mu}$
BA broken-axisymmetry	$\begin{pmatrix}\sqrt{\dfrac{(h-\chi)^2(h^2-\chi^2+2c\chi)}{8c\chi^3}}e^{i\phi_+}\\\sqrt{\dfrac{\chi^2-h^2(h^2+\chi^2+2c\chi)}{4c\chi^3}}e^{i\phi_0}\\\sqrt{\dfrac{(h+\chi)^2(h^2-\chi^2+2c\chi)}{8c\chi^3}}e^{i\phi_-}\end{pmatrix}$	$\dfrac{v}{2} - \dfrac{c}{2}\left(\dfrac{M}{n_0}\right)^2 + \dfrac{h^2-\chi^2+2c\chi}{2\chi} - \tilde{\mu}$

where h, χ are the field parameters, $\tilde{\mu}$ is chemical potential, n_0 is condensate density, $v = n_0\left(U^{[1]}(0) + \frac{4}{3}U^{[2]}(0)\right)$, $c = n_0\left(U^{[1]}(0) - U^{[2]}(0)\right)$, M is absolute value of magnetization.

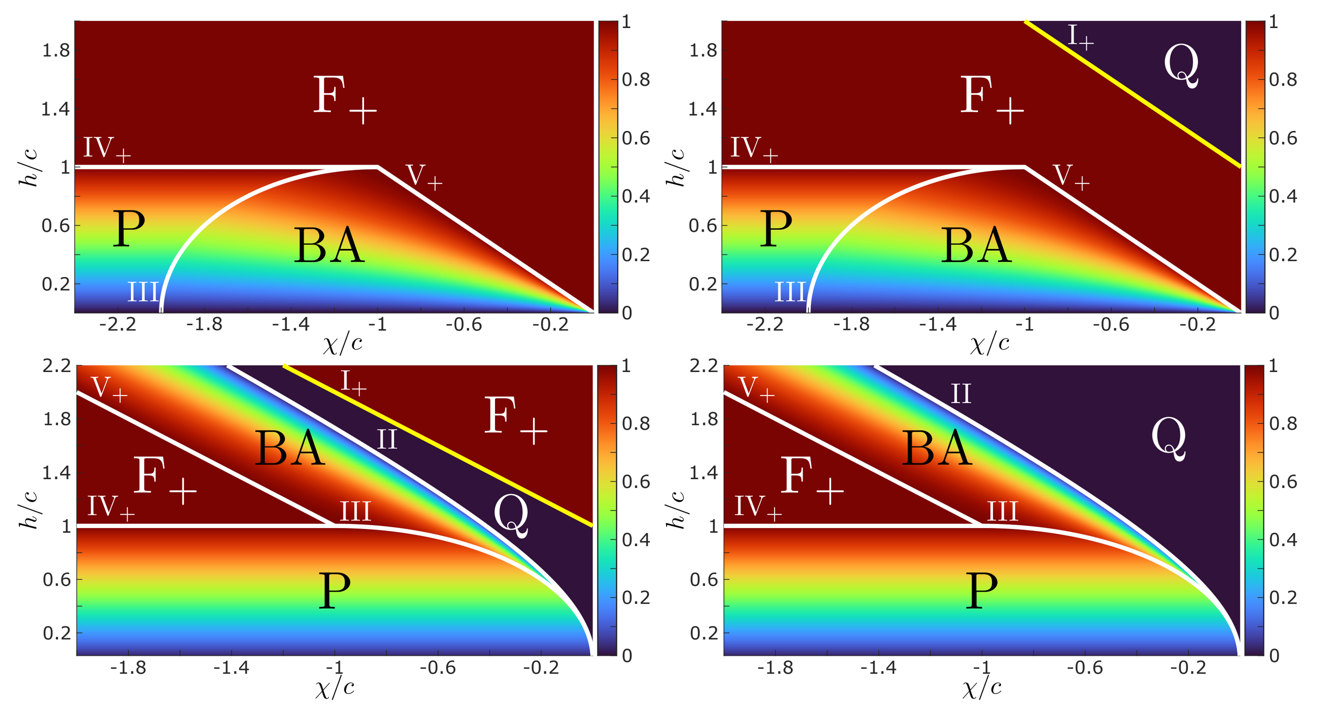

Fig. 2: Magnetic state diagrams of spin-1 BEC, showing magnetization M_z/n_0 (by colour) versus dimensionless magnetic fields χ/c and h/c for $c > 0$ (left panels) and $c < 0$ (right panels). Each row corresponds to certain regime of BA-state. The white and yellow lines denote the second- and first-order phase transitions, respectively.

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Summary

- We have obtained a many-body Hamiltonian of spin-F atoms, which includes the effects of multipolar exchange interaction and the coupling of a multipole moment with an external field.
- The Hamiltonian makes sense only for finite-range interatomic interaction. The zero-range (pseudo)potentials parameterized by s-wave scattering lengths do not give rise the multipolar exchange interactions.
- **Fermi gas.** We have employed the Hamiltonian with multipolar exchange interactions to derive the kinetic equation for spin-3/2 atoms in the collisionless approximation.
- We have applied the resulting equation to study high-frequency oscillations known as zero sound.
- We have shown that there are 8 zero-sound modes: 2 ('slow' and 'fast' waves) for each spin projections. The 'slow' waves are characterized by a larger damping factor than the 'fast' ones. The only one 'fast' mode corresponding to the spin projection $m_F = 3/2$ has a zero decrement.
- **Bose gas.** We have also applied the proposed Hamiltonian to study all possible magnetic states emerging in a gas of spin-1 atoms with BEC.
- We have found four magnetic states: ferromagnetic, quadrupolar, paramagnetic as well as broken-axisymmetry state. The latter is realized due to the coupling of a quadrupole moment with a magnetic field (quadratic Zeemean effect).

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