EXTREME POINTS IN THE ISOMETRIC EMBEDDING PROBLEM FOR MODEL SPACES

By

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Abstract. In 1996 A. Aleksandrov solved the isometric embedding problem for the model spaces K_Θ with an arbitrary inner function Θ. We find all extreme points of this convex set of measures in the case when Θ is a finite Blaschke product, and obtain some partial results for generic inner functions.

Introduction

In [4] A. Aleksandrov settled the isometric embedding problem for the model spaces $K_\Theta := H^2 \ominus \Theta H^2$. Precisely, let Θ be an arbitrary, nonconstant inner function on the unit disk $\mathbb{D}$, i.e., Θ belongs to the unit ball of H^∞ (the Schur class $\mathcal{S}$), and $|\Theta| = 1$ a.e. on the unit circle $\mathbb{T}$. Denote by $\mathcal{M}_+(\mathbb{T})$ the class of all finite, positive Borel measures on $\mathbb{T}$. The problem is to describe a subclass $\mathcal{M}(\Theta) \subset \mathcal{M}_+(\mathbb{T})$ of measures σ so that the identity operator (embedding) of the model space K_Θ to the space $L^2_\sigma(\mathbb{T})$ is isometric. In other words, the equality

$$\langle f, g\rangle_\sigma := \int_{\mathbb{T}} f(t)\overline{g(t)}\sigma(dt) = \int_{\mathbb{T}} f(t)\overline{g(t)}m(dt) = \langle f, g\rangle_m, \quad f, g \in K_\Theta \cap C(\mathbb{T})$$

holds for each continuous function $f, g \in K_\Theta$. Here m is the normalized Lebesgue measure on $\mathbb{T}$. As it turns out [3], for such measures and for each $f \in K_\Theta$ the boundary values exist a.e. with respect to the measure σ, so the equality can be extended to the whole model space K_Θ.

The result of Aleksandrov looks as follows.

Theorem A. $\sigma \in \mathcal{M}(\Theta)$ *if and only if there is a unique pair* (β, ω) *with a real number* β *and a Schur function* $\omega \in \mathcal{S}$ *so that*

$$\frac{1+\Theta(z)\omega(z)}{1-\Theta(z)\omega(z)} = i\beta + \int_{\mathbb{T}} \frac{t+z}{t-z}\sigma(dt). \tag{0.1}$$

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For an alternative proof see [12, Section 11.7].

Relation (0.1) can be viewed as a counterpart of the Nevanlinna parametrization for an indeterminate Hamburger moment problem; see [2, Theorem 3.2.2] and Section 3 below.

Remark 0.1. The function ω in (0.1) is an independent parameter, which runs over the class $\mathcal{S}$. Both β and σ in (0.1) are uniquely determined by ω,

$$\beta = \frac{2\mathrm{Im}(\omega(0)\Theta(0))}{|1-\Theta(0)\omega(0)|^2}.$$

Conversely, if two triplets $\{\omega_j, \beta_j, \sigma\}$, $j = 1, 2$, satisfy (0.1), then $\omega_1 = \omega_2$ and $\beta_1 = \beta_2$. Indeed, we have

$$\frac{1+\Theta\omega_2}{1-\Theta\omega_2} - \frac{1+\Theta\omega_1}{1-\Theta\omega_1} = i(\beta_2 - \beta_1),$$
$$2\Theta(\omega_2-\omega_1) = i(\beta_2-\beta_1)(1-\Theta\omega_2)(1-\Theta\omega_1).$$

The function on the right-hand side is outer, whereas the function on the left-hand side has a nontrivial inner factor. So, $\omega_1 = \omega_2$ and $\beta_1 = \beta_2$, as claimed. For instance, $\sigma = m$ enters the only triplet $\{0, 0, m\}$.

Thereby, the equality (0.1) generates a bijection $\mathcal{F}$

$$\mathcal{F} : \mathcal{S} \to \mathcal{M}(\Theta), \quad \mathcal{F}(\omega) = \sigma; \tag{0.2}$$

$\mathcal{F}$ is a homeomorphism with respect to *-weak topology on $\mathcal{M}(\Theta)$ and the topology of uniform convergence on compact subsets of $\mathbb{D}$ on $\mathcal{S}$. It is clear from (0.1) that ω is an inner function if and only if $\sigma = \mathcal{F}(\omega)$ is a singular measure in $\mathcal{M}(\Theta)$.

The isometric embedding problem has a long history, and there are at least two predecessors of Aleksandrov. L. de Branges [9, Theorem 32] solved the problem for meromorphic inner functions on the upper half-plane. Later on D. Sarason [16] proved the result on the isometric embedding for inner functions Θ with $\Theta(0) = 0$ and for measures of the form $|f|^2m, f \in H^2$. Since such measures form a dense set in $\mathcal{M}(\Theta)$, the result of Aleksandrov can be deduced from that of Sarason (at least for $\Theta(0) = 0$). So it seems reasonable referring to the measures from $\mathcal{M}(\Theta)$ as the **Aleksandrov–Sarason measures**.

Relations (0.1) with unimodular constants $\omega = \alpha \in \mathbb{T}$,

$$\frac{1+\alpha\Theta(z)}{1-\alpha\Theta(z)} = i\beta_\alpha + \int_{\mathbb{T}} \frac{t+z}{t-z}\sigma_\alpha(dt), \tag{0.3}$$

are well known in the theory of the model spaces [10, Chapter 9], [12, Chapter 11]. The measures σ_α in (0.3) are the **Clark measures** following D. Clark [11].

Given $n = 0, 1, \ldots,$ a measure $\sigma \in \mathcal{M}(\Theta)$ will be called a **Clark measure of order** n if the corresponding parameter ω in (0.1) is a finite Blaschke product (FBP) of order n.

The case $\Theta(z) = z$ in (0.1) arises in Geronimus' approach to the theory of orthogonal polynomials on the unit circle [17, Chapter 3]. The model space is now the one-dimensional space of constant functions, and $\mathcal{M}(\Theta)$ is the set of all probability measures on $\mathbb{T}$.

The set $\mathcal{M}(\Theta)$ is easily seen to be a convex, compact in *-weak topology of the space $\mathcal{M}_+(\mathbb{T})$ set. The study of the set $\mathcal{M}_{ext}(\Theta)$ of extreme points for $\mathcal{M}(\Theta)$ seems quite natural. This is exactly the problem we address here. A point $\sigma \in \mathcal{M}(\Theta)$ is said to be an **extreme point of** $\mathcal{M}(\Theta)$ if

$$\sigma = \frac{\sigma_1 + \sigma_2}{2}, \quad \sigma_j \in \mathcal{M}(\Theta) \Rightarrow \sigma_1 = \sigma_2 = \sigma. \tag{0.4}$$

To paraphrase, there is no nontrivial representation of σ as a convex linear combination of two Aleksandrov–Sarason measures.

The problem of the description of extreme points in the equivalent setting of the de Branges measures was put forward for the first time by A. Baranov [5, Section 7], and several important results were obtained therein.

We say that a measure $\sigma \in \mathcal{M}_+(\mathbb{T})$ has finite support if

$$\sigma = \sum_{j=1}^{p} s_j \delta(t_j), \quad s_j > 0, \quad \operatorname{supp}\sigma = \{t_j\}_{j=1}^{p}, \quad t_j = t_j(\sigma),$$

and write $|\operatorname{supp}\sigma| = p$ for the cardinality of the support. Denote by $\mathcal{M}_f(\Theta)$ the set of all measures in $\mathcal{M}(\Theta)$ with finite support. It is clear from (0.1) that $\mathcal{M}_f(\Theta)$ is nonempty if and only if both Θ and ω are FBP's.

The model space K_Θ is finite dimensional, $\dim K_\Theta < \infty$, if and only if $\Theta = B$ is a FBP. Our main result concerns this situation.

Theorem 0.2. *Let B be a FBP of order $n \geq 1$. A measure $\sigma \in \mathcal{M}_{ext}(B)$ if and only if $\sigma \in \mathcal{M}_f(B)$ and*

$$n \leq |\operatorname{supp}\sigma| \leq 2n - 1. \tag{0.5}$$

When studying subclasses of $\mathcal{M}(\Theta)$ it is sometimes reasonable to go to the $\mathcal{F}$-preimages and consider the corresponding subclasses of the Schur class instead. That is what we are going to do when dealing with the class $\mathcal{M}_{ext}(\Theta)$.

Denote by $\mathcal{S}_{ext}(\Theta) \subset \mathcal{S}$ the set of extreme Schur functions

$$\mathcal{S}_{ext}(\Theta) = \{\omega \in \mathcal{S} : \mathcal{F}(\omega) \in \mathcal{M}_{ext}(\Theta)\}. \tag{0.6}$$

The above result has an equivalent form.

Theorem 0.3. *Let B be a FBP of order $n \geq 1$. The set $\mathcal{S}_{ext}(B)$ agrees with the set of all FBP's of the order at most $n-1$.*

The case of generic inner functions Θ is much more delicate. Given two inner functions u, v, we say, following Sarason [6], that v **lurks within** K_u, if there is a nontrivial function $g \in H^2$ so that $vg \in K_u$. In other words, v is an inner divisor of some function from K_u.

Theorem 0.4. *Let Θ be a nonconstant inner function so that Θ does not lurk within K_{zI} for an inner function I. Then $I \in \mathcal{S}_{ext}(\Theta)$.*

Corollary 0.5. *Let Θ be a nonconstant inner function, which is not a FBP. Then each FBP belongs to $\mathcal{S}_{ext}(\Theta)$. Equivalently, each Clark measure of order $n = 0, 1, \dots$ belongs to $\mathcal{M}_{ext}(\Theta)$.*

We say that an inner function φ is a divisor of Θ if Θ/φ is again an inner function.

Theorem 0.6. *Let Θ be an arbitrary, nonconstant inner function. Then each divisor of Θ, distinct from Θ, belongs to $\mathcal{S}_{ext}(\Theta)$, but $\Theta \notin \mathcal{S}_{ext}(\Theta)$.*

The case of unimodular constant divisors corresponds to the Clark measures.

Corollary 0.7. *Let Θ be an arbitrary, nonconstant inner function. Then the Clark measures $\sigma_\alpha \in \mathcal{M}_{ext}(\Theta)$ for all $\alpha \in \mathbb{T}$.*

We examine the class $\mathcal{M}_f(B)$ of measures with finite support in Section 1, and prove Theorem 0.2 in Section 2. In Section 3, given an inner function Θ, we introduce, following [5, Section 7.1], a binary operation $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ (Θ-product). As it turns out, a Schur function $\omega \notin \mathcal{S}_{ext}(\Theta)$ if and only if ω admits a nontrivial factorization with respect to the Θ-product. Thereby, the "Θ-prime" functions ω constitute the class $\mathcal{S}_{ext}(\Theta)$. The results of Theorems 0.4 and 0.6 are obtained along this line of reasoning, by using some uniqueness conditions for the classical Nehari problem.

So far we have met only inner functions ω in $\mathcal{S}_{ext}(\Theta)$. By Theorem 0.3, for FBP's B all functions in $\mathcal{S}_{ext}(B)$ are inner. We show (see Proposition 3.9) that in the opposite case, that is, $\dim K_\Theta = \infty$, there are non-inner functions $\omega \in \mathcal{S}_{ext}(\Theta)$. In other words, the set $\mathcal{M}_{ext}(\Theta)$ contains measures with a nontrivial absolutely continuous part.

1 Some properties of the class $\mathcal{M}_f(B)$

Given a FBP B of order n, we denote by

$$\{(z_1, r_1), (z_2, r_2), \dots, (z_d, r_d)\}, \quad z_i \neq z_j, \quad i \neq j, \quad r_j \in \mathbb{N},$$

the set of its zeros, so that

$$B(z) := \prod_{k=1}^{d} \left(\frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z}_k z} \right)^{r_k}, \quad \deg B = r_1 + \cdots + r_d = n.$$

The model space

$$(1.1) \qquad K_B := H^2 \ominus BH^2 = \left\{ h(z) = \frac{P(z)}{\prod_{j=1}^{d}(1 - \overline{z}_j z)^{r_j}}, \quad \deg P \le n-1 \right\}$$

is the finite-dimensional space of all rational functions with the poles at the points $1/\overline{z_j}$ of degree at most r_j, $\dim K_B = n$. The case $z_d = 0$, i.e., $B(0) = 0$, will be of particular concern. Now

$$K_B = \left\{ h(z) = \frac{P(z)}{\prod_{j=1}^{d-1}(1 - \overline{z}_j z)^{r_j}}, \quad \deg P \le n-1 \right\},$$

and the monomials $1, z, \dots, z^{r_d - 1} \in K_B$. Put

$$\varphi_0(z) = 1, \quad \varphi_k(z) := \frac{1}{1 - \overline{z}_k z}, \quad k = 1, 2, \dots, d-1, \quad \varphi_d(z) = z,$$

so the standard basis in K_B is

$$(1.2) \qquad \{\varphi_1, \varphi_1^2, \dots, \varphi_1^{r_1}; \dots; \varphi_{d-1}, \varphi_{d-1}^2, \dots, \varphi_{d-1}^{r_{d-1}}; \varphi_d, \dots, \varphi_d^{r_d - 1}; \varphi_0\}.$$

It seems reasonable rearranging these functions in a unique sequence $\{e_l\}_{l=1}^n$, $e_n = 1$.

The following result is a consequence of Theorem A, but we give a simple, direct proof.

Proposition 1.1. *The support of each measure $\sigma \in \mathcal{M}(B)$ contains at least n points.*

Proof. If $|\operatorname{supp} \sigma| \le n-1$, then $\dim L^2_\sigma(\mathbb{T}) \le n-1$, and the functions $\{e_l\}_{l=1}^n$ in (1.2) are linearly dependent in $L^2_\sigma(\mathbb{T})$, so

$$\det \| \langle e_j, e_k \rangle_\sigma \|_{j,k=1}^n = 0.$$

On the other hand, the same system is linearly independent in $L^2_m(\mathbb{T})$, so $\det \| \langle e_j, e_k \rangle_m \|_{j,k=1}^n \ne 0$. The contradiction completes the proof. □

As we mentioned in the Introduction, a measure $\sigma \in \mathcal{M}_f(B)$ if and only if $\omega = \mathcal{F}^{(-1)}(\sigma)$ is a FBP. Moreover, $|\operatorname{supp} \sigma| = n + \deg \omega$, so $|\operatorname{supp} \sigma| = n$ if and only if $\sigma = \sigma_\alpha$ is the Clark measure (0.3).

It is not hard to display $\sigma \in \mathcal{M}_f(B)$ explicitly in terms of the corresponding parameters ω and B. Indeed, (0.1) now takes the form

$$\frac{1+B(z)\omega(z)}{1-B(z)\omega(z)} = i\beta + \sum_{k=1}^{p} \frac{t_k+z}{t_k-z} s_k, \tag{1.3}$$

and

$$\operatorname{supp}\sigma = \{t_j\}_{j=1}^{p}: \quad B(t_j)\omega(t_j) = 1, \quad j = 1, 2, \dots, p. \tag{1.4}$$

The weights s_j can be determined from the limit relations

$$2t_q s_q = (1+B(t_q)\omega(t_q)) \lim_{z\to t_q} \frac{t_q - z}{1-B(z)\omega(z)} = \frac{2}{[B\omega]'(t_q)},$$

or, in view of (1.4),

$$\frac{1}{s_q} = t_q[B\omega]'(t_q) = t_q\frac{B'(t_q)}{B(t_q)} + t_q\frac{\omega'(t_q)}{\omega(t_q)}.$$

A computation of the logarithmic derivative of a FBP is standard,

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{d} r_k \frac{1-|z_k|^2}{(1-\bar{z}_k z)(z-z_k)}.$$

So,

$$\frac{1}{s_q} = \sum_{k=1}^{d} r_k \frac{1-|z_k|^2}{|t_q - z_k|^2} + \sum_{j=1}^{m} \frac{1-|w_j|^2}{|t_q - w_j|^2}, \tag{1.5}$$

where $w_1, \dots, w_m$ are all zeros (counting multiplicity) of ω in (1.3).

The relation (1.5) provides an answer to the following "extremal mass problem": given a point $\tau \in \mathbb{T}$, find a measure $\sigma_{\max} \in \mathcal{M}_f(B)$ so that

$$\sigma_{\max}\{\tau\} = \max\{\sigma\{\tau\} : \sigma \in \mathcal{M}_f(B)\}.$$

Indeed, such a measure is exactly the Clark measure $\sigma = \sigma_\alpha$ with $\alpha = B^{-1}(\tau)$, $|\operatorname{supp}\sigma_{\max}| = n$, and

$$\frac{1}{\sigma_{\max}\{\tau\}} = \sum_{k=1}^{d} r_k \frac{1-|z_k|^2}{|\tau - z_k|^2}.$$

Remark 1.2. As a matter of fact, the above Clark measure solves the same extremal problem within the whole class $\mathcal{M}(B)$. Relation (1.5) holds in the form

$$\frac{1}{s_q} = \sum_{k=1}^{d} r_k \frac{1-|z_k|^2}{|\tau - z_k|^2} + |\omega'(\tau)|,$$

where ω' is the angular derivative of ω (cf. [10, Section 9.2]).

Here is another simple property of measures $\sigma \in \mathcal{M}_f(B)$.

Proposition 1.3. *Let $\{t_j\}_{j=1}^p$ be an arbitrary set of distinct points on $\mathbb{T}$. There is a measure $\sigma \in \mathcal{M}_f(B)$ such that*

(1) $\{t_j\} \in \operatorname{supp}\sigma$;

(2) $|\operatorname{supp}\sigma| \le n+p-1$.

Proof. The proof is based on the interpolation with FBP's (see, e.g., [15, Theorem 1]): there is a FBP ω so that $\deg\omega \le p-1$ and

$$\omega(t_j) = B^{-1}(t_j), \quad j = 1, \dots, p.$$

The corresponding measure σ in (1.3) is the one we need. □

It turns out that the intersection of supports of two different measures from $\mathcal{M}_f(B)$ can not be too large. Put

$$\mathcal{M}_{n+k}(B) := \{\sigma \in \mathcal{M}_f(B) : |\operatorname{supp}\sigma| = n+k\}, \quad k = 0, 1, \dots.$$

Lemma 1.4. *Let $\sigma_j \in \mathcal{M}_{n+p_j}(B)$, $j = 1, 2$, and let*

$$|\operatorname{supp}\sigma_1 \cap \operatorname{supp}\sigma_2| \ge p_1 + p_2 + 1.$$

Then $\sigma_1 = \sigma_2$.

Proof. Let ω_j be the FBP's for σ_j in (1.3), $\deg\omega_j = p_j$, $j = 1, 2$. Let $\zeta_1, \dots, \zeta_{p_1+p_2+1} \in \operatorname{supp}\sigma_1 \cap \operatorname{supp}\sigma_2$, so, by (1.4),

$$\omega_1(\zeta_l) = \omega_2(\zeta_l), \quad l = 1, 2, \dots, p_1 + p_2 + 1.$$

Note that

$$\omega_j(z) = \gamma_j \frac{Q_j(z)}{Q_j^*(z)}, \quad j = 1, 2,$$

where γ_j are unimodular constants, Q_j are algebraic polynomials, Q_j^* are the reversed polynomials, and

$$\deg Q_j = p_j, \quad \deg Q_j^* \le p_j, \quad j = 1, 2.$$

We see that for the polynomial

$$Q(z) = \gamma_1 Q_1(z) Q_2^*(z) - \gamma_2 Q_2(z) Q_1^*(z), \quad \deg Q \le p_1 + p_2,$$

the relations

$$Q(\zeta_l) = 0, \quad l = 1, 2, \dots, p_1 + p_2 + 1$$

hold, so $Q = 0$, $\omega_1 = \omega_2$, and $\sigma_1 = \sigma_2$ (see Remark 0.1). □

Corollary 1.5. *If $\sigma_j \in \mathcal{M}_n(B)$, $j = 1, 2$, and $\operatorname{supp}\sigma_1 \cap \operatorname{supp}\sigma_2 \ne \emptyset$, then $\sigma_1 = \sigma_2$. If $\sigma_j \in \mathcal{M}_{n+k}(B)$, $k = 0, 1, \dots, n-1$, and $\operatorname{supp}\sigma_1 = \operatorname{supp}\sigma_2$, then $\sigma_1 = \sigma_2$.*

2 Extreme points of $\mathcal{M}(\Theta)$ for finite dimensional model spaces

We begin with the result which provides an upper bound in (0.5). It can be viewed as a counterpart of [2, Theorem 2.3.4] for the classical moment problem.

Proposition 2.1. *Let* $\sigma \in \mathcal{M}_{ext}(B)$. *Then* $\sigma \in \mathcal{M}_f(B)$ *and* $|\operatorname{supp}\sigma| \le 2n-1$.

Proof. Assume first that $z_d = 0$. Define a system of real valued, linearly independent functions on $\mathbb{T}$, accompanying (1.2)

$$\begin{aligned} &x_{k,j}(t) := \operatorname{Re}\varphi_k^j(t), \quad y_{k,j}(t) := \operatorname{Im}\varphi_k^j(t), \quad j = 1,\dots,r_k, \qquad k = 1,\dots,d-1,\\ &x_{d,j}(t) := \operatorname{Re} t^j, \qquad y_{d,j}(t) := \operatorname{Im} t^j, \qquad j = 1,\dots,r_d-1, \quad x_{d,0} = 1. \end{aligned}$$

We arrange them in a sequence $\{v_l\}_{l=1}^{2n-1}$, and denote by E their complex, linear span

$$E := \operatorname{span}_{1\le l\le 2n-1}\{v_l\}, \qquad \dim E = 2n-1.$$

Clearly, $t^l \in E$ for $|l| \le r_d - 1$, and

$$\varphi_k^j = x_{k,j} + iy_{k,j} \in E, \qquad \overline{\varphi_k^j} = x_{k,j} - iy_{k,j} \in E$$

(or $e_m, \overline{e_m} \in E$) for the appropriate values of k, j, m. It is a matter of a direct computation to make sure that the product $e_m\overline{e_l} \in E$, $m, l = 1,\dots,n$. For instance,

$$\begin{aligned} \varphi_p(t)\overline{\varphi_q(t)} &= \frac{1}{(1-\bar z_p t)(1-z_q\bar t)} = \frac{\varphi_p(t)+\overline{\varphi_q(t)}-1}{1-\bar z_p z_q},\\ \varphi_p^2(t)\overline{\varphi_q(t)} &= \frac{1}{(1-\bar z_p t)^2(1-z_q\bar t)} = \frac{\varphi_p^2(t)+\varphi_p(t)\overline{\varphi_q(t)}-\varphi_p(t)}{1-\bar z_p z_q}, \end{aligned}$$

etc. The rest is a simple induction. We conclude, thereby, that $f\overline{g} \in E$ for each $f, g \in K_B$.

Assume next that $|\operatorname{supp}\sigma| \ge 2n$. Then the inclusion $E \subset L^1_\sigma(\mathbb{T})$ is proper, so there is a nontrivial, linear functional Φ_0 on $L^1_\sigma(\mathbb{T})$, $\|\Phi_0\| \le 1$, vanishing on E. Equivalently, there is a function $\varphi_0 \in L^\infty_\sigma(\mathbb{T})$ such that $|\varphi_0| \le 1$ $[\sigma]$-almost everywhere, and

$$\int_{\mathbb{T}} x_{k,j}(t)\varphi_0(t)\sigma(dt) = \int_{\mathbb{T}} y_{k,j}(t)\varphi_0(t)\sigma(dt) = 0$$

for all appropriate values of j, k. Since the functions $x_{j,k}$, $y_{j,k}$ are real valued, the function φ_0 can be taken real valued as well.

Consider now two measures $\sigma_\pm(dt) := (1 \pm \varphi_0)\sigma(dt)$, $\sigma_\pm \in \mathcal{M}_+(\mathbb{T})$. By the construction, $\sigma_\pm \in \mathcal{M}(B)$, and the representation $2\sigma = \sigma_+ + \sigma_-$ is nontrivial. Hence, σ is not an extreme point of $\mathcal{M}(B)$, as claimed.

It remains to examine the general case when $B(0) \neq 0$. The standard trick with the change of variables (see, e.g., [17, pp. 140–141]) reduces this case to the one discussed above. Given $a \in \mathbb{D}$, put

$$b_a(z) := \frac{z+a}{1+\bar{a}z}, \quad B_a(z) := B(b_a(z)), \quad \omega_a(z) := \omega(b_a(z)).$$

If we replace z with $b_a(z)$ in (0.1), we have

$$\frac{1+B_a(z)\omega_a(z)}{1-B_a(z)\omega_a(z)} = i\beta + \int_{\mathbb{T}} \frac{t+b_a(z)}{t-b_a(z)}\sigma(dt),$$

and since

$$\frac{t+b_a(z)}{t-b_a(z)} = i\beta_{a,t} + \frac{1-|a|^2}{|t-a|^2}\frac{b_a(t)+z}{b_a(t)-z},$$

we come to

$$\frac{1+B_a(z)\omega_a(z)}{1-B_a(z)\omega_a(z)} = i\beta_a + \int_{\mathbb{T}} \frac{1-|a|^2}{|t-a|^2}\frac{b_a(t)+z}{b_a(t)-z}\sigma(dt) = i\beta_a + \int_{\mathbb{T}} \frac{\tau+z}{\tau-z}\sigma^{(a)}(d\tau).$$

It is clear that the map $\sigma \to \sigma^{(a)}$ is a bijection of $\mathcal{M}(B)$ onto $\mathcal{M}(B_a)$, which is also the bijection between $\mathcal{M}_{ext}(B)$ and $\mathcal{M}_{ext}(B_a)$. Obviously, it is a bijection between $\mathcal{M}_f(B)$ and $\mathcal{M}_f(B_a)$, and in this case $|\operatorname{supp}\sigma| = |\operatorname{supp}\sigma^{(a)}|$. But $B_a(0) = 0$ with $a = z_d$, so the above argument applies. The proof is complete. □

Proof of Theorem 0.2. It remains to show that each measure $\sigma \in \mathcal{M}_{n+k}(B)$, $k = 0, 1, \dots, n-1$ is the extreme point of $\mathcal{M}(B)$. Indeed, let $2\sigma = \sigma_1 + \sigma_2$, then

$$\sigma_j \in \mathcal{M}_{n+p_j}, \quad j = 1, 2, \quad 0 \le p_1, p_2 \le k.$$

Since $\operatorname{supp}\sigma = \operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$, we have

$$|\operatorname{supp}\sigma| = |\operatorname{supp}\sigma_1| + |\operatorname{supp}\sigma_2| - |\operatorname{supp}\sigma_1 \cap \operatorname{supp}\sigma_2|,$$

or

$$|\operatorname{supp}\sigma_1 \cap \operatorname{supp}\sigma_2| = n + p_1 + n + p_2 - n - k = n + p_1 + p_2 - k \ge p_1 + p_2 + 1.$$

By Lemma 1.4, $\sigma_1 = \sigma_2$, so σ is the extreme point of $\mathcal{M}(B)$, as claimed. □

3 Extreme points of $\mathcal{M}(\Theta)$ for generic inner functions

The case of generic inner functions is much more delicate.

Let us define a binary operation in the Schur class, which corresponds to taking a half-sum of measures under transformation $\mathcal{F}$. Precisely, given two Schur functions s_1, s_2, we denote by $s := (s_1 \circ s_2)_\Theta$ the operation so that

$$\mathcal{F}((s_1 \circ s_2)_\Theta) = \frac{\mathcal{F}(s_1) + \mathcal{F}(s_2)}{2}. \tag{3.1}$$

It is a matter of elementary computation based on the relation

$$\frac{1+\Theta(z)s(z)}{1-\Theta(z)s(z)} = \frac{1}{2}\Big(\frac{1+\Theta(z)s_1(z)}{1-\Theta(z)s_1(z)} + \frac{1+\Theta(z)s_2(z)}{1-\Theta(z)s_2(z)}\Big), \qquad s=(s_1\circ s_2)_\Theta,$$

to check that

$$(s_1\circ s_2)_\Theta := \frac{s_0-\Theta s_1 s_2}{1-\Theta s_0}, \qquad s_0 := \frac{s_1+s_2}{2}; \tag{3.2}$$

$(s_1\circ s_2)_\Theta$ will be called a **Θ-product** of s_1 and s_2.

The operation (3.2) was introduced in [5, formula (26)] for the model spaces K_Θ in the upper half-plane with $\Theta = E^*/E$, where E is an entire function of de Branges. In fact, the argument in [5, Section 7.1] works for arbitrary model spaces.

We list the main properties of the Θ-product in the statement below.

Proposition 3.1. *Let* Θ *be a nonconstant, inner function.*

(i) $\circ$ *is a binary operation on the Schur class, which is idempotent, that is,* $(s\circ s)_\Theta = s$.

(ii) $s=(s_1\circ s_2)_\Theta$ *is an inner function if and only if so are both* s_1 *and* s_2.

(iii) *The following equality holds:*

$$s = s_0 + \Theta h, \qquad s_0 = \frac{s_1+s_2}{2}, \qquad h\in H^\infty. \tag{3.3}$$

Proof. (i) Since $1-\Theta s_0$ is an outer function [13, Corollary II.4.8], $(s_1\circ s_2)_\Theta$ belongs to the Smirnov class, so one has to verify that

$$|(s_1\circ s_2)_\Theta(t)| \le 1$$

for a.e. $t\in\mathbb{T}$. Indeed,

$$|1-\Theta s_0|^2 = 1+|s_0|^2 - \mathrm{Re}(\Theta s_1+\Theta s_2),$$
$$|s_0-\Theta s_1 s_2|^2 = |s_0|^2 + |s_1 s_2|^2 - |s_1|^2\mathrm{Re}(\Theta s_2) - |s_2|^2\mathrm{Re}(\Theta s_1),$$

so

$$\begin{aligned} &|1-\Theta s_0|^2 - |s_0-\Theta s_1 s_2|^2 \\ &\quad = 1-|s_1s_2|^2 - \mathrm{Re}(\Theta s_1)(1-|s_2|^2) - \mathrm{Re}(\Theta s_2)(1-|s_1|^2) \\ &\quad = 1-|s_1s_2|^2 - |s_1|(1-|s_2|^2) - |s_2|(1-|s_1|^2) \\ &\qquad + (|s_1|-\mathrm{Re}(\Theta s_1))(1-|s_2|^2) + (|s_2|-\mathrm{Re}(\Theta s_2))(1-|s_1|^2) \\ &\quad = (1-|s_1s_2|)(1-|s_1|)(1-|s_2|) + (|s_1|-\mathrm{Re}(\Theta s_1))(1-|s_2|^2) \\ &\qquad + (|s_2|-\mathrm{Re}(\Theta s_2))(1-|s_1|^2) \ge 0, \end{aligned}$$

as needed.

By definition (3.2), $(s\circ s)_\Theta = s$ for each $s\in\mathcal{S}$, so the operation is idempotent.

(ii) If both s_1 and s_2 are inner functions, then, by the above calculation, so is $(s_1 \circ s_2)_\Theta$. Conversely, assume that $(s_1 \circ s_2)_\Theta$ is an inner function, but $|s_1| < 1$ a.e. on a set $E \subset \mathbb{T}$ of positive measure. It follows from the above calculation that

$$|s_2| = 1, \quad |s_2| - \mathrm{Re}(\Theta s_2) = 0$$

a.e. on E. Hence $\Theta s_2 = 1$ a.e. on the set of positive measure, so Θ is a unimodular constant.

(iii) It follows directly from the definition that

$$s_0 = s + \Theta \frac{s_1 s_2 - s_0^2}{1 - \Theta s_0} = s + \Theta h,$$
$$h = \frac{s_1 s_2 - s_0^2}{1 - \Theta s_0} = -\frac{1}{4} \frac{(s_1 - s_2)^2}{1 - \Theta s_0}.$$

Since $1 - \Theta s_0$ is an outer function, h belongs to the Smirnov class, and moreover,

$$|h(t)| = |s_0(t) - s(t)| \le 2$$

a.e. on $\mathbb{T}$, so $h \in H^\infty$, as claimed. □

Remark 3.2. The Θ-product is a nontrivial operation already for $\Theta = 1$. It is clear from the definition that for $s_2 = \Theta = 1$ one has $(s_1 \circ s_2)_\Theta = 1$ for any $s_1 \in \mathcal{S}$.

As a matter of fact, one can define a whole one-parameter family of binary operations in the Schur class, which corresponds to an arbitrary convex linear combination of the two measures, not just their half-sum; see [5, formula (25)],

$$(s_1 \circ s_2)_\Theta^t := \frac{ts_1 + (1-t)s_2 - \Theta s_1 s_2}{1 - \Theta(ts_2 + (1-t)s_1)}, \quad 0 \le t \le 1. \tag{3.4}$$

So, (3.2) is a particular case of (3.4) with $t = 1/2$. We have

$$\mathcal{F}((s_1 \circ s_2)_\Theta^t) = t\mathcal{F}(s_1) + (1-t)\mathcal{F}(s_2).$$

Definition 3.3. A function $s \in \mathcal{S}$ is called **Θ-prime** if

$$s = (s_1 \circ s_2)_\Theta \quad \Rightarrow \quad s = s_1 = s_2.$$

It is clear from (3.1) that $\sigma = \mathcal{F}(s) \in \mathcal{M}_{ext}(\Theta)$ if and only if s is Θ-prime. Equivalently, $s \in \mathcal{S}_{ext}(\Theta)$ if and only if s is Θ-prime. Moreover, if at least for one $t \in (0,1)$ a function s admits a nontrivial factorization $s = (s_1 \circ s_2)_\Theta^t$, then $s \notin \mathcal{S}_{ext}(\Theta)$.

The Lebesgue measure m is the only one which knowingly belongs to $\mathcal{M}(\Theta)$ for any inner function Θ. We show that it is never in $\mathcal{M}_{ext}(\Theta)$. The following result is borrowed from [5, Corollary 7.2].

Proposition 3.4. *If $s \in \mathcal{S}_{ext}(\Theta)$, then $\|s\|_\infty = 1$.*

Proof. Assume, on the contrary, that $\|s\|_\infty = 1 - \delta$, $\delta > 0$. Put $t = \delta/2$,

$$s_1 = 0, \quad s_2 = \frac{s}{1 - t(1 - \Theta s)}.$$

We see that

$$|s_2| \le \frac{|s|}{1 - 2t} \le \frac{1 - \delta}{1 - \delta} = 1,$$

so $s_2 \in \mathcal{S}$. By (3.4), $s = (s_1 \circ s_2)^t_\Theta$, and the factorization is nontrivial as long as $s \not\equiv 0$. For $s \equiv 0$ a possible nontrivial factorization can be taken as

$$0 = (s_1 \circ s_2)_\Theta, \quad s_1 = \frac{1}{3}, \quad s_2 = \frac{1}{2\Theta - 3}.$$

Hence, $s \notin \mathcal{S}_{ext}(\Theta)$. The contradiction completes the proof. □

Example 3.5. It is easy to see that the Clark measures $\sigma_\alpha = \mathcal{F}(\alpha)$ are the extreme measures for each $\alpha \in \mathbb{T}$ and any inner function Θ. Indeed, assume that

$$\alpha = \frac{\omega_0 - \Theta\omega_1\omega_2}{1 - \Theta\omega_0}, \quad \omega_0 = \frac{\alpha + \Theta\omega_1\omega_2}{1 + \alpha\Theta},$$

and so

$$(\alpha - \omega_0)(1 + \alpha\Theta) = \Theta(\alpha^2 - \omega_1\omega_2).$$

But both functions $\alpha - \omega_0 = \alpha(1 - \bar{\alpha}\omega_0)$ and $1 + \alpha\Theta$ are outer, and so is their product, whereas the left-hand side has a nontrivial inner factor. Hence

$$\alpha^2 = \omega_1\omega_2, \quad \omega_0 = \alpha,$$

which implies $\omega_1 = \omega_2 = \omega = \alpha$ is Θ-prime, as claimed.

To determine whether a function $\omega \in \mathcal{S}$ is Θ-prime, we apply uniqueness conditions in the celebrated Nehari problem, with the notion of a minifunction playing a key role.

Given a function $g \in \mathcal{B}$, the unit ball of $L^\infty(\mathbb{T})$, the Nehari problem concerns a set

$$N(g) := \{g + H^\infty\} \cap \mathcal{B},$$

which is nonempty, since $g \in N(g)$. Following Adamjan–Arov–Krein [1], we call g an **undeformable minifunction** if $N(g) = \{g\}$. Clearly, $\|g\|_\infty = 1$ for each such g.

Lemma 3.6. *Given a nonconstant inner function Θ and $s \in \mathcal{S}$, let $s\bar{\Theta}$ be the undeformable minifunction. Then s is Θ-prime.*

Proof. Let $s=(s_1\circ s_2)_\Theta$, by Proposition 3.1, (iii),

$$s\bar\Theta = s_0\bar\Theta + h, \quad h\in H^\infty.$$

But $s\bar\Theta$ is an undeformable minifunction, so $s=s_0$ a.e. Next,

$$s_0=\frac{s_0-\Theta s_1s_2}{1-\Theta s_0}, \quad s_0^2=\left(\frac{s_1+s_2}{2}\right)^2=s_1s_2, \tag{3.5}$$

and so $s_1=s_2$ a.e., and s is Θ-prime, as needed. □

Proof of Theorem 0.4. In view of Lemma 3.6, it suffices to show that $I\bar\Theta$ is the undeformable minifunction. According to the result in [1, Remark 3.2], a unimodular function f, i.e., $|f|=1$ a.e. on $\mathbb{T}$, is not an undeformable minifunction if and only if it admits the following representation

$$f(t)=\frac{\xi_+(t)}{t\xi_-(t)}, \quad \xi_\pm\in H^2_\pm, \tag{3.6}$$

a.e. on $\mathbb{T}$.

Assume that $I\bar\Theta$ does not belong to the set of undeformable minifunctions. Then, by (3.6) with $f=I\bar\Theta$, one has $tI\xi_-=\Theta\xi_+$, so $\Theta\xi_+\in K_{tI}$, and Θ is lurking within K_{tI}. The contradiction completes the proof. □

The result in Corollary 0.5 is immediate from the above theorem, as for a FBP b and an inner function Θ, which is not a FBP, Θ does not lurk within K_{zb}. Indeed, the model space K_{zb} consists of rational functions, see (1.1), and Θg, $g\in H^2$, is not such.

A special case of Corollary 0.5 (for an atomic singular function Θ) was proved in [5, Theorem 7.1].

It might be worth changing the perspective by looking for Θ such that a given $s\in\mathcal{S}_{ext}(\Theta)$. For instance, it is clear from Theorem 0.3 and Corollary 0.5 that given a FBP $s=b$, all such Θ are the inner functions of the order exceeding the order of b (ord$\Theta=+\infty$ off the class of FBP's), cf. [6, Theorem 2.1].

Remark 3.7. For particular inner functions Θ the result can be obtained by simpler means. For instance, let Θ have infinitely many zeros $\{z_j\}_{j\ge1}$. Let a FBP $b=(b_1\circ b_2)_\Theta$. Then, by (3.3), we have $b=b_0+\Theta h$ and so

$$b(z_j)=b_0(z_j), \quad j=1,2,\dots.$$

The uniqueness condition in the Nevanlinna–Pick interpolation problem (b is the FBP), see [13, Theorem I.2.2 and Corollary I.2.3], implies

$$b=b_0=\frac{b_1+b_2}{2},$$

which leads to $b_1=b_2=b$, as needed.

The same argument applies in the case when the underlying singular measure for Θ contains an atom, with the boundary Nevanlinna–Pick problem instead; see [8, Theorem 1.3]. So, the extremal property of the Schur functions $s \in \mathcal{S}_{ext}(\Theta)$ is tightly related to uniqueness in some classical interpolation problems.

There is a striking similarity between the Aleksandrov formula (0.1) and the Nevanlinna parametrization for an indeterminate Hamburger moment problem; see [2, Theorem 3.2.2]. Recall that the Hamburger moment problem deals with measures μ on the real line having prescribed moments of all orders. It is called indeterminate if the set of such measures is infinite. It turns out that all measures μ arise from the Nevanlinna formula

$$-\frac{A(z)u(z) - C(z)}{B(z)u(z) - D(z)} = \int_{\mathbb{R}} \frac{\mu(dx)}{x - z}, \quad z \in \mathbb{C}_+, \tag{3.7}$$

when the independent parameter u runs over the class of analytic on the upper half-plane $\mathbb{C}_+$ functions with the nonnegative imaginary part ($u = \infty$ is included). A measure μ is called a **canonical solution of order** $n = 0, 1, \ldots$ if the parameter u in (3.7) is a real rational function of order n.

The set of all solutions is clearly convex, so the extreme points show up. The result of Glazman–Naiman [14], see [2, Corollary 3.4.3], states that each canonical solution of order n is the extreme measure for the indeterminate Hamburger moment problem. Corollary 0.5 is a direct counterpart of the latter result for the isometric embedding problem. Furthermore, if Θ is not a FBP, the set $\mathcal{M}_{ext}(\Theta)$ is dense in $\mathcal{M}(\Theta)$, since, by the Carathéodory theorem, the FBP's are dense in $\mathcal{S}$, and $\mathcal{F}$ is the homeomorphism. This is a counterpart of [7, Theorem 1].

Proof of Theorem 0.6. To show that each divisor φ of Θ, $\varphi \not\equiv \Theta$, is Θ-prime, we again apply Lemma 3.6 with $g = \varphi\overline{\Theta} = \overline{\psi}$, ψ is a nonconstant inner function. Indeed, assume, on the contrary, that $\overline{\psi}$ is not an undeformable minifunction. Then there is a nonzero function $f \in H^\infty$ such that

$$\|\overline{\psi} - f\|_\infty = \|1 - \psi f\|_\infty \le 1.$$

Hence, $\mathrm{Re}\,\psi f \ge 0$ on the disk $\mathbb{D}$, and, by [13, Corollary II.4.8], the function ψf is outer. But it has a nontrivial inner factor ψ, so the contradiction justifies the claim.

For any nonzero $f \in \mathcal{S}$ the function $\Theta f^2 = (f \circ (-f))_\Theta \notin \mathcal{S}_{ext}(\Theta)$, so in particular, $\Theta \notin \mathcal{S}_{ext}(\Theta)$. The proof is complete. □

Remark 3.8. I thank the referee who pointed out a simple proof of this result which has nothing to do with AAK minifunctions. Indeed, let $\Theta = \varphi\psi$ as above, and assume, on the contrary, that there is a nontrivial factorization $\varphi = (\varphi_1 \circ \varphi_2)_\Theta$. Then

$$\varphi(1 - \Theta\varphi_0) = \varphi_0 - \Theta\varphi_1\varphi_2, \quad \varphi_0 = \frac{\varphi_1 + \varphi_2}{2},$$

so φ_0 is divisible by φ in H^∞: $\varphi_0 = \varphi h$, $h \in \mathcal{S}$. If $h \equiv 1$, then φ is Θ-prime, see (3.5). Otherwise,

$$\varphi - \varphi_0 = \Theta(\varphi\varphi_0 - \varphi_1\varphi_2), \quad 1 - h = \psi(\varphi\varphi_0 - \varphi_1\varphi_2),$$

and the nontrivial outer function on the left-hand side has a nontrivial inner factor ψ. The contradiction completes the proof.

Finally, we show that there are non-inner Schur functions $w \in \mathcal{S}_{ext}(\Theta)$, as long as Θ is not a FBP.

Proposition 3.9. *Assume that Θ is not a FBP. Then there are outer Schur functions $w \in \mathcal{S}_{ext}(\Theta)$.*

Proof. There is a point $\zeta \in \mathbb{T}$ so that Θ does not admit an analytic continuation across any open arc Γ centered at ζ. Fix such an arc, and take an outer Schur function w such that $w^{-1} \in H^\infty$ and $|w| = 1$ a.e. on Γ. Following the line of reasoning from [13, Example IV.4.2], we show first that $w\overline{\Theta}$ is an undeformable minifunction. Indeed, assume that there is a nonzero function $g \in H^\infty$ so that

$$\|w\overline{\Theta} - g\|_\infty = \|w - \Theta g\|_\infty \le 1,$$

and hence

$$|w - \Theta g| = |w||1 - \Theta g/w| = |1 - \Theta g/w| \le 1$$

a.e. on Γ. So $\mathrm{Re}\Theta g/w \ge 0$ a.e. on Γ, and, by [13, Exercise II.14 (a)], the inner factor of the function $\Theta g/w \in H^\infty$ admits an analytic continuation across Γ, that contradicts our assumption.

By Lemma 3.6, w is Θ-prime, as claimed. □

Note that the above function w can be taken infinitely smooth in the closed unit disk.

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