

# ZERO ENERGY SCATTERING FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATORS AND APPLICATIONS TO DISPERSIVE ESTIMATES

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ABSTRACT. We show that for a one-dimensional Schrödinger operator with a potential, whose  $(j + 1)$ 'th moment is integrable, the  $j$ 'th derivative of the scattering matrix is in the Wiener algebra of functions with integrable Fourier transforms. We use this result to improve the known dispersive estimates with integrable time decay for the one-dimensional Schrödinger equation in the resonant case.

## 1. INTRODUCTION

This paper is concerned with the one-dimensional Schrödinger equation

$$i\dot{\psi}(x, t) = H\psi(x, t), \quad H = -\frac{d^2}{dx^2} + V(x), \quad (x, t) \in \mathbb{R}^2, \quad (1.1)$$

with a real-valued potential  $V$  contained in one of the spaces  $L^1_\sigma = L^1_\sigma(\mathbb{R})$ ,  $\sigma \in \mathbb{R}$ , associated with the norms

$$\|\psi\|_{L^p_\sigma} = \begin{cases} \left( \int_{\mathbb{R}} (1 + |x|)^{p\sigma} |\psi(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{x \in \mathbb{R}} (1 + |x|)^\sigma |\psi(x)|, & p = \infty. \end{cases}$$

We recall (e.g., [15] or [16, Sect. 9.7]) that for  $V \in L^1_1$  the operator  $H$  has a purely absolutely continuous spectrum on  $[0, \infty)$  and a finite number of eigenvalues in  $(-\infty, 0)$ . Associated with the absolutely continuous spectrum is the scattering matrix

$$\mathcal{S}(k) = \begin{pmatrix} T(k) & R_-(k) \\ R_+(k) & T(k) \end{pmatrix},$$

which maps incoming to outgoing states at a given energy  $\omega = k^2 \geq 0$ . Here  $T$  is the transmission coefficient and  $R_\pm$  are the reflection coefficients with respect to right and left incident. At the edge of the continuous spectrum  $k = 0$  the scattering matrix generically looks like

$$\mathcal{S}(0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

More precisely, this happens when the zero energy is non-resonant, that is, if the equation  $Hf_0 = 0$  has no bounded (distributional) solution. In the resonant situation the behavior of the scattering matrix is more delicate. For  $V \in L^1_1$  it is already nontrivial to establish continuity of the scattering matrix at  $k = 0$  (for  $V \in L^1_2$  de

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l'Hospital's rule suffices). This question arose around 1985 in an attempt to clarify whether the low-energy asymptotics of the scattering matrix, obtained for  $V \in L^1_2$  in [6], are valid for  $V \in L^1_1$ . The problem was solved independently by Guseinov [12] and Klaus [13] (for a refined version see [1] and [9, Theorem 2.1]). It also plays an important role in semiclassical analysis, see [5] and the references therein. Furthermore, if  $V \in L^1_{j+1}$  with  $j > 0$ , then, away from 0, one can take derivatives of the scattering matrix up to order  $j$ , and in the non-resonant case it is again easy to see that they are continuous at  $k = 0$ . This clearly raises the question about continuity at  $k = 0$  of the  $j$ 'th derivative in the resonant case. We will establish this as one of our main results in Theorem 2.2. This result is new even for the first derivative. We remark, that if  $V$  decays exponentially, then  $S(k)$  is analytic in a neighborhood of  $k = 0$ , and the full Taylor expansion can be obtained [2, 3].

It is important to emphasize that this question is not only of interest in scattering theory, but also plays a role in solving the Korteweg–de Vries equation via the inverse scattering transform (see e.g., [11], where continuity of higher derivatives is needed), and in deriving dispersive estimates for (1.1). The latter case has attracted considerable interest (e.g. [9, 10] and the references therein) due to its importance for proving asymptotic stability of solitons for the associated nonlinear evolution equations (see e.g. [4, 14]).

As an application of our results, we will establish the following dispersive decay estimate with integrable time decay in the resonant case:

**Theorem 1.1.** *Suppose  $V \in L^1_3(\mathbb{R})$  and suppose there is a bounded solution  $f_0$  of  $Hf_0 = 0$  satisfying the normalization  $\lim_{x \rightarrow \infty} (|f_0(x)|^2 + |f_0(-x)|^2) = 2$ . Denote by  $P_0 : L^1_2 \rightarrow L^\infty_2$  the operator given by the kernel  $[P_0](x, y) = f_0(x)f_0(y)$ . By  $P_{ac}$  we denote the projector onto the absolutely continuous subspace of  $H$ . Then  $e^{-itH}P_{ac}$  extends to a bounded operator  $L^1_2 \rightarrow L^\infty_2$  satisfying the following decay estimate:*

$$\|e^{-itH}P_{ac} - (4\pi it)^{-\frac{1}{2}}P_0\|_{L^1_2 \rightarrow L^\infty_2} = \mathcal{O}(t^{-3/2}), \quad t \rightarrow \infty. \quad (1.2)$$

This theorem is an improvement of an earlier result by Goldberg [10], who established it for  $V \in L^1_4(\mathbb{R})$ . If there is no resonance (i.e. no bounded solution) this result (with  $P_0 = 0$ ) was shown for  $V \in L^1_2(\mathbb{R})$  in [9, Theorem 1.2]. For extensions to discrete one-dimensional Schrödinger equations (Jacobi operators) see [7], [8].

## 2. LOW ENERGY SCATTERING

In this section we establish some properties of the scattering data for our operator  $H$  with  $V \in L^1_{j+1}$ ,  $j \geq 0$ . To this end we introduce the Banach algebra  $\mathcal{A}$  of Fourier transforms of integrable functions:

$$\mathcal{A} = \left\{ f(k) : f(k) = \int_{\mathbb{R}} e^{ikp} \hat{f}(p) dp, \hat{f}(\cdot) \in L^1(\mathbb{R}) \right\} \quad (2.1)$$

with the norm  $\|f\|_{\mathcal{A}} = \|\hat{f}\|_{L^1}$ , plus the corresponding unital Banach algebra  $\mathcal{A}_1$ :

$$\mathcal{A}_1 = \left\{ f(k) : f(k) = c + \int_{\mathbb{R}} e^{ikp} \hat{g}(p) dp, \hat{g}(\cdot) \in L^1(\mathbb{R}), c \in \mathbb{C} \right\} \quad (2.2)$$

with the norm  $\|f\|_{\mathcal{A}_1} = |c| + \|\hat{g}\|_{L^1}$ . We also use the fact, which is known as Wiener's lemma [17], that if  $f \in \mathcal{A}_1 \setminus \mathcal{A}$  and  $f(k) \neq 0$  for all  $k \in \mathbb{R}$  then  $f^{-1}(k) \in \mathcal{A}_1$ .

First we recall a few facts from scattering theory [6], [15]. If  $V \in L^1_1$ , there exist Jost solutions  $f_{\pm}(x, k)$  of  $Hf = k^2 f$ ,  $k \in \overline{\mathbb{C}}_+$ , which asymptotically behave like  $f_{\pm}(x, k) \sim e^{\pm ikx}$  as  $x \rightarrow \pm\infty$ . These solutions are given by

$$f_{\pm}(x, k) = e^{\pm ikx} h_{\pm}(x, k), \quad h_{\pm}(x, k) = 1 \pm \int_0^{\pm\infty} B_{\pm}(x, y) e^{\pm 2iky} dy, \quad (2.3)$$

where  $B_{\pm}(x, y)$  are real-valued and satisfy (see [6, §2] or [15, §3.1])

$$|B_{\pm}(x, y)| \leq e^{\gamma_{\pm}(x)} \eta_{\pm}(x + y), \quad (2.4)$$

$$\left| \frac{\partial}{\partial x} B_{\pm}(x, y) \pm V(x + y) \right| \leq 2e^{\gamma_{\pm}(x)} \eta_{\pm}(x + y) \eta_{\pm}(x), \quad (2.5)$$

with

$$\gamma_{\pm}(x) = \int_x^{\pm\infty} (y - x) |V(y)| dy, \quad \eta_{\pm}(x) = \pm \int_x^{\pm\infty} |V(y)| dy. \quad (2.6)$$

Since  $\eta_{\pm} \in L^1(\mathbb{R}_{\pm})$ , we conclude that

$$h_{\pm}(x, \cdot) - 1, \quad h'_{\pm}(x, \cdot) \in \mathcal{A}, \quad \forall x \in \mathbb{R}. \quad (2.7)$$

Here and throughout the rest of this paper a prime will always denote a derivative with respect to the spatial variable  $x$ . As an immediate consequence of the estimates (2.4) and (2.5) we have the following strengthening of (2.7):

**Lemma 2.1.** *Let  $V \in L^1_{j+1}$ . Then  $\frac{\partial^l}{\partial k^l} (h_{\pm}(x, k) - 1)$ ,  $\frac{\partial^l}{\partial k^l} h'_{\pm}(x, k)$  are contained in  $\mathcal{A}$  for  $0 \leq l \leq j$ . Moreover, for  $\pm x \geq 0$ , the  $\mathcal{A}$ -norms of these expressions do not depend on  $x$ .*

The fact that  $f_{\pm}(x, -k)$  also solve  $Hf = k^2 f$  for  $k \in \mathbb{R}$  leads to the scattering relations

$$T(k) f_{\pm}(x, k) = R_{\mp}(k) f_{\mp}(x, k) + f_{\mp}(x, -k), \quad (2.8)$$

where the transmission coefficient  $T$  and the reflection coefficients  $R_{\pm}$  can be expressed in terms of Wronskians. To this end we denote by

$$W(f(x), g(x)) = f(x)g'(x) - f'(x)g(x)$$

the usual Wronskian and set

$$W(k) = W(f_-(x, k), f_+(x, k)), \quad W_{\pm}(k) = W(f_{\mp}(x, k), f_{\pm}(x, -k)).$$

Then

$$T(k) = \frac{2ik}{W(k)}, \quad R_{\pm}(k) = \mp \frac{W_{\pm}(k)}{W(k)}. \quad (2.9)$$

The transmission and reflection coefficients are elements of the Wiener algebra, which was established in [9, Theorem 2.1]. Here we extend this result to the derivatives of the scattering data.

**Theorem 2.2.** *If  $V \in L^1_{j+1}$ , then  $\frac{d^l}{dk^l} (T(k) - 1) \in \mathcal{A}$  and  $\frac{d^l}{dk^l} R_{\pm}(k) \in \mathcal{A}$  for  $0 \leq l \leq j$ .*

*Proof.* We only focus on the resonant case, since the other case is straightforward.

First of all, we abbreviate  $h_{\pm}(k) = h_{\pm}(0, k)$ ,  $h'_{\pm}(k) = h'_{\pm}(0, k)$ . Then (2.3) leads us to

$$W(k) = 2ikh_+(k)h_-(k) + h_-(k)h'_+(k) - h'_-(k)h_+(k), \quad (2.10)$$

$$W_{\pm}(k) = h_{\mp}(k)h'_{\pm}(-k) - h_{\pm}(-k)h'_{\mp}(k). \quad (2.11)$$

Following [9] we introduce

$$\Phi_{\pm}(k) = h_{\pm}(k)h'_{\pm}(0) - h'_{\pm}(k)h_{\pm}(0), \quad (2.12)$$

$$K_{\pm}(x, y) = \pm \int_y^{\pm\infty} B_{\pm}(x, z)dz, \quad D_{\pm}(x, y) = \pm \int_y^{\pm\infty} \frac{\partial}{\partial x} B_{\pm}(x, z)dz, \quad (2.13)$$

where  $B_{\pm}(x, y)$  are given by (2.3). Again  $K_{\pm}(0, y)$  is denoted by  $K_{\pm}(y)$  and similar for  $D_{\pm}(y)$ . In [9, Theorem 2.1], the following equation for  $\Phi_{\pm}(k)$  was obtained:

$$\Phi_{\pm}(k) = 2ik\Psi_{\pm}(k), \quad \Psi_{\pm}(k) = \int_0^{\pm\infty} H_{\pm}(y)e^{\pm 2iky}dy, \quad (2.14)$$

where

$$H_{\pm}(x) = D_{\pm}(x)h_{\pm}(0) - K_{\pm}(x)h'_{\pm}(0), \quad \pm x \geq 0. \quad (2.15)$$

Moreover,  $H_{\pm}$  satisfies an estimate similar to (2.4), as we will show in Lemma 2.3 below. As a consequence,  $\Psi_{\pm}$  and its derivatives up to order  $j$  will be in the Wiener algebra.

Next, a straightforward computation (cf. [9]) shows

$$\frac{W(k)}{2ik} = h_{-}(k)h_{+}(k) + \begin{cases} \frac{h_{+}(k)}{h_{-}(0)}\Psi_{-}(k) - \frac{h_{-}(k)}{h_{+}(0)}\Psi_{+}(k), & h_{+}(0)h_{-}(0) \neq 0, \\ \frac{h'_{+}(k)}{h'_{-}(0)}\Psi_{-}(k) - \frac{h'_{-}(k)}{h'_{+}(0)}\Psi_{+}(k), & h_{+}(0) = h_{-}(0) = 0, \end{cases}$$

and since  $\frac{W(k)}{2ik} = T(k)^{-1} \neq 0$  for all  $k \in \mathbb{R}$ , and  $T(k) \rightarrow 1$  as  $k \rightarrow \infty$ , we conclude that  $\frac{d^l}{dk^l}(T(k) - 1) \in \mathcal{A}$  for  $0 \leq l \leq j$ . Analogously,

$$\frac{W_{\pm}(k)}{2ik} = \begin{cases} \frac{h_{\pm}(-k)}{h_{\mp}(0)}\Psi_{\mp}(k) - \frac{h_{\mp}(k)}{h_{\pm}(0)}\Psi_{\pm}(-k), & h_{+}(0)h_{-}(0) \neq 0, \\ \frac{h'_{\pm}(-k)}{h'_{\mp}(0)}\Psi_{\mp}(k) - \frac{h'_{\mp}(k)}{h'_{\pm}(0)}\Psi_{\pm}(-k), & h_{+}(0) = h_{-}(0) = 0, \end{cases}$$

and hence  $R_{\pm}(k) = \mp \frac{W_{\pm}(k)}{2ik}T(k)$  has the claimed properties.  $\square$

To complete the proof of Theorem 2.2 we need the following result, which is an extension of [9, Lemma 2.2]:

**Lemma 2.3.** *Let  $H_{\pm}(x)$  be given by (2.15). For  $V \in L^1_1$  the following estimate is valid:*

$$|H_{\pm}(x)| \leq \hat{C}\eta_{\pm}(x), \quad \pm x \geq 0, \quad (2.16)$$

with some constant  $\hat{C} > 0$  and  $\eta_{\pm}(x)$  given by (2.6). Moreover, for  $\Psi_{\pm}(k)$  defined by (2.14) and for  $V \in L^1_{j+1}$  we have

$$\frac{d^l}{dk^l}\Psi_{\pm}(k) \in \mathcal{A}, \quad 0 \leq l \leq j.$$

*Proof.* It suffices to prove the estimate for  $H_{\pm}$ . The Marchenko equation (§3.5 in [15]) states that the kernels  $B_{\pm}(x, y)$  solve the equations

$$F_{\pm}(x+y) + B_{\pm}(x, y) \pm \int_0^{\pm\infty} B_{\pm}(x, t)F_{\pm}(x+y+z)dz = 0, \quad (2.17)$$

where the functions  $F_{\pm}(x)$  are absolutely continuous with  $x F'_{\pm}(x) \in L^1(\mathbb{R}_{\pm})$  and

$$|F_{\pm}(x)| \leq C\eta_{\pm}(x), \quad \pm x \geq 0, \quad (2.18)$$

with  $\eta_{\pm}$  from (2.6). Now the calculations in [9, Lemma 2.2] lead to the following integral equation for  $H_{\pm}(x)$ :

$$\begin{aligned} H_{\pm}(x) \mp \int_0^{\pm\infty} H_{\pm}(y) F_{\pm}(x+y) dy &= G_{\pm}(x), \\ G_{\pm}(x) &= h_{\pm}(0) \left( \int_0^{\pm\infty} B_{\pm}(0, y) F_{\pm}(x+y) dy \pm F_{\pm}(x) \right). \end{aligned} \quad (2.19)$$

The estimates (2.4) and (2.18) imply

$$|G_{\pm}(x)| \leq \tilde{C} \eta_{\pm}(x), \quad \pm x \geq 0. \quad (2.20)$$

Now, for a given potential  $V$  fix  $N > 0$  such that

$$D(N) := \max_{\pm} \left( \pm C \int_{\pm N}^{\pm\infty} \eta_{\pm}(y) dy \right) \in (0, 1),$$

where  $C$  is given by (2.18). Then we can rewrite (2.19) in the form

$$\begin{aligned} H_{\pm}(x) \mp \int_{\pm N}^{\pm\infty} H_{\pm}(y) F_{\pm}(x+y) dy &= G_{\pm}(x, N), \\ G_{\pm}(x, N) &= G_{\pm}(x) \pm \int_0^{\pm N} H_{\pm}(y) F_{\pm}(x+y) dy. \end{aligned} \quad (2.21)$$

From (2.13) and the estimates (2.4)–(2.5) we deduce  $H_{\pm} \in L^{\infty}(\mathbb{R}_{\pm}) \cap C(\mathbb{R})$ . We also have

$$|G_{\pm}(x, N)| \leq C(N) \eta_{\pm}(x) \quad (2.22)$$

by the boundedness of  $H_{\pm}$ , the estimate (2.20), and monotonicity of  $\eta_{\pm}(x)$ . Applying to (2.21) the method of successive approximations we get

$$|H_{\pm}(x)| \leq C(N) \eta_{\pm}(x) \sum_{n=0}^{\infty} \left( \pm C \int_{\pm N}^{\pm\infty} \eta_{\pm}(y) dy \right)^n \leq \hat{C}(N) \eta_{\pm}(x),$$

with  $\hat{C}(N) = C(N)(1 - D(N))^{-1}$ . This implies (2.16). The rest follows from (2.14).  $\square$

For later use we note that in the resonant case the Jost solutions are dependent at  $k = 0$ . If we define  $\gamma$  via

$$f_+(x, 0) = \gamma f_-(x, 0), \quad (2.23)$$

then a straightforward calculation using the scattering relations (2.8) as well as  $|T(k)|^2 + |R_{\pm}(k)|^2 = 1$  shows

$$T(0) = \frac{2\gamma}{1 + \gamma^2}, \quad R_{\pm}(0) = \pm \frac{1 - \gamma^2}{1 + \gamma^2}. \quad (2.24)$$

In particular, all three quantities are real-valued since  $f_{\mp}(x, 0) \in \mathbb{R}$  and hence  $\gamma \in \mathbb{R}$ .

To establish Theorem 1.1 we need the following generalization of Lemma 2.1:

**Lemma 2.4.** *Let  $V \in L^1_{j+1}$  with  $j \geq 1$ . Then  $\frac{\partial^l}{\partial k^l} \left( \frac{h_{\pm}(x, k) - h_{\pm}(x, 0)}{k} \right)$  as well as  $\frac{\partial^l}{\partial k^l} \left( \frac{h'_{\pm}(x, k) - h'_{\pm}(x, 0)}{k} \right)$  are contained in  $\mathcal{A}$  for  $0 \leq l \leq j - 1$ . Moreover, for  $\pm x \geq 0$  the  $\mathcal{A}$ -norms of these expressions do not depend on  $x$ .*

*Proof.* Using (2.3) and Fubini, a little calculation shows

$$\begin{aligned} \frac{h_{\pm}(x, k) - h_{\pm}(x, 0)}{k} &= \pm 2i \int_0^{\pm\infty} B_{\pm}(x, z) \left( \int_0^z e^{\pm 2iky} dy \right) dz \\ &= \pm 2i \int_0^{\pm\infty} e^{\pm 2iky} \left( \int_y^{\pm\infty} B_{\pm}(x, z) dz \right) dy = 2i \int_0^{\pm\infty} K_{\pm}(x, y) e^{\pm 2iky} dy. \end{aligned}$$

Now after differentiating with respect to  $k$  the claim follows by (2.4). For the second item we can proceed exactly in the same way.  $\square$

Similarly,

**Lemma 2.5.** *Let  $V \in L_{j+1}^1$  with  $j \geq 1$ . Then  $\frac{d^l}{dk^l} \left( \frac{\Psi_{\pm}(k) - \Psi_{\pm}(0)}{k} \right) \in \mathcal{A}$  for  $0 \leq l \leq j - 1$ .*

*Proof.* This follows as in the previous lemma using the estimate for  $H_{\pm}$  from Lemma 2.3.  $\square$

Combining the last two lemmas we obtain:

**Theorem 2.6.** *Let  $V \in L_{j+1}^1$  with  $j \geq 1$ . Then  $\frac{d^l}{dk^l} \left( \frac{T(k) - T(0)}{k} \right)$  and  $\frac{d^l}{dk^l} \left( \frac{R_{\pm}(k) - R_{\pm}(0)}{k} \right)$  are elements of  $\mathcal{A}$  for  $0 \leq l \leq j - 1$ .*

### 3. DISPERSIVE DECAY ESTIMATES

In this section we prove the integrable dispersive decay estimate (1.2) for the Schrödinger equation (1.1) in the resonant case. For the one-parameter group of (1.1) the spectral theorem and Stone's formula imply

$$e^{-itH} P_{ac} = \frac{1}{2\pi i} \int_0^{\infty} e^{-it\omega} (\mathcal{R}(\omega + i0) - \mathcal{R}(\omega - i0)) d\omega, \quad (3.1)$$

where  $\mathcal{R}(\omega) = (H - \omega)^{-1}$  is the resolvent of the Schrödinger operator  $H$ , and the limit is understood in the strong sense [16, Problem 4.3]. Given the Jost solutions, we can express the kernel of the resolvent  $\mathcal{R}(\omega)$  for  $\omega = k^2 \pm i0$ ,  $k > 0$ , as [16, Lemma 9.7]

$$[\mathcal{R}(k^2 \pm i0)](x, y) = -\frac{f_+(y, \pm k)f_-(x, \pm k)}{W(\pm k)} = \mp \frac{f_+(y, \pm k)f_-(x, \pm k)T(\pm k)}{2ik} \quad (3.2)$$

for all  $x \leq y$  (and the positions of  $x, y$  reversed if  $x > y$ ). Therefore, in the case  $x \leq y$ , the integral kernel of  $e^{-itH} P_{ac}$  is given by

$$[e^{-itH} P_{ac}](x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(tk^2 - |y-x|k)} h_+(y, k) h_-(x, k) T(k) dk, \quad (3.3)$$

where the integral has to be understood as an improper integral. Another result that we need in order to obtain our decay estimates is the following variant of the van der Corput lemma [9, Lemma 5.4]:

**Lemma 3.1.** *Consider the oscillatory integral  $I(t) = \int_a^b e^{it\phi(k)} f(k) dk$ , where  $\phi(k)$  is a real-valued function. If  $\phi''(k) \neq 0$  in  $[a, b]$  and  $f \in \mathcal{A}_1$ , then we have  $|I(t)| \leq C_2 [t \min_{a \leq k \leq b} |\phi''(k)|]^{-1/2} \|f\|_{\mathcal{A}_1}$  for  $t \geq 1$ , where  $C_2 \leq 2^{8/3}$  is the optimal constant from the van der Corput lemma.*

Now we come to the proof of our main Theorem 1.1. We first give an alternate representation of our projection operator  $(4\pi it)^{-\frac{1}{2}}P_0$ :

**Lemma 3.2** ([10]). *The integral kernel of  $(4\pi it)^{-\frac{1}{2}}P_0$ , which is (per definition) given by  $(4\pi it)^{-\frac{1}{2}}f_0(x)f_0(y)$ , can also be written in the form*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} T(0) f_-(x, 0) f_+(y, 0) dk.$$

*Proof.* It is clear that  $f_0(x) = c_{\pm} f_{\pm}(x, 0)$  and by our normalization  $c_-^{-2} + c_+^{-2} = 2$ . Using (2.23) we have  $c_- = \gamma c_+$  and hence  $c_{\pm}^2 = \frac{1+\gamma^{\mp 1}}{2}$ . Moreover, (2.24) implies  $c_+ c_- T(0) = 1$ , and using  $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} dk = (4\pi it)^{-\frac{1}{2}}$  the claim follows.  $\square$

Finally we have all the ingredients needed to obtain Theorem 1.1:

*Proof of Theorem 1.1.* For the kernel of  $e^{-itH} P_{ac}$  we use (3.3). Then by Lemma 3.2, the kernel  $G(x, y, t)$  of  $e^{-itH} P_{ac} - (4\pi it)^{-\frac{1}{2}} P_0$  can now be written as

$$G(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} (e^{i|y-x|k} h_+(y, k) h_-(x, k) T(k) - h_+(y, 0) h_-(x, 0) T(0)) dk.$$

Integrating this formula by parts, we obtain

$$G(x, y, t) = \frac{1}{4\pi it} \int_{-\infty}^{\infty} e^{-itk^2} S(x, y, k) dk,$$

where

$$S(x, y, k) = \frac{\partial}{\partial k} \left( \frac{e^{i|y-x|k} h_+(y, k) h_-(x, k) T(k) - h_+(y, 0) h_-(x, 0) T(0)}{k} \right).$$

Now we apply Lemma 3.1 to get the desired  $t^{-\frac{3}{2}}$  time-decay. So in order to finish our proof, it remains to bound the  $\mathcal{A}$ -norm of  $S(x, y, k)$  which follows from the lemma below.  $\square$

**Lemma 3.3.** *Assume  $V \in L^1_3$ . Then*

$$\|S(x, y, \cdot)\|_{\mathcal{A}} \leq C(|x| + |y|)^2.$$

*Proof.* We assume w.l.o.g.  $x \leq y$  and distinguish the cases (i)  $x \leq 0 \leq y$ , (ii)  $0 \leq x \leq y$  and (iii)  $x \leq y \leq 0$ . Introduce the function  $g(x, y, k) := T(k) h_+(y, k) h_-(x, k)$ . Then  $S(x, y, k)$  can be written as:

$$\begin{aligned} S(x, y, k) &= \frac{\partial}{\partial k} \left( \frac{e^{i(y-x)k} - 1}{k} \right) g(x, y, k) \\ &\quad + \frac{e^{i(y-x)k} - 1}{k} \frac{\partial}{\partial k} g(x, y, k) + \frac{\partial}{\partial k} \left( \frac{g(x, y, k) - g(x, y, 0)}{k} \right). \end{aligned}$$

The  $\mathcal{A}$ -norm of  $\frac{e^{i(y-x)k} - 1}{k}$  is bounded by  $C(|x| + |y|)$  and that of its derivative by  $C(|x| + |y|)^2$ . So it remains to consider the  $\mathcal{A}_1$ -norms of  $g(x, y, k)$  and  $\frac{\partial}{\partial k} g(x, y, k)$ , and  $\mathcal{A}$ -norm of  $\frac{\partial}{\partial k} P(x, y, k)$ , where

$$P(x, y, k) = \frac{g(x, y, k) - g(x, y, 0)}{k}.$$

We start with case (i). Then  $g(x, y, k) \in \mathcal{A}_1$  with  $\mathcal{A}_1$ -norm independent of  $x$  and  $y$ . After applying the product rule, Lemmas 2.1 and 2.6 imply  $\|\frac{\partial}{\partial k}g(x, y, k)\|_{\mathcal{A}} \leq C$ . Moreover,

$$P(x, y, k) = \frac{T(k) - T(0)}{k} h_+(y, k) h_-(x, k) + \frac{h_+(y, k) - h_+(y, 0)}{k} h_-(x, k) T(0) + \frac{h_-(x, k) - h_-(x, 0)}{k} h_+(y, 0) T(0).$$

Taking here the derivative with respect to  $k$  and invoking Lemma 2.1, Lemma 2.4, and Lemma 2.6, we are done in this case. In the cases (ii) and (iii) we use the scattering relations (2.8) to see that the following representations are valid:

$$g(x, y, k) = \begin{cases} h_+(y, k) (R_+(k) h_+(x, k) e^{2ixk} + h_+(x, -k)), & 0 \leq x \leq y, \\ h_-(x, k) (R_-(k) h_-(y, k) e^{-2iyk} + h_-(y, -k)), & x \leq y \leq 0. \end{cases}$$

Thus  $g(x, y, k)$  has an  $\mathcal{A}_1$ -norm independent of  $x$  and  $y$ , since for any function  $f(k) \in \mathcal{A}$  and any real  $s$  we have  $f(k)e^{iks} \in \mathcal{A}$  with the norm independent of  $s$ . If we take the derivative with respect to  $k$ , again everything is contained in  $\mathcal{A}$  by Lemma 2.1 and Lemma 2.6, however we get additional terms from the derivatives of  $e^{2ixk}$  and  $e^{2iyk}$ . So it follows that the  $\mathcal{A}$ -norm of  $\frac{\partial}{\partial k}g(x, y, k)$  is at most proportional to  $|x|$  or  $|y|$  respectively. Finally, let us have a look at  $\frac{\partial}{\partial k}P(x, y, k)$ . We observe that in case (ii) one can represent  $P$  as

$$P(x, y, k) = \frac{h_+(y, k) - h_+(y, 0)}{k} h_+(x, -k) + \frac{h_+(x, -k) - h_+(x, 0)}{k} h_+(y, 0) + \frac{R_+(k) - R_+(0)}{k} h_+(x, k) h_+(y, k) e^{2ixk} + \frac{e^{2ixk} - 1}{k} R_+(0) h_+(x, k) h_+(y, k) + \frac{h_+(x, k) - h_+(x, 0)}{k} h_+(y, k) R_+(0) + \frac{h_+(y, k) - h_+(y, 0)}{k} h_+(x, 0) R_+(0).$$

Here again every summand is an element of  $\mathcal{A}$  by Lemma 2.1, Lemma 2.4, and Theorem 2.6. Since the derivative of  $\frac{e^{2ixk} - 1}{k}$  also occurs here, we conclude that the  $\mathcal{A}$ -norm of  $\frac{\partial}{\partial k}P(x, y, k)$  is at most proportional to  $|x|^2$ . From the same reasons in the case (iii) this derivative will be proportional to  $|y|^2$ .  $\square$

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#### REFERENCES

- [1] T. Aktosun and M. Klaus, *Small energy asymptotics for the Schrödinger equation on the line*, Inverse Problems **17** (2001), 619–632.
- [2] D. Bollé, F. Gesztesy, and M. Klaus, *Scattering theory for one-dimensional systems with  $\int dx V(x) = 0$* , J. Math. Anal. Appl. **122** (1987), 496–518.
- [3] D. Bollé, F. Gesztesy, and S.F.J. Wilk, *A complete treatment of low-energy scattering in one dimension*, J. Operator Theory **13** (1985), 3–31.
- [4] V.S. Buslaev and C. Sulem, *On asymptotic stability of solitary waves for nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré Anal. Non Linéaire **20** (2003), 419–475.
- [5] O. Costin, W. Schlag, W. Staubach, and S. Tanveer, *Semiclassical analysis of low and zero energy scattering for one-dimensional Schrödinger operators with inverse square potentials*, J. Funct. Anal. **255** (2008), 2321–2362.
- [6] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), 121–251.

- [7] I. Egorova, M. Holzleitner, and G. Teschl, *Properties of the scattering matrix and dispersion estimates for Jacobi operators*, J. Math. Anal. Appl. **434** (2016), 956–966.
- [8] I. Egorova, E. Kopylova, and G. Teschl, *Dispersion estimates for one-dimensional discrete Schrödinger and wave equations*, J. Spectr. Theory **5** (2015), 663–696.
- [9] I. Egorova, E. Kopylova, V. Marchenko, and G. Teschl, *Dispersion estimates for one-dimensional Schrödinger and Klein-Gordon equations revisited*, Russian Math. Surveys (to appear).
- [10] M. Goldberg, *Transport in the one dimensional Schrödinger equation*, Proc. Amer. Math. Soc. **135** (2007), 3171–3179.
- [11] K. Grunert and G. Teschl, *Long-time asymptotics for the Korteweg–de Vries equation via nonlinear steepest descent*, Math. Phys. Anal. Geom. **12** (2009), 287–324.
- [12] I. M. Guseinov, *On the continuity of the reflection coefficient for one-dimensional Schrödinger equation*, Diff. Equations **21** (1985), 1993–1995.
- [13] M. Klaus, *Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line*, Inverse Problems **4** (1988), 505–512.
- [14] E. Kopylova and A. Komech, *On asymptotic stability of kink for relativistic Ginsburg-Landau equation*, Arch. Rat. Mech. and Analysis **202** (2011), 213–245.
- [15] V. A. Marchenko, *Sturm-Liouville Operators and Applications*, rev. ed., Amer. Math. Soc., Providence, 2011.
- [16] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, 2nd ed., Amer. Math. Soc., Rhode Island, 2014.
- [17] N. Wiener, *Tauberian theorems*, Ann. of Math. (2) **33** (1932), 1–100.

## ADDENDUM

After publication of this paper we learned that Theorem 1.1 has been obtained for  $V \in L^1_4(\mathbb{R})$  by Mizutani in

- H. Mizutani, *Dispersive estimates and asymptotic expansions for Schrödinger equations in dimension one*, J. Math. Soc. Japan **63** (2011), 239–261.

This paper also contains the non-resonant case (with  $P_0 = 0$ ) for  $V \in L^1_2(\mathbb{R})$  plus some further extensions.

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