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On the Critical Capacity of the Hopfield Model

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Abstract: We estimate the critical capacity of the zero-temperature Hopfield model by using a novel and rigorous method. The probability of having a stable fixed point is one when $\alpha \leq 0.113$ for a large number of neurons. This result is an advance on all rigorous results in the literature and the relationship between the capacity α and retrieval errors obtained here for small α coincides with replica calculation results.

1. Introduction and Main Results

The Hopfield model is one of the most important models in the theory of spin glasses and neural networks [H], [M-P-V]. It has been intensively investigated in the past few years (see e.g. book [M-P-V] and references therein). One of the main problems is the critical capacity which has been studied by means of the replica trick [A], [A-G-S]. Here the value $\alpha_c = 0.138...$ (coinciding also with numerical experiments) was found. But this result is nonrigorous from the mathematical point of view. There are few rigorous approaches in the literature to estimate the critical capacity of the Hopfield model [N], [L], [T]. Here we introduce a novel approach based upon analysis of the Fourier transform of the joint distribution of the effective fields. It enables us to obtain a new bound for the critical capacity and also allows us to prove rigorously, for small α, the results obtained in terms of the extreme value theory [F-T].

Consider the sequential dynamics of the Hopfield model in the form

$$\sigma_k(t+1) = \text{sign}\{ \sum_{j=1, j\neq k}^{N} \tilde{J}_{kj}\sigma_j(t)\}, \tag{1.1}$$

where, as usually,

$$\tilde{J}_{jk} = \frac{1}{N}\sum_{\mu=1}^{p+1}\tilde{\xi}_j^\mu\tilde{\xi}_k^\mu, \quad \frac{p}{N}\to\alpha, \quad as \quad N\to\infty, \tag{1.2}$$

and $\tilde{\xi}_k^\mu$ $(j = 1,\dots,N)$, $(\mu = 1,\dots,p+1)$ are i.i.d. random variables assuming values ± 1 with probability $\frac{1}{2}$. This dynamical system is determined by the energy function

$$\mathcal{H}(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_{j\neq k}^{N}\tilde{J}_{jk}\sigma_j\sigma_k, \tag{1.3}$$

where we denote $\boldsymbol{\sigma}\equiv(\sigma_1,\dots,\sigma_N)$. It is easily seen that the function $\mathcal{H}(\boldsymbol{\sigma})$ does not increase in the process of evolution. Thus, the dynamics of the model depends on the "energy landscape" of the function $\mathcal{H}(\boldsymbol{\sigma})$ and the local minima of the function are the fixed points of dynamics (1.1). Newman [N] was the first, who proved, that for $\alpha\leq 0.056.$, an "energy barrier" exists with probability 1 around every point $\boldsymbol{\sigma}^\mu = \boldsymbol{\xi}^\mu \equiv (\tilde{\xi}_1^\mu,\dots,\tilde{\xi}_N^\mu)$, i.e. there exist some positive numbers δ and ε, such that for any $\boldsymbol{\sigma}$, belonging to

$$\Omega_\delta^\mu \equiv \{\boldsymbol{\sigma} : ||\boldsymbol{\sigma}-\boldsymbol{\xi}^\mu||^2 = 2[\delta N]\},$$

the following inequality holds

$$\mathcal{H}(\boldsymbol{\sigma}) - \mathcal{H}(\boldsymbol{\xi}^\mu) \geq \varepsilon N$$

(here and below the norm $||...||$ corresponds to the usual scalar product $(...,...)$ in $\mathbf{R}^N$). In other words, it means that

$$\min_{\boldsymbol{\sigma}\in\Omega_\delta^\mu}\mathcal{H}(\boldsymbol{\sigma}) - \mathcal{H}(\boldsymbol{\xi}^\mu) \geq \varepsilon^2 N. \tag{1.4}$$

This result was improved by Loukianova [L], who proved the existence of the "energy barriers" for $\alpha\leq 0.071$ and then by Talagrand [T]. One can show, that if such a "barrier" exists, then inside each open ball

$$B_\delta^\mu \equiv \{\boldsymbol{\sigma} : ||\boldsymbol{\sigma}-\boldsymbol{\xi}^\mu||^2 < 2[\delta N]\}$$

there exists a point of local minimum of the function $\mathcal{H}(\boldsymbol{\sigma})$, which, as it was mentioned above, is the fixed point of dynamics (1.1).

Thus, it is clear that the point $\boldsymbol{\sigma}^*$ in which $\mathcal{H}(\boldsymbol{\sigma}^*) = \min_{\boldsymbol{\sigma}\in\Omega_\delta^\mu}\mathcal{H}(\boldsymbol{\sigma})$ plays an important role in dynamics (1.1). We shall study the probability of the event, that the point $\boldsymbol{\sigma}^{(1,\delta)}\in\Omega_\delta^1$ with

$$\sigma_k^{(1,\delta)} = -\tilde{\xi}_k^1,\ (k=1,\dots,[\delta N]), \quad \sigma_k^{(1,\delta)} = \tilde{\xi}_k^1,\ (k=1+[\delta N],\dots,N) \tag{1.5}$$

is a local minimum of the function $\mathcal{H}(\boldsymbol{\sigma})$ on Ω_δ^1. This means that $\mathcal{H}(\boldsymbol{\sigma}^{(1,\delta)})$ must be less then the value of $\mathcal{H}(\boldsymbol{\sigma})$ for any $\boldsymbol{\sigma}\in\Omega_\delta^1$ which is the "nearest neighbor" of $\boldsymbol{\sigma}^{(1,\delta)}$ in Ω_δ^1. It is easy to see that, it is so if and only if for any $k=1,\dots,[\delta N]$ and $j=[\delta N]+1,\dots,N$

$$-2\tilde{J}_{kj}\sigma_j^{(1,\delta)}\sigma_k^{(1,\delta)} + \sigma_k^{(1,\delta)}\sum_{i=1,i\neq k}^{N}\tilde{J}_{ki}\sigma_i^{(1,\delta)} + \sigma_j^{(1,\delta)}\sum_{i=1,i\neq j}^{N}\tilde{J}_{ji}\sigma_i^{(1,\delta)} \geq 0. \tag{1.6}$$

It is useful to introduce at this point the definition of "effective fields".

Definition 1. *The effective fields generated by the configuration* $\boldsymbol{\sigma}$ *on the neuron* k *is*

$$z_k \equiv \sigma_k \sum_{i=1, i\neq k}^{N} \tilde{J}_{ki}\sigma_i$$

Our approach is based on the analysis of the joint probability distribution of the variables z_k $(k = 1, \ldots, N)$.

Since with probability larger than $1 - e^{-N \,\mathrm{const}\, \tilde{\varepsilon}^2}$ all matrix elements $\tilde{J}_{kj}$ satisfy the inequality

$$|\tilde{J}_{kj}| \leq \frac{\tilde{\varepsilon}}{2} \qquad (k, j = 1, ..., N), \tag{1.7}$$

one can derive from (1.6) that, if we denote by $\tilde{x}_k^0$ the effective fields, generated by the configuration $\boldsymbol{\sigma}^{(1,\delta)}$

$$\tilde{x}_k^0 = \sigma_k^{(1,\delta)} \sum_{i=1, i\neq k}^{N} \tilde{J}_{ki}\sigma_i^{(1,\delta)}, \tag{1.8}$$

the necessary condition for $\boldsymbol{\sigma}^{(1,\delta)}$ to be a local minimum point is

$$\min_{k=1,\ldots,[\delta N]} \tilde{x}_k^0 + \min_{j=[\delta N]+1,\ldots,N} \tilde{x}_j^0 \geq -\tilde{\varepsilon}, \tag{1.9}$$

and the sufficient condition has the same form with $+\tilde{\varepsilon}$ in the r.h.s. Thus, if we consider the events

$$\mathcal{A}_k^0(q) = \{\tilde{x}_k^0 \geq q\}, \tag{1.10}$$

then the event $\mathcal{M}$ that $\boldsymbol{\sigma}^{(1,\delta)}$ is a local minimum point satisfies the relations

$$\begin{array}{l} \cup_{q+q'\geq\tilde{\varepsilon}}(\cap_{k=1}^{[\delta N]}\mathcal{A}_k^0(q) \cap_{k=[\delta N]+1}^{N} \mathcal{A}_k^0(q') \subset \mathcal{M} \\ \subset \cup_{q+q'\geq -\tilde{\varepsilon}}(\cap_{k=1}^{[\delta N]}\mathcal{A}_k^0(q) \cap_{k=[\delta N]+1}^{N} \mathcal{A}_k^0(q')). \end{array} \tag{1.11}$$

So we should study the behaviour of

$$P_N(q, q') \equiv \mathrm{Prob}\{\cap_{k=1}^{[\delta N]}\mathcal{A}_k^0(q) \cap_{k=[\delta N]+1}^{N} \mathcal{A}_k^0(q')\}. \tag{1.12}$$

Observe that, in particular, $P_N(0, 0)$ is the probability to have a fixed point of dynamics (1.1) at the point $\boldsymbol{\sigma}^{(1,\delta)}$. Now let us introduce the new notation:

$$\xi_k^\mu \equiv \sigma_k^{(1,\delta)}\tilde{\xi}_k^{\mu+1}, \quad (\mu = 1, ...p,\ k = 1, ...N). \tag{1.13}$$

Then ξ_k^μ $(k = 1, \ldots, N)$, $(\mu = 1, \ldots, p)$ are also i.i.d. random variables assuming the values ± 1 with probability $\frac{1}{2}$. Denote

$$\tilde{x}_k = \frac{1}{N}\sum_{\mu=1}^{p}\sum_{j=1}^{N}\xi_k^\mu\xi_j^\mu = \tilde{x}_k^0 + \alpha_N \pm (1 - 2\delta_N), \quad \alpha_N = \frac{p+1}{N}, \quad \delta_N = \frac{[\delta N]}{N}. \tag{1.14}$$

Here α_N appears because we include in the summation the term with $j = k$, the term $\pm(1 - 2\delta_N)$ is due to the term $N^{-1}(\boldsymbol{\xi}^1, \boldsymbol{\sigma}^{(1,\delta)})$, and the sign here depends on k: it is plus for $k = 1, \ldots, [\delta N]$ and minus for $k = [\delta N] + 1, \ldots N$.

To simplify formulae we introduce also

$$\begin{array}{c} a_1 \equiv \alpha_N + 1 - 2\delta_N + q \to a_1^*, \quad a_1^* \equiv \alpha + 1 - 2\delta + q \\ a_2 \equiv \alpha_N - 1 + 2\delta_N + q' \to a_2^*, \quad a_2^* \equiv \alpha - 1 + 2\delta + q' \end{array} \tag{1.15}$$

which yield

$$P_N(q,q') \equiv \langle \prod_{k=1}^{[\delta N]} \theta(\tilde{x}_k - a_1) \prod_{k=1+[\delta N]}^{N} \theta(\tilde{x}_k - a_2) \rangle. \tag{1.16}$$

Here and below the symbol $\langle \dots \rangle$ denotes averaging with respect to all $\{\xi_k^\mu\}$ $(k = 1, \cdots, N, \mu = 1, \cdots, p+1)$.

In order to formulate the main results of the paper we need some other definitions.

Consider the function $\mathcal{F}_0(U,V;\alpha,\delta,q,q')$ of the form

$$\begin{array}{c} \mathcal{F}_0(U,V;\alpha,\delta,q,q') \equiv \delta \log H(\frac{a_1^*}{U} - V) + (1-\delta)\log H(\frac{a_2^*}{U} - V) \\ -UV + \frac{1}{2}V^2 + \alpha \log U, \end{array} \tag{1.17}$$

where

$$H(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt. \tag{1.18}$$

Define also

$$A(x) \equiv -\frac{d}{dx} \log H(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi} H(x)}, \tag{1.19}$$

$$\begin{array}{c} A_{1,2}(U,V) \equiv \frac{1}{U} A(\frac{a_{1,2}^*}{U} - V), \\ D(U,V) \equiv \frac{1}{2} - \delta A_1(U,V) - (1-\delta) A_2(U,V) \\ -\frac{1}{2}\delta(1-\delta)(A_1(U,V) - A_2(U,V))^2, \end{array} \tag{1.20}$$

and

$$\mathcal{F}_0^D(U,V;\alpha,\delta,q,q') \equiv \begin{cases} \mathcal{F}_0(U,V;\alpha,\delta,q,q'), & \text{if } D(U,V) \geq 0 \\ \frac{1}{1-2D(U,V)}[\delta \log H(\frac{a_1^*}{U} - V) & \\ +(1-\delta)\log H(\frac{a_2^*}{U} - V)] - UV + \frac{V^2}{2} + \alpha \log U, & \text{if } D(U,V) < 0. \end{cases} \tag{1.21}$$

Theorem 1.

$$\begin{array}{c} \limsup_{N\to\infty} \frac{1}{N} \log \langle \prod_{k=1}^{[\delta N]} \theta(\tilde{x}_k - a_1) \prod_{k=1+[\delta N]}^{N} \theta(\tilde{x}_k - a_2) \rangle \\ \leq \max_{U>0} \min_V \mathcal{F}_0^D(U,V;\alpha,\delta,q,q') - \frac{\alpha}{2} \log \alpha + \frac{\alpha}{2}. \end{array} \tag{1.22}$$

Remark 1. Note that in all interesting cases (see Theorems 2 and 3 below)

$$\max_{U>0}\min_{V}\mathcal{F}_0^D(U,V;\alpha,\delta,q,q')=\max_{U>0}\min_{V}\mathcal{F}_0(U,V;\alpha,\delta,q,q')$$

and one can substitute $\mathcal{F}_0^D$ by $\mathcal{F}_0$ in the r.h.s. of (1.22).

Remark 2. The proof of Theorem 1 can be generalized almost literally to the case (cf. (1.16))

$$P_{N,[\delta_1 N]}(q,q')\equiv\langle\prod_{k=1+[\delta_1 N]}^{[\delta N]}\theta(\tilde{x}_k-a_1)\prod_{k=1+[\delta N]}^{N}\theta(\tilde{x}_k-a_2)\rangle. \tag{1.23}$$

We obtain

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{N}\log P_{N,[\delta_1 N]}(q,q')\le\\ &\max_{U>0}\min_{V}\mathcal{F}_1^D(U,V;\alpha,\delta,\delta_1,q,q')-\frac{\alpha}{2}\log\alpha+\frac{\alpha}{2},\end{aligned} \tag{1.24}$$

with (cf. (1.17)-(1.21))

$$\mathcal{F}_1^D(U,V;\alpha,\delta,\delta_1,q,q')\equiv\begin{cases}\mathcal{F}_1(U,V;\alpha,\delta,\delta_1,q,q'), & \text{if } D^1(U,V)\ge 0;\\ \frac{1}{1-2D^1(U,V)}[(\delta-\delta_1)\log H(\frac{a_1^*}{U}-V) & \\ +(1-\delta)\log H(\frac{a_2^*}{U}-V)]-UV+\frac{1}{2}V^2+\alpha\log U, & \\ & \text{if } D^1(U,V)\le 0;\end{cases} \tag{1.25}$$

where

$$\begin{aligned}&\mathcal{F}_1(U,V;\alpha,\delta,\delta_1,q,q')\equiv(\delta-\delta_1)\log H(\frac{a_1^*}{U}-V)\\ &+(1-\delta)\log H(\frac{a_2^*}{U}-V)-UV+\frac{1}{2}V^2+\alpha\log U,\end{aligned} \tag{1.26}$$

and

$$\begin{aligned}D^1(U,V)\equiv&(1-\delta_1)^{-1}[\frac{1}{2}-(\delta-\delta_1)A_1(U,V)-(1-\delta)A_2(U,V)\\ &-\frac{1}{2}(\delta-\delta_1)(1-\delta)(A_1(U,V)-A_2(U,V))^2)]\end{aligned} \tag{1.27}$$

with $A_{1,2}(U,V)$ defined in (1.20).

Theorem 2. *If α is small enough, $\delta << \alpha^3\log\alpha^{-1}$ and $q=q'=0$, then*

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\log\langle\prod_{k=1}^{[\delta N]}\theta(\tilde{x}_k-a_1)\prod_{k=1+[\delta N]}^{N}\theta(\tilde{x}_k-a_2)\rangle\le\delta\log H(\frac{1-2\delta}{\sqrt{\alpha}})\\ +(1-\delta)\log H(-\frac{1-2\delta}{\sqrt{\alpha}})+O(e^{-1/\alpha})+o(\delta\log\alpha^{-1}).\end{aligned} \tag{1.28}$$

Thus, $P^*_N(\delta,\alpha)$ - the probability to have a fixed point of the dynamics of the Hopfield model at the distance δ from the first pattern has an upper bound of the form:

$$P^*_N(\delta,\alpha) \le \exp\{N[-\delta\log\delta-(1-\delta)\log(1-\delta)+\delta\log H(\frac{1-2\delta}{\sqrt{\alpha}})$$
$$+(1-\delta)\log H(-\frac{1-2\delta}{\sqrt{\alpha}})+O(e^{-1/\alpha})+o(\delta\log\alpha^{-1})+o(1)]\}.$$

Remark 3. It follows from Theorem 2, that $\delta_c(\alpha)$- the minimal δ for which $P^*_N(\delta,\alpha)$ does not decay exponentially in N, as $N\to\infty$, has the asymptotic behaviour

$$\delta_c(\alpha)\sim\frac{\sqrt{\alpha}}{\sqrt{2\pi}}e^{-1/2\alpha}.$$

This result coincides with the formula found by Amit at al. with replica calculations [A-G-S] and the one, obtained by J.Feng and B.Tirozzi in [F-T], using the extreme value theory.

Theorem 3. *Denote by $\mathcal{A}$ the event that there exist some $\delta,\varepsilon>0$ and some point $\boldsymbol{\sigma}^0\in B^1_\delta$, such that $\min_{\boldsymbol{\sigma}\in\Omega^1_\delta}\mathcal{H}(\boldsymbol{\sigma})-\mathcal{H}(\boldsymbol{\sigma}^0)>\varepsilon^2N$.*

Then if for some α and δ

$$\max_{0\le q}\max_{U}\min_{V}\{\mathcal{F}^D_0(U,V;\alpha,\delta,q,-q)\}-\frac{\alpha}{2}\log\alpha+\frac{\alpha}{2}+C^*(\delta)<0, \tag{1.29}$$

then there exists some $C(\alpha)>0$ such that

$$Prob\{\overline{\mathcal{A}}\}\le e^{-NC(\alpha)}. \tag{1.30}$$

Here and below

$$C^*(\delta)\equiv-\delta\log\delta-(1-\delta)\log(1-\delta). \tag{1.31}$$

Numerical calculations show that condition (1.29) is fulfilled for any $\alpha\le\alpha_c=0.113...$.

The paper is organized as follows. In Section 2 we prove Theorems 1, 2 and 3. In the process of the proof we shall need some auxiliary facts which we formulate there as Lemmas 1-4 and Propositions 1-4. Section 3 is devoted to the proof of the auxiliary results.

2. Proof of Main Results

Proof of Theorem 1

To make the idea of the proof more understandable we first carry out all computations when $\{\xi^\mu_j\}$ are Gaussian random variables. Since this part has no connection with the rigorous proof of Theorem 1, we just sketch the proof, without going into details.

To find P_N^g which corresponds to P_N (see (1.16)) in the Gaussian case, we study the Fourier transform of the joint probability distribution of the variables $\tilde{x}_k$.

$$
\begin{aligned}
F(\zeta_1,...,\zeta_N) &\equiv (2\pi)^{-N/2}\langle \exp\{i\sum_{k=1}^{p}\tilde{x}_k\zeta_k\}\rangle \\
= (2\pi)^{-N/2}&\langle \exp\{i\sum_{\mu=1}^{p}(N^{-1/2}\sum_{k=1}^{N}\xi_k^\mu\zeta_k)(N^{-1/2}\sum_{j=1}^{N}\xi_j^\mu)\}\rangle \\
&= (2\pi)^{-N/2}\prod_{\mu=1}^{p}\langle e^{i\tilde{u}^\mu\tilde{v}^\mu}\rangle,
\end{aligned}
\tag{2.1}
$$

where we use notations

$$
\tilde{u}^\mu \equiv N^{-1/2}\sum_{k=1}^{N}\xi_k^\mu\zeta_k, \quad \tilde{v}^\mu \equiv N^{-1/2}\sum_{j=1}^{N}\xi_j^\mu. \tag{2.2}
$$

It is easy to see that

$$
\langle e^{i\tilde{u}^\mu\tilde{v}^\mu}\rangle = (2\pi)^{-1}\int du^\mu dv^\mu \langle e^{i(u^\mu\tilde{u}^\mu+v^\mu\tilde{v}^\mu)}\rangle e^{-iu^\mu v^\mu}. \tag{2.3}
$$

Thus, using the inverse Fourier transform for the function $F(\zeta_1,...,\zeta_N)$, we get

$$
\begin{aligned}
P_N^g &= \frac{1}{(2\pi)^{N/2}}\int\prod_{k=1}^{N}\theta(x_k-a_k)dx_k\int(\prod_{j=1}^{N}d\zeta_j)\exp\{-i\sum_{k=1}^{N}x_k\zeta_k\}F(\zeta_1,...,\zeta_N) \\
&= \frac{1}{(2\pi)^{(N+p)}}\int(\prod_{\mu=1}^{p}e^{-iu^\mu v^\mu}du^\mu dv^\mu)\prod_{k=1}^{N}\int dx_k\theta(x_k-a_k)\int d\zeta_k\langle\exp\{-i\zeta_k x_k \\
&+\sum_{\mu=1}^{p}i(u^\mu\tilde{u}^\mu+v^\mu\tilde{v}^\mu)\}\rangle = \frac{1}{(2\pi)^{N+p}}\int(\prod_{\mu=1}^{p}e^{-iu^\mu v^\mu}du^\mu dv^\mu)\prod_{k=1}^{N}\int dx_k\theta(x_k-a_k) \\
&\cdot\int d\zeta_k e^{-i\zeta_k x_k}\int(\prod_{\mu=1}^{p}\frac{e^{-(\xi_k^\mu)^2/2}}{\sqrt{2\pi}})\exp\{i(N^{-1/2}\sum_{\mu=1}^{p}u^\mu\xi_k^\mu\zeta_k+N^{-1/2}\sum_{\mu=1}^{p}v^\mu\xi_k^\mu)\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{(N+p)}}\int(\prod_{\mu=1}^{p}e^{-iu^\mu v^\mu}du^\mu dv^\mu)\prod_{k=1}^{N}\int dx_k\theta(x_k-a_k) \\
&\quad\cdot\int d\zeta_k\cdot e^{-i\zeta_k x_k}(\prod_{\mu=1}^{p}\exp\{-\frac{(u^\mu\zeta_k+v^\mu)^2}{2N}\}) \\
&= \frac{1}{(2\pi)^{(\frac{N}{2}+p)}}\int(\prod_{\mu=1}^{p}\exp\{-iu^\mu v^\mu-\frac{(v^\mu)^2}{2}\}du^\mu dv^\mu) \\
&\cdot\prod_{k=1}^{N}\int dx_k\frac{\theta(x_k-a_k)}{U}\exp\{\frac{(ix_k+N^{-1}\sum_{\mu=1}^{p}u^\mu v^\mu)^2}{2U^2}\},
\end{aligned}
\tag{2.4}
$$

where $U \equiv (N^{-1}\sum_{\mu=1}^p (u^\mu)^2)^{1/2}$. Therefore we have

$$\begin{aligned} P_N^g = (2\pi)^{-p} \int (\prod_{\mu=1}^p du^\mu dv^\mu) \exp\{-i\sum_{\mu=1}^p u^\mu v^\mu - \frac{1}{2}\sum_{\mu=1}^p (v^\mu)^2\} \cdot \\ \cdot \prod_{k=1}^N H(\frac{a_k - iN^{-1}\sum_{\mu=1}^p u^\mu v^\mu}{U}). \end{aligned} \tag{2.5}$$

Now let us fix $\overline{u} = \{u^\mu\}_{\mu=1}^p$ and change variables in the integral with respect to $\overline{v} = \{v^\mu\}_{\mu=1}^p$

$$v_1 = \frac{1}{\sqrt{N}}(\overline{e}_1, \overline{v}),\ v_2 = (\overline{e}_2, \overline{v}), ..., v_p = (\overline{e}_p, \overline{v}), \tag{2.6}$$

where $\{\overline{e}_i\}_{i=1}^p$ is the orthonormal system of vectors in $\mathbf{R}^p$ such that $e_1^\mu = (U\sqrt{N})^{-1}u^\mu$. Then, integrating with respect $v_2, ..., v_p$, we obtain

$$\begin{aligned} P_N^g = (2\pi)^{-(p-1)/2} \int (\prod_{\mu=1}^p du^\mu) \int dv_1 \exp\{-iNUv_1 - \frac{N}{2}(v_1)^2 \\ +[N\delta]\log H(\frac{a_1}{U} - iv_1) + (N - [N\delta])\log H(\frac{a_2}{U} - iv_1)\}. \end{aligned} \tag{2.7}$$

Using the spherical coordinates in the integral with respect to $\overline{u}$ and integrating with respect to angular variables, we get

$$\begin{aligned} P_N^g = \Gamma(p) \int_0^\infty dU \int dv_1 \exp\{(p-1)\log U - iNUv_1 - \frac{N}{2}(v_1)^2 \\ +[N\delta]\log H(\frac{a_1}{U} - iv_1) + (N - [N\delta])\log H(\frac{a_2}{U} - iv_1)\}. \end{aligned} \tag{2.8}$$

Let $V(U)$ be the point of minimum with respect to V of the function $\mathcal{F}_0(U,V)$ defined by (1.17). Let us change the path of integration with respect to v_1 in (2.8) from the real axis to the line L which is parallel to it, but contains the point $z = -iV(U)$. Then, following the saddle point method, we divide the integral into two parts

$$\begin{aligned} P_N^g = \Gamma(p) \int_0^\infty dU (\int_{|t|>N^{-1/3}} + \int_{|t|\le N^{-1/3}}) dt \exp\{(p-1)\log U \\ -NUV(U) + \frac{N}{2}(V(U))^2 - iNUt - \frac{N}{2}t^2 \\ +[N\delta]\log H(\frac{a_1}{U} - V(U) - it) + (N - [N\delta])\log H(\frac{a_2}{U} - V(U) - it)\}. \end{aligned} \tag{2.9}$$

Due to the simple inequality

$$|H(a+ic)| \le H(a)e^{c^2/2}, \tag{2.10}$$

valid for any real numbers a and c, we conclude, that the second integral is $o(1)\exp\{N\mathcal{F}_0(U,V;\alpha,\delta,q,q')\}$. Replacing in the first integral $\mathcal{F}_0(U,V(U)-it)$ by its Taylor expansion up to the second order term (the first order term is zero due to the choice $V(U)$) and then performing the Gaussian integration, we see that

$$P_N^g \le \Gamma(p) \int_0^\infty dU \exp\{N(\mathcal{F}_0(U,V(U);\delta,q,q') + o(1))\}. \tag{2.11}$$

Applying the standard Laplace method, we conclude that for the Gaussian random variables ξ_k^μ Eq. (1.22) can be replaced by the following stronger statement:

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N^g = \max_{U>0} \mathcal{F}_0(U, V(U); \delta, q, q') - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2}.$$

The difference of non-Gaussian case from the Gaussian one is that we have, in the sixth line of (2.4), $\prod_{\mu=1}^p \cos\frac{u^\mu\zeta_k+v^\mu}{\sqrt{N}}$ instead of $\prod_{\mu=1}^p \exp\{-\frac{(u^\mu\zeta_k+v^\mu)^2}{2N}\}$. To replace the former term by the latter one we have to estimate the difference between them for different $\overline{u}$, $\overline{v}$ and $\overline{\zeta}$. To this end we introduce some smoothing factors in the integration (2.4).

Lemma 1.

$$\langle\prod_{k=1}^N \theta(\tilde{x}_k - a_k)\rangle \leq P_N^1 l_N^{p/2}(1-e^{-h^2/2\lambda})^{-N} e^{No(1)} + e^{-\,const\,N(\varepsilon_N^*)^{-1/2}}, \quad (2.12)$$

where

$$P_N^1 \equiv \frac{1}{(2\pi)^{N+p}} \int d\overline{u}d\overline{v} \exp\{-il_N(\overline{u},\overline{v}) - \varepsilon_N^*\frac{(\overline{u},\overline{u})}{2N} - \varepsilon_N^*\frac{(\overline{v},\overline{v})}{2N}\} \cdot \prod_{k=1}^N \int d\zeta_k \hat{\chi}_{N,h}(\zeta_k) e^{-\lambda\zeta_k^2/2 - ia_k\zeta_k} \prod_{\mu=1}^p \cos\frac{u^\mu\zeta_k + v^\mu}{\sqrt{N}},$$

$\varepsilon_N^* = (\log\log N)^{-1}$, $l_N \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4(\varepsilon_N^*)^2}$, *$\lambda$ is a fixed positive number and*

$$\hat{\chi}_{N,h}(\zeta) = \frac{2}{\zeta}\sin\zeta\frac{N^{1/2+d}+2h}{2}\exp\{-i\zeta\frac{N^{1/2+d}}{2}\}$$

is the complex conjugate of the Fourier transform of $\chi_{N,h}(x)$ -the characteristic function of the interval $(-h, N^{1/2+d}+h)$ with some positive d and $h > (\frac{2\lambda}{\pi})^2$. Here and below $\overline{v} = (v^1,\dots,v^p)$, $\overline{u} = (u^1,\dots,u^p)$, $d\overline{v} = \prod_{\mu=1}^p dv^\mu$ and $d\overline{u} = \prod_{\mu=1}^p du^\mu$.

Remark 4. In fact we can take $\varepsilon_N^* \to 0$ as slowly as we want, we can even fix $\varepsilon_N^* = \varepsilon$ with ε being small enough. However, in this case we have to be more careful to control the constants which will appear in our estimates.

Now we start to prove Theorem 1. Denote

$$F_{N,k}(\overline{u},\overline{v}) = \frac{1}{2\pi}\int d\zeta_k \hat{\chi}_{N,h}(\zeta_k) e^{-\lambda\zeta_k^2/2 - ia_k\zeta_k} \prod_{\mu=1}^p \cos\frac{u^\mu\zeta_k+v^\mu}{\sqrt{N}}; \qquad \tilde{F}(\overline{u},\overline{v}) = \prod_k F_{N,k}(\overline{u},\overline{v}). \quad (2.13)$$

To simplify formulae in the places where it is not important, we confine ourselves to the case $a_k = a$. Since in this case all $F_{N,k}(\overline{u},\overline{v})$ are identical, we could omit the index k.

To replace the product term of *cos* in Eq. (2.13) by the exponent we modify a method originally proposed by Lyapunov. He employed it to prove that the distribution of the sum of independent variables uniformly converges to the normal distribution (see [Lo]). To ensure the method to work, the second and the third moments of the random variables must be bounded. Since in our setting the random variables have the form $u^\mu\xi_k^\mu$ and $v^\mu\xi_k^\mu$ and their moments coincide with $|u^\mu|^{2,3}$ and $|v^\mu|^{2,3}$, we need to remove large $|u^\mu|$ and $|v^\nu|$ in the integrals. For this purpose we take $\varepsilon_N=(\log N)^{-1}$ and denote

$$\chi_{\varepsilon_N}(u^\mu,v^\mu)=\theta(\varepsilon_N^2\sqrt{N}-|u^\mu|)\theta(\varepsilon_N\sqrt{N}-|v^\mu|). \tag{2.14}$$

Note, that the different powers of ε_N in the θ-functions for u and v are necessary in our estimates below.

Rewrite

$$\begin{aligned} P_N^1 &= \frac{1}{(2\pi)^p}\int e^{-il_N(\overline{u},\overline{v})}\tilde{F}(\overline{u},\overline{v})\exp\{-\frac{\varepsilon_N^*}{2}(\overline{v},\overline{v})-\frac{\varepsilon_N^*}{2}(\overline{u},\overline{u})\}d\overline{u}d\overline{v}\\ &=\sum_{m=0}^{p}C_p^m\int d\overline{u}d\overline{v}\prod_{\mu=1}^{m}(1-\chi_{\varepsilon_N}(u^\mu,v^\mu))\prod_{\nu=m+1}^{p}\chi_{\varepsilon_N}(u^\nu,v^\nu)\\ &\quad\cdot e^{-\frac{\varepsilon_N^*}{2}(\overline{u},\overline{u})}e^{-\frac{\varepsilon_N^*}{2}(\overline{v},\overline{v})}e^{-il_N(\overline{u},\overline{v})}\tilde{F}(\overline{u},\overline{v})\equiv\sum_{m=0}^{p}C_p^m I_m. \end{aligned} \tag{2.15}$$

Let us first estimate I_m in the above equation

$$\begin{aligned} |I_m| &\le \frac{1}{(2\pi)^p}\int d\overline{u}d\overline{v}\prod_{\mu=1}^{m}(1-\chi_{\varepsilon_N}(u^\mu,v^\mu))\\ &\cdot\prod_{\nu=m+1}^{p}\chi_{\varepsilon_N}(u^\nu,v^\nu)e^{-\frac{\varepsilon_N^*}{2}(\overline{u},\overline{u})}e^{-\frac{\varepsilon_N^*}{2}(\overline{v},\overline{v})}\prod_{k=1}^{N}\int d\zeta_k|\hat{\chi}_{N,h}(\zeta_k)|e^{-\lambda\zeta_k^2/2}. \end{aligned} \tag{2.16}$$

Now, using the bound

$$\begin{aligned} &\int d\zeta_k|\hat{\chi}_{N,h}(\zeta_k)|e^{-\lambda\zeta_k^2/2}\\ &=\int d\zeta_k|\frac{2}{\zeta_k}\sin(\zeta_k\frac{N^{1/2+d}}{2})|e^{-\lambda\zeta_k^2/2}\le \text{const}\,\log N, \end{aligned} \tag{2.17}$$

we arrive at

$$|I_m|\le e^{\text{const}\,N\log\log N}(\varepsilon_N^*)^{-p}e^{-mN\varepsilon_N^*\varepsilon_N^4/2}.$$

Thus,

$$|\sum_{m=m_0}^{p}C_p^m I_m|\le e^{-\text{const}\,N\log\log N}, \tag{2.18}$$

where $m_0=[(\log N)^5]>>(\varepsilon_N^*)^{-1}\varepsilon_N^{-4}\log\log N$.

In the following it would be more convenient to have the integration with respect to $u^1,\dots,u^m$ and $v^1,\dots,v^m$ in the whole $\mathbf{R}$. Therefore, we perform the first product in (2.15) and rewrite $\sum_{m=0}^{m_0}C_p^m I_m$ in the form

$$\sum_{m=0}^{m_0} C_p^m I_m = \sum_{m=0}^{m_0} \tilde{C}_m \tilde{I}_m, \tag{2.19}$$

where

$$\tilde{I}_m \equiv \frac{1}{(2\pi)^p} \int d\overline{u} d\overline{v} \prod_{\nu=m+1}^{p} \chi_{\varepsilon_N}(u^\mu, v^\mu) e^{-\frac{\varepsilon_N^*}{2}(\overline{u},\overline{u})} e^{-\frac{\varepsilon_N^*}{2}(\overline{v},\overline{v})} e^{-il_N(\overline{u},\overline{v})} \tilde{F}(\overline{u},\overline{v}) \tag{2.20}$$

and $\tilde{C}_m$ are some combinatorial coefficients. These coefficients are not important, because for our choice of m $(m \le m_0 = o(N))$ all of them are of the order $e^{o(N)}$ and after taking the logarithm and dividing by N give us $o(1)$-terms. Thus, we have

$$P_N^1 = \sum_{m=0}^{m_0} \tilde{C}_m \tilde{I}_m + O(e^{-\operatorname{const} N \log\log N}). \tag{2.21}$$

To proceed further we define

$$\begin{aligned} F^{(m)}(\overline{u}_1, \overline{v}_1; \overline{u}_2, \overline{v}_2) \equiv \exp\{-\frac{(\overline{v}_2, \overline{v}_2)}{2N}\} \\ \cdot\langle H_{N,h,\tilde{U}}(\frac{a - h - i\frac{(\overline{u}_2,\overline{v}_2)}{N} - \frac{(\overline{u}_1,\overline{\xi}_1)}{\sqrt{N}}}{\sqrt{\tilde{U}^2+\lambda}}) \exp\{i\frac{(\overline{v}_1,\overline{\xi}_1)}{\sqrt{N}}\}\rangle \\ = \int d\zeta \hat{\chi}_{N,h}(\zeta) e^{-\lambda\zeta^2/2 - ia\zeta} \prod_{\mu=1}^{m} \cos\frac{u^\mu\zeta + v^\mu}{\sqrt{N}} \\ \exp\{-\frac{1}{2N}(\overline{v}_2,\overline{v}_2) - \frac{1}{N}(\overline{u}_2,\overline{v}_2)\zeta - \frac{1}{2N}(\overline{u}_2,\overline{u}_2)\zeta^2\}, \end{aligned} \tag{2.22}$$

where

$$H_{N,h,\tilde{U}}(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \theta(\frac{N^{1/2+d} + 2h}{\sqrt{\tilde{U}^2+\lambda}} - t) \exp\{-\frac{1}{2}(t+x)^2\} dt. \tag{2.23}$$

Here and below $\overline{u}_1 \equiv (u^1, ..., u^m)$ and $\overline{v}_1 \equiv (v^1, ..., v^m)$, $\overline{u}_2 \equiv (u^{m+1}, ..., u^p)$, $\overline{v}_2 \equiv (v^{m+1}, ..., v^p)$, so that $\overline{u} = \{\overline{u}_1, \overline{u}_2\}$, $\overline{v} = \{\overline{u}_1, \overline{u}_2\}$, $\overline{\xi}_1 \equiv (\xi_1^1, ..., \xi_1^m)$ is the random vector with independent components, assuming values ± 1 with probability $\frac{1}{2}$, $\langle ... \rangle$ means the average with respect to $\overline{\xi}_1$ and $\tilde{U} \equiv [\frac{1}{N}(\overline{u}_2, \overline{u}_2)]^{1/2}$. Expression (2.22) is obtained from (2.13) by changing *cos* in the product $\prod_{\mu=m+1}^{p}$ by the correspondent exponent and then by integration with respect to ζ_k.

The main technical tool at this step is a lemma, which is a modification of the Lyapunov theorem.

Lemma 2. *For any $\overline{u}_2, \overline{v}_2, \overline{\lambda}_2$ such that $|u^\nu|, |v^\nu|, |\lambda^\nu| \le \varepsilon_N \sqrt{N}$ and any $\overline{u}_1, \overline{v}_1, \overline{\lambda}_1$ the function*

$$R^{(m)}(\overline{u}_1, \overline{w}_1; \overline{u}_2, \overline{w}_2) \equiv F_N(\overline{u}_1, \overline{w}_1; \overline{u}_2, \overline{w}_2) - F^{(m)}(\overline{u}_1, \overline{w}_1; \overline{u}_2, \overline{w}_2)$$

admits the bound

$$|R^{(m)}(\overline{u}_1,\overline{w}_1;\overline{u}_2,\overline{w}_2)| \le \ const\,\varepsilon_N^2(1+\frac{(\overline{\lambda}_2,\overline{\lambda}_2)}{N})(\tilde{U}^2+\lambda)^{1/2}$$
$$\cdot\exp\{-\frac{\lambda(\overline{v}_2,\overline{v}_2)}{4N(\tilde{U}^2+\lambda)}+\frac{(\overline{\lambda},\overline{\lambda})}{N}\}+\exp\{-\ const\,\varepsilon_N^{-4}+\frac{(\overline{\lambda},\overline{\lambda})}{N}\}. \tag{2.24}$$

Here and below $\overline{w}\equiv\overline{v}+i\overline{\lambda}$.

This lemma allows us to replace in our formulae F_N by $F^{(m)}$ in the following sense. Let us write

$$\tilde{I}_m \equiv \frac{1}{(2\pi)^p}\int d\overline{u}d\overline{v}\prod_{\nu=m+1}^{p}\chi_{\varepsilon_N}(u^\mu,v^\mu)\exp\{-il_N(\overline{u},\overline{v})\}$$
$$\cdot(F^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u},\overline{v})+R^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u}_2,\overline{v}_2))^N e^{-\frac{\varepsilon_N^*}{2}(\overline{v},\overline{v})}e^{-\frac{\varepsilon_N^*}{2}(\overline{u},\overline{u})} \tag{2.25}$$
$$\equiv\sum_{k=0}^{N}C_N^k I_{m,k}\ ,$$

where

$$I_{m,k}\equiv\frac{1}{(2\pi)^p}\int d\overline{u}d\overline{v}\prod_{\nu=m+1}^{p}\chi_{\varepsilon_N}(u^\mu,v^\mu)e^{-il_N(\overline{u},\overline{v})}(F^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u},\overline{v}))^{N-k}$$
$$\cdot(R^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u}_2,\overline{v}_2))^k e^{-\frac{\varepsilon_N^*}{2}(\overline{v},\overline{v})}e^{-\frac{\varepsilon_N^*}{2}(\overline{u},\overline{u})}.$$

Lemma 3. *For* $k>k_0\equiv[N\log^{-1/2}\varepsilon_N^{-1}]$

$$|I_{m,k}|\le e^{N\,const}(\varepsilon_N)^{2k}(\varepsilon_N^*)^{-2p}\exp\{-k\,const\log\varepsilon_N^{-1}\}.$$

Thus, we get that for $k>k_0$ $I_{m,k}$ have the order $e^{-N\,\mathrm{const}\,\log^{1/2}\varepsilon_N^{-1}}$ and so we can neglect these terms in (2.25).

Now we shall study the leading terms in the r.h.s. of Eq. (2.25) ($I_{m,k}$ with $k<k_0$). In fact, the next step is a version of the saddle point method (cf.(2.8)-(2.11)).

Let us take any real fixed V and change the path of integration w.r. to $\overline{v}_2$ from the product of intervals $(-\varepsilon_N\sqrt{N},\varepsilon_N\sqrt{N})$ to the product of the paths $L_1^\nu\cup L_2^\nu$, with $L_1^\nu=(-\varepsilon_N\sqrt{N}-\frac{iVu^\nu}{\tilde{U}},\varepsilon_N\sqrt{N}-\frac{iVu^\nu}{\tilde{U}})$ and $L_2^\nu=(-\varepsilon_N\sqrt{N},-\varepsilon_N\sqrt{N}-\frac{iVu^\nu}{\tilde{U}})\cup(\varepsilon_N\sqrt{N}-\frac{iVu^\nu}{\tilde{U}},\varepsilon_N\sqrt{N})$ ($\nu=m+1,...N$). It can be done, since all our functions are analytical w.r.to v^ν,

Then take any real λ^μ, such that $(\overline{\lambda}_1,\overline{\lambda}_1)\le N\,\mathrm{const}$ and choose the paths of integration with respect to $\overline{v}_1$ as $L^\mu=\{w^\mu=t^\mu-i\lambda^\mu,t^\mu\in\mathbf{R}\}$. Finally, we get

$$I_{m,k}=\frac{1}{(2\pi)^p}\sum_{n=1}^{p-m}C_{p-m}^n\int d\overline{u}_1\int_{\prod_{\mu=1}^m L^\mu}d\overline{w}_1\int_{\prod_{\nu=m+1}^{p-n}L_1^\nu}d\overline{w}_3$$
$$\cdot\int_{\prod_{\nu=p-n+1}^{p}L_2^\nu}d\overline{w}_4\int_{-\varepsilon_N^2\sqrt{N}}^{\varepsilon_N^2\sqrt{N}}d\overline{u}_3d\overline{u}_4e^{-\varepsilon_N^*(\overline{w},\overline{w})/2}e^{-\varepsilon_N^*(\overline{u},\overline{u})/2} \tag{2.26}$$
$$\cdot e^{-il_N(\overline{u},\overline{w})}(F^{(m)}(\overline{u},\overline{w}))^{N-k}(R^{(m)}(\overline{u},\overline{w}))^k\equiv\sum_{n=1}^{p-m}C_{p-m}^n I_{m,k,n}.$$

Here and below $\overline{u} = \{\overline{u}_1, \overline{u}_3, \overline{u}_4\}$, $\overline{w} = \{\overline{w}_1, \overline{w}_3, \overline{w}_4\}$, where $\overline{u}_1, \overline{w}_1$ are the same as before and we divide vectors $\overline{w}_2$ and $\overline{u}_2$ in two sub-vectors $\overline{u}_2 = \{\overline{u}_3, \overline{u}_4\}$, $\overline{w}_2 = \{\overline{w}_3, \overline{w}_4\}$ in such a way, that $\overline{u}_4, \overline{w}_4$ include the last n components of $\overline{u}_2$ and $\overline{w}_2$ respectively.

Now let us get rid of $I_{m,k,n}$ with sufficiently large n. Similarly to the proof of Lemma 3 on the basis of Lemma 2, we get

$$|I_{m,k,n}| \le e^{N \,\mathrm{const}} (\varepsilon_N^*)^{-p} e^{-\,\mathrm{const}\, nN\varepsilon_N^2} \exp\{(\overline{\lambda}_1, \overline{\lambda}_1) + NV^2\}. \tag{2.27}$$

So, taking $n > n_0 = [\varepsilon_N^{-5/2}]$, on the basis of (2.27) one can conclude that we need to study only the first n_0 terms in (2.26).

Remark, that starting from this moment, we shall distinguish the terms with a_1 and a_2. Denote

$$\begin{gathered} G_m^*(U, V, \overline{u}_1, \overline{\lambda}_1) \equiv \\ \langle H(\frac{a_1 - h - VU - N^{-1/2}(\overline{u}_1, \overline{\xi}_1)}{\sqrt{U^2+\lambda}}) \exp\{\frac{(\overline{\lambda}_1, \overline{\xi}_1)}{\sqrt{N}}\}\rangle^{\delta} \\ \cdot\langle H(\frac{a_2 - h - VU - N^{-1/2}(\overline{u}_1, \overline{\xi}_1)}{\sqrt{U^2+\lambda}}) \exp\{\frac{(\overline{\lambda}_1, \overline{\xi}_1)}{\sqrt{N}}\}\rangle^{1-\delta} \\ \cdot \exp\{-\frac{l_N}{N}(\overline{u}_1, \overline{\lambda}_1) - l_N UV + \frac{1}{2}V^2\}. \end{gathered} \tag{2.28}$$

Lemma 4. *Let $G_{m,k,n}(V, \overline{u}_1, \overline{\lambda}_1, \overline{u}_3)$ be the function which we get, if in (2.26) integrate with respect to $\overline{w}_1$, $\overline{w}_3$, $\overline{u}_4$ and $\overline{w}_4$. Then*

$$\begin{gathered} |G_{m,k,n}(V, \overline{u}_1, \overline{\lambda}_1, \overline{u}_3)| \\ \le (2\pi)^{-p/2} (G_m^*(U, V, \overline{u}_1, \overline{\lambda}_1))^N e^{-\frac{\varepsilon_N^*}{2}(\overline{u}_1, \overline{u}_1) + No(1)}. \end{gathered} \tag{2.29}$$

Here and below $U = [N^{-1}(\overline{u}_3, \overline{u}_3)]^{1/2}$, so that $\tilde{U}^2 = U^2 + N^{-1}(\overline{u}_4, \overline{u}_4)$.

Once we have an upper bound for $G_{m,k,n}$ we can estimate all the $\tilde{I}_m$ in (2.21). Let us study first the term with $m = 0$. Consider the function

$$\begin{gathered} \mathcal{F}_{\lambda,h}(U, V) \equiv \delta \log H(\frac{a_1^* - h - VU}{\sqrt{U^2+\lambda}}) + (1-\delta) \log H(\frac{a_2^* - h - VU}{\sqrt{U^2+\lambda}}) \\ -UV + \frac{1}{2}V^2. \end{gathered} \tag{2.30}$$

Let $V(U)$ be chosen from the condition

$$\mathcal{F}_0(U, V(U); \alpha, \delta, q, q') = \min_V \mathcal{F}_0(U, V; \alpha\delta, q, q'). \tag{2.31}$$

The function $\mathcal{F}_{\lambda,h}(U, V(U))$ and the functions which appear in the exponent of (2.29) for $m = 0$ satisfy the inequalities of the type

$$\mathcal{F}_{\lambda,h}(U, V(U)) \le \alpha \log U - \frac{U^2}{2}$$

(it follows from $\log H(x) \leq 0$ and $V(U) \leq U$). Thus, since $a_{1,2} \to a^*_{1,2}$ and $l_N \to 1$ as $N \to \infty$, on the basis of (2.29) for $m = 0$, we get

$$|\tilde{I}_0| \leq (2\pi)^{-p/2} \int d\overline{u}_3 \exp\{N[\mathcal{F}_{\lambda,h}(U, V(U)) + o(1)]\},$$

where $\tilde{I}_0$ is defined by formula (2.20) for $m = 0$.

Remark 5. Let us note, that here we have use the following simple statement. If the continuous functions $\phi(U)$, $\phi_N(U)$ $(N = 1, 2, ..)$ $(U \in \mathbf{R}_+)$ satisfy the inequalities

$$\begin{array}{ll} \phi(U), \phi_N(U) \leq -C_1 U^2, & U \geq L \\ \phi(U), \phi_N(U) \leq C_2 \log U, & U \leq \varepsilon \end{array} \tag{2.32}$$

with some positive C_1 and C_2 and $\phi_N(U) \to \phi(U)$, as $N \to \infty$, uniformly in each compact set in $\mathbf{R}_+$, then $\int \exp\{N\phi_N(U)\}dU = e^{o(N)} \int \exp\{N\phi(U)\}dU$. The proof of this statement is very simple, and we omit it.

Below we shall use this remark without additional comments.

Performing the spherical change of variables and using the Laplace method, we get now

$$|\tilde{I}_0| \leq \exp\{N[\max_U \mathcal{F}_{\lambda,h}(U, V(U)) + \alpha \log U - \frac{\alpha}{2} \log \alpha + \frac{\alpha}{2} + o(1)]\}. \tag{2.33}$$

To study the terms with $m \neq 0$ we chose $\overline{\lambda}_1(U, V, \overline{u}_1)$ in such a way that

$$G^*_m(U, V, \overline{u}_1, \overline{\lambda}_1(U, V, \overline{u}_1)) = \min_{\overline{\lambda}_1 \in \mathbf{R^m}} G^*_m(U, V, \overline{u}_1, \overline{\lambda}_1), \tag{2.34}$$

where the function G^*_m is defined by (2.28). Then we use the inequality, which follows from the fact that $(\log H(x))'' \leq 0$.

$$H(x + y) \leq H(x) e^{-A(x)y} \tag{2.35}$$

with the function $A(x)$ defined by (1.19). On the basis of this inequality we get

$$\begin{aligned} &\langle H(\frac{a_{1,2} - h - VU - N^{-1/2} \sum_{\mu=1}^m u^\mu \xi_1^\mu}{\sqrt{U^2 + \lambda}}) \exp\{\sum_{\mu=1}^m \lambda^\mu \frac{\xi_1^\mu}{\sqrt{N}}\}\rangle \\ &\leq \langle H(\frac{a_{1,2} - h - VU}{\sqrt{U^2 + \lambda}}) \exp\{\sum_{\mu=1}^m (A_{1,2}^{(\lambda,h)} u^\mu + \lambda^\mu) \frac{\xi_1^\mu}{\sqrt{N}}\}\rangle \\ &= H(\frac{a_{1,2} - h - VU}{\sqrt{U^2 + \lambda}}) \prod_{\mu=1}^m \cosh \frac{A_{1,2}^{(\lambda,h)} u^\mu + \lambda^\mu}{\sqrt{N}} \\ &\leq H(\frac{a_{1,2} - h - VU}{\sqrt{U^2 + \lambda}}) \exp\{\frac{1}{2N} \sum_{\mu=1}^m (A_{1,2}^{(\lambda,h)} u^\mu + \lambda^\mu)^2\}, \end{aligned} \tag{2.36}$$

where

$$A_{1,2}^{(\lambda,h)} = (U^2 + \lambda)^{-1/2} A(\frac{a_{1,2} - h - VU}{\sqrt{U^2 + \lambda}}). \tag{2.37}$$

Thus,

$$
\begin{aligned}
G^*_m(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))| \le \exp\{\delta\log H(\frac{a_1-h-UV}{\sqrt{U^2+\lambda}})\\
+(1-\delta)\log H(\frac{a_2-h-UV}{\sqrt{U^2+\lambda}}) - l_N UV + \frac{1}{2}V^2\\
+\min_{\lambda^\mu}[\frac{\delta}{2N}\sum_{\mu=1}^m (A_1^{(\lambda,h)}u^\mu+\lambda^\mu)^2 + \frac{1-\delta}{2N}\sum_{\mu=1}^m (A_2^{(\lambda,h)}u^\mu+\lambda^\mu)^2\\
-\frac{l_N}{N}\sum_{\mu=1}^m \lambda^\mu u^\mu]\},
\end{aligned}
\tag{2.38}
$$

Taking $\lambda^\mu = (1-A_1^{(\lambda,h)}\delta - A_2^{(\lambda,h)}(1-\delta))u^\mu$, which give us the minimum of the expression in the r.h.s. of (2.38), we get

$$
\begin{aligned}
|G^*_m(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))| \le \exp\{N[\delta\log H(\frac{a_1-h-UV}{\sqrt{U^2+\lambda}})\\
+(1-\delta)\log H(\frac{a_2-h-UV}{\sqrt{U^2+\lambda}}) - UV + \frac{1}{2}V^2] - D^{(\lambda,h)}(U,V)(\overline{u}_1,\overline{u}_1)\},
\end{aligned}
\tag{2.39}
$$

where $D^{(\lambda,h)}(U,V)$ is defined by (1.20) if we substitute there $A_{1,2}(U,V)$ by $A_{1,2}^{(\lambda,h)}(U,V)$. From (2.39) it is easy to see, that if $D^{(\lambda,h)}(U,V)\ge 0$, then

$$
\begin{aligned}
|\int d\overline{u}_1 G^*_m(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))\exp\{-\frac{\varepsilon_N^*}{2}(\overline{u}_1,\overline{u}_1)\}|\\
\le e^{No(1)}\exp\{N[\delta\log H(\frac{a_1^*-h-UV}{\sqrt{U^2+\lambda}}) + (1-\delta)\log H(\frac{a_2^*-h-UV}{\sqrt{U^2+\lambda}})\\
-UV+\frac{1}{2}V^2]\}.
\end{aligned}
\tag{2.40}
$$

If $D^{(\lambda,h)}(U,V)$ is negative, we use

Proposition 1. *If $D^{(\lambda,h)}(U,V)<0$, λ and h are small enough, then*

$$
\begin{aligned}
|\int d\overline{u}_1 G^*_m(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))\exp\{-\frac{\varepsilon_N^*}{2}(\overline{u}_1,\overline{u}_1)\}|\\
\le \exp\{N[\frac{\delta}{1-2D^{(\lambda,h)}(U,V)}\log H(\frac{a_1^*-h-UV}{\sqrt{U^2+\lambda}})\\
+\frac{1-\delta}{1-2D^{(\lambda,h)}(U,V)}\log H(\frac{a_2^*-h-UV}{\sqrt{U^2+\lambda}}) - UV + \frac{1}{2}V^2 + o(1)]\}.
\end{aligned}
\tag{2.41}
$$

Thus, on the basis of (2.39) and (2.41), we have got that for any n-independent finite V

$$
\begin{aligned}
|\int d\overline{u}_1 G^*_m(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))\exp\{-\frac{\varepsilon_N^*}{2}((\overline{u}_1,\overline{u}_1)+(\overline{u}_3,\overline{u}_3))\}|\\
\le \exp\{N[\mathcal{F}^D_{\lambda,h}(U,V)+o(1)]\},
\end{aligned}
$$

where for $D^{(\lambda,h)}(U,V)<0$, $\mathcal{F}^D_{\lambda,h}(U,V)$ is defined by the expression in the exponent in the r.h.s. of (2.41) and for $D^{(\lambda,h)}(U,V)\ge 0$, it coincides with $\mathcal{F}_{\lambda,h}(U,V)$. Then, choosing V to minimise this estimate for any U, we get

$$\begin{aligned}\int d\overline{u}_3\Big|\int d\overline{u}_1 G^*_m(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))\exp\{-\frac{\varepsilon^*_N}{2}((\overline{u}_1,\overline{u}_1)+(\overline{u}_3,\overline{u}_3))\}\Big|\\ \le\int dU\exp\{N[\min_V \mathcal{F}^D_{\lambda,h}(U,V)+\alpha\log U-\frac{\alpha}{2}\log\alpha+\frac{\alpha}{2}+o(1)]\},\end{aligned}\tag{2.42}$$

Thus, for any $m\le m_0=o(N)$

$$|\tilde{I}_m|\le\exp\{N[\max_U\{\min_V\mathcal{F}^D_{\lambda,h}(U,V)+\alpha\log U\}-\frac{\alpha}{2}\log\alpha+\frac{\alpha}{2}+o(1)]\}.$$

Hence,

$$P_N\le\exp\{N[\max_U\min_V\{\mathcal{F}^D_{\lambda,h}(U,V)+\alpha\log U\}-\frac{\alpha}{2}\log\alpha+\frac{\alpha}{2}+o(1)]\}.$$

Therefore, on the basis of Lemma 1, we have

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\log\langle\prod_{k=1}^{[\delta N]}\theta(\tilde{x}_k-a_1)\prod_{k=1+[\delta N]}^{N}\theta(\tilde{x}_k-a_2)\rangle\\ \le\max_U\min_V\{\mathcal{F}^D_{\lambda,h}(U,V)\}+\alpha\log U\}-\frac{\alpha}{2}\log\alpha+\frac{\alpha}{2}+o(1).\end{aligned}$$

We get the conclusions of Theorem 1, after taking the limits $\lambda\to 0$ and then $h\to 0$.

Proof of Theorem 2
To prove Theorem 2 let us show that if α is small enough to satisfy the condition

$$e^{-\frac{1}{2\alpha}}<\alpha^4,\tag{2.43}$$

then

$$\begin{aligned}\max_U\min_V\mathcal{F}^D_0(U,V;\alpha,\delta,0,0)&\le\log H(\frac{a^*_1-\alpha}{\sqrt{\alpha}})+(1-\delta)\log H(\frac{a^*_2-\alpha}{\sqrt{\alpha}})\\ &+\frac{\alpha}{2}\log\alpha-\frac{\alpha}{2}+O(\delta^2\alpha^{-3})+O(e^{-1/\alpha})\\ &=\mathcal{F}_0(\sqrt{\alpha},\sqrt{\alpha};\alpha,\delta,0,0)+O(\delta^2\alpha^{-3})+O(e^{-1/\alpha}).\end{aligned}\tag{2.44}$$

By virtue of the condition $\delta<<\alpha^3\log\alpha^{-1}$, we get then the statement (1.28) of Theorem 2.

We start, proving (2.44) for $U>2\sqrt{\alpha}$.

Proposition 2. *If $U>2\sqrt{\alpha}$, and $V(U)$ is defined by condition (2.31), then $\sqrt{\alpha}\le V(U)\le U$.*

On the basis of Proposition 2, we get

$$\begin{aligned}\mathcal{F}_0^D(U,V(U);\alpha,\delta,0,0) &\le \alpha\log U - V(U)U + \frac{1}{2}(V(U))^2 \\ &\le \alpha\log U - \sqrt{\alpha}U + \frac{\alpha}{2} \le \alpha\log 2\sqrt{\alpha} - 2\alpha + \frac{\alpha}{2}.\end{aligned} \tag{2.45}$$

Here the first inequality is due to $\log H(x) \le 0$, while the second and the third follow from Proposition 2. But, using the asymptotic formulae

$$\begin{aligned}H(x) &= \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}(1+O(1/x^2))\ (x>>1), \\ H(x) &= 1+\frac{1}{x\sqrt{2\pi}}e^{-x^2/2}(1+O(1/x^2))\ (x<<-1)\end{aligned} \tag{2.46}$$

and condition $\delta << \alpha^3\log\alpha^{-1}$, it is easy to get that the r.h.s. of (2.44) is

$$\mathcal{F}_0(\sqrt{\alpha},\sqrt{\alpha};\alpha,\delta,0,0) \sim \alpha\log\sqrt{\alpha} - \frac{\alpha}{2} + o(\alpha^2) > \alpha\log 2\sqrt{\alpha} - 2\alpha + \frac{\alpha}{2}. \tag{2.47}$$

This inequality and (2.45) prove (2.44) for $U > 2\sqrt{\alpha}$.

Now let us check (2.44) for $U < 0.5\sqrt{\alpha}$. To this end let us write equation for $V(U)$ which follows from (2.31).

$$U = V + \delta A(\frac{\alpha+1-2\delta}{U} - V) + (1-\delta)A(\frac{\alpha-(1-2\delta)}{U} - V), \tag{2.48}$$

where function $A(x)$ is defined by (1.19). By using asymptotic formulae

$$A(x) = x(1+O(1/x))\ (x>>1), \quad A(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}(1+O(1/x))\ (x<<-1), \tag{2.49}$$

we get that in this case

$$V(U) = U + o(\alpha^2).$$

Therefore

$$\begin{aligned}\mathcal{F}_0^D(U,V(U);\alpha,\delta,0,0) &\le \alpha\log U - V(U)U + \frac{1}{2}(V(U))^2 \\ &\le \alpha\log U - \frac{U^2}{2} \le \alpha\log 0.5\sqrt{\alpha} - \frac{\alpha}{8}.\end{aligned} \tag{2.50}$$

Now, using again (2.47), we obtain (2.44) for $U \le 0.5\sqrt{\alpha}$.

Now we are left to prove (2.44) for $0.5\sqrt{\alpha} \le U \le 2\sqrt{\alpha}$. Let us prove first, that for those U the function $D(U,V(U))$ defined by (1.20) is positive. To this end we use again asymptotic formulae (2.49). Then we get

$$\begin{aligned}A_1(U,V(U)) &= U^{-2} + o(\alpha^2) = O(\alpha^{-1}), \\ A_2(U,V(U)) &= O(\alpha^{-1/2}e^{-1/8\alpha}) = O(\sqrt{\alpha}).\end{aligned}$$

Here in the last equality we have used (2.43). Using these estimates, it is easy to obtain that $D(U,V(U)) > 0$ and therefore for $0.5\sqrt{\alpha} \le U \le 2\sqrt{\alpha}$,

$$\mathcal{F}_0^D(U,V(U);\alpha,\delta,0,0) = \min_V \mathcal{F}_0(U,V;\alpha,\delta,0,0).$$

But

$$\begin{aligned}&\max_U \min_V \mathcal{F}_0(U,V;\alpha,\delta,0,0) \le \max_U \mathcal{F}_0(U,U;\alpha,\delta,0,0)\\ &= \max_U\{\alpha \log U - \frac{U^2}{2} + \delta \log H(\frac{a_1^*}{U} - U) + (1-\delta)\log H(\frac{a_2^*}{U} - U)\}.\end{aligned} \tag{2.51}$$

Taking the derivative of the r.h.s. of (2.51) with respect to U we get:

$$\begin{aligned}&\frac{\partial}{\partial U}\mathcal{F}_0(U,U;\alpha,\delta,0,0) =\\ &\frac{\alpha}{U} - U + \delta(\frac{a_1^*}{U^2}+1)A(\frac{a_1^*}{U} - U) + (1-\delta)(\frac{a_2^*}{U^2}+1)A(\frac{a_2^*}{U} - U).\end{aligned} \tag{2.52}$$

Using asymptotic formulae (2.49) we get the equation for U^* which is the maximum point of the r.h.s. of (2.51):

$$\frac{\alpha}{U^*} - U^* + O(\frac{\delta}{\alpha^{3/2}}) + O(e^{-1/2\alpha}) = 0,$$

so

$$U^* = \sqrt{\alpha} + O(\frac{\delta}{\alpha^{3/2}}) + O(e^{-1/2\alpha}).$$

But since $\frac{d}{dU}(\alpha \log U - \frac{1}{2}U^2)\Big|_{U=\sqrt{\alpha}} = 0$, the Taylor expansion for this function starts from the term $(U-\sqrt{\alpha})^2$ and we get

$$\mathcal{F}_0(U^*,U^*;\alpha,\delta,0,0) = \mathcal{F}_0(\sqrt{\alpha},\sqrt{\alpha};\alpha,\delta,0,0) + O(\delta^2\alpha^{-3}) + O(e^{-1/\alpha}).$$

Hence, we have proved (2.44) and so (1.28) is proven.

Now one can easily derive the estimate for $P_N^*(\delta,\alpha)$ from the inequality

$$P_N^*(\delta,\alpha) \le C_N^{[\delta N]} P_N(0,0),$$

where $P_N(q,q')$ is defined by (1.12). Thus, we have finished the proof of Theorem 2.

Proof of Theorem 3
It is easy to see, that, if for some $\varepsilon > 0$ for any local minimum point $\boldsymbol{\sigma}^*$ in Ω_δ^1 we can find a point $\overline{\sigma}^{**}$ inside the ball B_δ^1, such that

$$\mathcal{H}(\boldsymbol{\sigma}^*) - \mathcal{H}(\boldsymbol{\sigma}^{**}) \ge \varepsilon^2 N, \tag{2.53}$$

then the event $\mathcal{A}$ takes place. Let $\{x_k^*\}_{k=1}^N$ be the effective field generated by the configuration $\boldsymbol{\sigma}^*$. Consider $I(\boldsymbol{\sigma}^*) \subset \{1,2,\ldots,N\}$ - the set of indexes $i_1,\ldots,i_{[N\delta]}$ such that $\sigma_i^*\tilde{\xi}_i^1 = -1$. Assume that the number N_ε of indexes $i \in I(\boldsymbol{\sigma}^*)$ for which $x_k^* \le -(\frac{1}{2}+\alpha)\varepsilon$, is larger then εN (we denote the set of these indexes by $I_\varepsilon(\boldsymbol{\sigma}^*)$). Then consider the point $\boldsymbol{\sigma}^{**}$, which differ from $\boldsymbol{\sigma}^*$ in the components with $[\varepsilon N]+1$ first indexes $i \in I_\varepsilon(\boldsymbol{\sigma}^*)$, and coincides with $\boldsymbol{\sigma}^*$ in all the other components. Since we have changed only the components of $\boldsymbol{\sigma}^*$ with indexes $i \in I_\varepsilon(\boldsymbol{\sigma}^*) \subset I(\boldsymbol{\sigma}^*)$, $\boldsymbol{\sigma}^{**} \in B^1_\delta$. On the other hand,

$$\begin{aligned} \mathcal{H}(\boldsymbol{\sigma}^*) - \mathcal{H}(\boldsymbol{\sigma}^{**}) = \frac{1}{2}(\tilde{\mathbf{J}}^0(\boldsymbol{\sigma}^{**} - \boldsymbol{\sigma}^*), (\boldsymbol{\sigma}^{**} + \boldsymbol{\sigma}^*)) = \\ -2 \sum_{i \in I_\varepsilon(\boldsymbol{\sigma}^*)} x_i^* + \frac{1}{2}(\tilde{\mathbf{J}}^0(\boldsymbol{\sigma}^{**} - \boldsymbol{\sigma}^*), (\boldsymbol{\sigma}^{**} - \boldsymbol{\sigma}^*)) \\ \geq (1+2\alpha)\varepsilon^2 N - \frac{\alpha}{2}((\boldsymbol{\sigma}^{**} - \boldsymbol{\sigma}^*), (\boldsymbol{\sigma}^{**} - \boldsymbol{\sigma}^*)) \\ \geq (1+2\alpha)\varepsilon^2 N - 2\alpha\varepsilon^2 N \geq \varepsilon^2 N, \end{aligned} \tag{2.54}$$

where $\tilde{\mathbf{J}}^0$ is defined by (1.2) with zero diagonal elements and we have used the inequality $\tilde{\mathbf{J}}^0 + \alpha \mathbf{I} = \tilde{\mathbf{J}} \geq 0$.

So, we have proved that

$$\mathcal{A} \supset \cup_{\varepsilon > 0} \mathcal{B}_\varepsilon, \tag{2.55}$$

where $\mathcal{B}_\varepsilon$ denotes the event, that for any extreme point $\boldsymbol{\sigma}^* \in \Omega^1_\delta$, the number N_ε of indexes in the set $I_\varepsilon(\boldsymbol{\sigma}^*)$ is larger then εN. Hence,

$$\overline{\mathcal{A}} \subset \cap_{\varepsilon > 0} \overline{\mathcal{B}}_\varepsilon, \quad \mathrm{Prob}(\overline{\mathcal{A}}) \leq \inf_{\varepsilon > 0} \mathrm{Prob}(\overline{\mathcal{B}}_\varepsilon \cap \mathcal{K}_{\tilde{\varepsilon}}) + \mathrm{Prob}\{\overline{\mathcal{K}}_{\tilde{\varepsilon}}\}, \tag{2.56}$$

where the event $\mathcal{K}_{\tilde{\varepsilon}}$ means that inequalities (1.7) hold. Let us note now, that $\overline{\mathcal{B}}_\varepsilon$ corresponds to the event, that there exists a local minimal point $\boldsymbol{\sigma}^* \in \Omega^1_\delta$, such that $N_\varepsilon \leq N\varepsilon$. Thus,

$$\mathrm{Prob}(\overline{\mathcal{B}}_\varepsilon \cap \mathcal{K}_{\tilde{\varepsilon}}) \leq \sum_{k=0}^{[\varepsilon N]} C_N^{[\delta N]} C_{[\delta N]}^k \mathrm{Prob}(\mathcal{B}^0_{\varepsilon,k} \cap \mathcal{K}_{\tilde{\varepsilon}}), \tag{2.57}$$

where $\mathcal{B}^0_{\varepsilon,k}$ denotes the event, that the point $\boldsymbol{\sigma}^{(1,\delta)}$ of the form (1.14) is a local minimal point in Ω^1_δ, and $\tilde{x}^0_i \leq -(\frac{1}{2} + \alpha)\varepsilon$ for $i = 1, \ldots, k$. Taking into account that under condition (1.7) the necessary condition for $\boldsymbol{\sigma}^{(1,\delta)}$ to be a minimum point is (1.9), we obtain that for $k \neq 0$

$$\begin{aligned} \mathrm{Prob}(\mathcal{B}^0_{\varepsilon,k} \cap \mathcal{K}_{\tilde{\varepsilon}}) \leq \mathrm{Prob}\{\tilde{x}^0_i \geq -(\frac{1}{2} + \alpha)\varepsilon, \ i = k+1, \ldots, [\delta N]; \\ \tilde{x}^0_j \geq -\tilde{\varepsilon}, \ j = [\delta N]+1, \ldots, N\} = P_{N,k}(-(\frac{1}{2} + \alpha)\varepsilon, -\tilde{\varepsilon}). \end{aligned} \tag{2.58}$$

And for $k = 0$

$$\mathcal{B}^0_{\varepsilon,0} \cap \mathcal{K}_{\tilde{\varepsilon}} \subset (\cap_{i=1}^{[\delta N]} \mathcal{A}^0_i(-(\frac{1}{2} + \alpha)\varepsilon) \cap_{j=[\delta N]+1}^N \mathcal{A}^0_j(-\tilde{\varepsilon})) \cup (\cup_{q > -\varepsilon(0.5+\alpha)} \mathcal{C}(\tilde{q})), \tag{2.59}$$

where $\mathcal{A}^0_j(\tilde{q})$ is defined by (1.10) and

$$\mathcal{C}(q) \equiv \{ \min_{i=1,\ldots,[\delta N]} \tilde{x}^0_i \geq q, \min_{j=[\delta N]+1,\ldots,N} \tilde{x}^0_j = -q - \tilde{\varepsilon}\}. \tag{2.60}$$

But it is easy to see that for any $\Delta > 0$, if we denote

$$\mathcal{A}(q, -q-\Delta) \equiv \cap_{i=1}^{[\delta N]} \mathcal{A}^0_i(q) \cap_{j=[\delta N]+1}^N \mathcal{A}^0_j(-q-\Delta-\tilde{\varepsilon}),$$

then

$$\begin{aligned} \cup_{0 \leq t \leq 1} \mathcal{C}(q + t\Delta) \subset \mathcal{A}(q, -q-\Delta-\tilde{\varepsilon}) \Rightarrow \\ \mathrm{Prob}\{\cup_{0 \leq t \leq 1} \mathcal{C}(q + t\Delta)\} \leq P_N(q, -q-\Delta-\tilde{\varepsilon}). \end{aligned} \tag{2.61}$$

To have an upper bound for the value of q which we need to consider we use

Proposition 3. *For any positive* $\alpha \le 0.113$ $\delta \le 0.6\alpha^2$ *there exists* $q_0(\alpha,\delta)$*, such that for any* $\tilde d > 0$

$$Prob\{\cup_{q>q_0+\tilde d}\mathcal{C}(q)\} \le \exp\{-NC_{\tilde d}\},$$

where $C_{\tilde d} > C^*(\delta)$ *with* $C^*(\delta)$ *defined in (1.31).*
For $\alpha \le 0.113$*,* $\delta \le 0.00645$ *and* $\delta \le 0.6\alpha^2$ $\quad q_0(\alpha,\delta) \le 0.13$*.*

On the basis of this proposition, we can restrict ourselves by $0 \le q \le q_0 + \tilde d$ and, using (2.59)-(2.61), write

$$\begin{aligned}\mathrm{Prob}\{\mathcal{B}\cap\mathcal{K}_{\tilde\varepsilon}\} &\le P_N(-(\frac{1}{2}+\alpha)\varepsilon,-\tilde\varepsilon) + \sum_{l=1}^{M} P_N(l\Delta,-\tilde\varepsilon-(l+1)\Delta)\\ &\le P_N(-(\frac{1}{2}+\alpha)\varepsilon,\tilde\varepsilon) + M \max_{0\le q\le q_0+\tilde d}\tilde P_N(q,-q-\Delta-\tilde\varepsilon) + e^{-NC_{\tilde d}},\end{aligned} \tag{2.62}$$

where $M = \frac{q_0+\tilde d+\varepsilon(0.5+\alpha)}{\Delta}$. Now, using Theorem 1, we get from (2.56), (2.57) and (2.62)

$$\begin{aligned}\mathrm{Prob}(\overline{\mathcal{A}}\cap\mathcal{K}_{\tilde\varepsilon}) \le \exp\{-NC_{\tilde d}\}\\ (M+1)C_N^{[\delta N]}C_{[\delta N]}^{[\varepsilon N]}(\exp\{N[C(\alpha,\delta,\tilde\varepsilon,\varepsilon,\Delta)+o(1)]\},\end{aligned} \tag{2.63}$$

where

$$\begin{aligned}C(\alpha,\delta,\tilde\varepsilon,\varepsilon,\Delta) = \max[&\max_{0\le\delta_1\le\varepsilon}\max_U \mathcal{F}_1^D(U;\alpha,\delta,\delta_1,-(\frac{1}{2}+\alpha)\varepsilon,-\tilde\varepsilon) - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2};\\ &\max_U\min_V \mathcal{F}_0^D(U,V;\alpha,\delta,-(\frac{1}{2}+\alpha)\varepsilon,-\tilde\varepsilon) - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2};\\ &\max_{q>\varepsilon(o.5+\alpha)}\max_U\min_V \mathcal{F}_0^D(U,V;\alpha,\delta,q,-q-\Delta-\tilde\varepsilon) - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2}].\end{aligned}$$

Since $\mathcal{F}_0^D$ and $\mathcal{F}_1^D$ are continuous with respect to q, q', δ_1, we get for $\Delta,\varepsilon\to 0$

$$\mathrm{Prob}(\overline{\mathcal{A}}\cap\mathcal{K}_{\tilde\varepsilon}) \le \exp\{N[C(\alpha,\delta,\tilde\varepsilon,\tilde d)+o(1)]\} + \exp\{-N(C_{\tilde d}-C^*(\delta))\}, \tag{2.64}$$

where

$$C(\alpha,\delta,\tilde d,\tilde\varepsilon) = \max_{0\le q\le q_0+\tilde d}\max_U\min_V\{\mathcal{F}_0^D(U,V;\alpha,\delta,q,-q-\tilde\varepsilon) - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2} + C^*(\delta)\}, \tag{2.65}$$

and therefore

$$\begin{aligned}\mathrm{Prob}(\overline{\mathcal{A}}) &\le \exp\{N[C(\alpha,\delta,\tilde\varepsilon,\tilde d)+o(1)]\} + \exp\{-N(C_{\tilde d}-C^*(\delta))\}\\ &+\mathrm{Prob}\{\overline{\mathcal{K}}_{\tilde\varepsilon}\} \le \exp\{N[C(\alpha,\delta,\tilde\varepsilon,\tilde d)+o(1)]\}\\ &+\exp\{-N(C_{\tilde d}-C^*(\delta))\} + \exp\{-\mathrm{const}\,N\tilde\varepsilon^2\}.\end{aligned} \tag{2.66}$$

Since $(C_{\tilde d} - C^*(\delta)) > 0$ for all $\tilde d > 0$, we conclude, that if for some $\delta > 0$ $C(\alpha,\delta,0,0) < 0$, then we always can choose $\tilde d$ and $\tilde\varepsilon$ small enough to provide that all the exponents in the r.h.s. of (2.66) are negative. Thus, we obtain the statement of Theorem 3.

Proposition 4. *Consider the functions*

$$\begin{aligned}\Phi(U,q,\alpha,\delta) &\equiv \min_V\{\mathcal{F}_0(U,V;\alpha,\delta,q,-q)-\frac{\alpha}{2}\log\alpha+\frac{\alpha}{2}+C^*(\delta)\}\\ \Phi_0(q,\alpha,\delta) &\equiv \max_U \Phi(U,q,\alpha,\delta)\equiv \Phi(U(q,\alpha,\delta),q,\alpha,\delta).\end{aligned} \tag{2.67}$$

If for some $0.071 \le \alpha_1 \le \alpha_2 \le \alpha_c$, $0.0035 \le \delta \le \delta_c = 0.00778$

$$\Phi_0(0,\alpha_2,\delta)<0, \quad \frac{\partial\Phi}{\partial q}(U_2,0,\alpha_2,\delta)<0, \quad \frac{\partial\Phi}{\partial\alpha}(U_1,0,\alpha_2,\delta)>0, \tag{2.68}$$

then $\Phi_0(q,\alpha,\delta) < 0$ *for any* $\alpha_1 \le \alpha \le \alpha_2$ *and* $0 \le q \le q_0$. *Here* $U_1 = U(0,\alpha_1,\delta) < U_2 = U(q_0,\alpha_2,\delta)$.

If also $\delta \le k_c\alpha^2$ ($k_c \equiv \frac{\delta_c}{\alpha_c^2}$) *and*

$$\max_{U\le\sqrt{\alpha}}\min_V \mathcal{F}_0^D(U,V;\alpha,\delta)+C^*(\delta)-\frac{\alpha}{2}\log\alpha+\frac{\alpha}{2}<0, \tag{2.69}$$

then $C(\alpha,\delta,0,0)$ *defined by (2.65) is negative.*

From (1.29) it is easy to see, that to find α_c and δ_c we should study the field of parameters α, δ where $\Phi_0(0,\alpha,\delta) < 0$. Let us fix for the moment α and study the behaviour of the function $\Phi_0(0,\alpha,\delta)$ as a function of δ. We find, that it is negative for $0 \le \delta \le \delta_1(\alpha)$ and $\delta_2(\alpha) \le \delta \le \delta_3(\alpha)$. But for $0 \le \delta \le \delta_1(\alpha)$ $C(\alpha,\delta,0,0)$ defined by (2.65) cannot be negative, because if it is so, then according to Theorem 3, there exists a minimum point inside the ball $B^1_{\delta_1}$. But by the virtue of Theorem 1, the probability to have the minimum point in Ω^1_δ ($\delta < \delta_1$) vanishes, as $N \to \infty$, because $\Phi_0(0,\alpha,\delta) < 0$. Thus we should study $\delta_2(\alpha) \le \delta \le \delta_3(\alpha)$. When α increases, $|\delta_3(\alpha)-\delta_2(\alpha)|$ decreases and for $\alpha=\alpha_c$ $\delta_3(\alpha_c)=\delta_2(\alpha_c)=\delta_c$. Then evidently

$$\Phi_0(0,\alpha_c,\delta_c)=0, \quad \frac{\partial\Phi_0}{\partial\delta}(0,\alpha_c,\delta_c)=0.$$

So we find from these equations, that $\alpha_c = 0.11326...$, $\delta_c = 0.00777...$ Unfortunately, for this (α_c,δ_c) condition (2.69) is not fulfilled. So we take a bit smaller $\alpha = 0.113$ and $\delta = 0.00645$, for wich (2.69) is fulfilled. Then, using (2.68), we obtain the statement of Theorem 3 for all $0.071 \le \alpha \le 0.113$ in three steps:
(1) $0.1105 \le \alpha \le 0.113$, $\delta = 0.00645$;
(2) $0.095 \le \alpha \le 0.1105$, $\delta = 0.0042$;
(3) $0.071 \le \alpha \le 0.095$, $\delta = 0.0035$.

For $\alpha \le 0.071$ the statement of Theorem 3 follows from the result of [L].

3. Auxiliary Results

Proof of Lemma 1.
At the first step we check that, if $\tilde{x}_k$ are defined by relations (1.14), then

$$\langle\theta(\tilde{x}_k-(a_k+N^{1/2+d}))\rangle \le e^{-\text{const}\,N^{1+2d}}.$$

To this end we use the Chebyshev inequality, according to which

$$\langle\theta(\tilde{x}_k-(a_k+N^{1/2+d})\rangle \le \min_{\tau>0}\langle\exp\{\tau\tilde{x}_k-\tau(a_k+N^{1/2+d})\}\rangle$$

$$=\min_{\tau>0}e^{-\tau(a_k+N^{1/2+d})}\prod_{\mu=1}^{p}\langle\exp\{\frac{\tau}{N}\sum_{j=1}^{N}\xi_k^\mu\xi_j^\mu\}\rangle=\min_{\tau>0}e^{-\tau(a_k+N^{1/2+d})}(\cosh\frac{\tau}{N})^{(pN)}$$

$$\le\min_{\tau>0}\exp\{-\tau(a_k+N^{1/2+d})+\alpha\frac{\tau^2}{2}\}\le e^{-\,\mathrm{const}\,N^{1+2d}}.$$

Thus,

$$\begin{aligned}\langle\prod_{k=1}^{N}\theta(\tilde{x}_k-a_k)\rangle&=\langle\prod_{k=1}^{N}\theta(\tilde{x}_k-a_k)(\theta(a_k+N^{1/2+d}-\tilde{x}_k)\\&+\theta(\tilde{x}_k-(a_k+N^{1/2+d})))\rangle\le\langle\prod_{k=1}^{N}\theta(\tilde{x}_k-a_k)\theta(a_k+N^{1/2+d}-\tilde{x}_k)\rangle\\&2^N\sum_{k=1}^{N}\langle\theta(\tilde{x}_k-(a_k+N^{1/2+d}))\rangle\\&\le\langle\prod_{k=1}^{N}\theta(\tilde{x}_k-a_k)\theta(a_k+N^{1/2+d}-\tilde{x}_k)\rangle+e^{-\,\mathrm{const}\,N^{1+2d}}.\end{aligned}\tag{3.1}$$

Consider

$$D_{\lambda,\varepsilon_N^*}(x^1,...,x_N)\equiv\frac{\exp\{-\frac{1}{2}\sum_{j,k=1}^{N}(\lambda\mathbf{I}+\varepsilon_N^*l_N^{-1}\mathbf{J})_{jk}^{-1}x_jx_k-\frac{1}{2}\sum_{j,k=1}^{N}\varepsilon_N^*l_N^{-1}J_{jk}\}}{l_N^{p/2}(2\pi)^{N/2}\det^{1/2}\{\lambda\mathbf{I}+\varepsilon_N^*l_N^{-1}\mathbf{J}\}},$$

where $\mathbf{I}$ is a unit matrix and $\mathbf{J}$ is a matrix with entries

$$J_{jk}=\frac{1}{N}\sum_{\mu=1}^{p}\xi_j^\mu\xi_k^\mu.$$

We study the composition $D_{\lambda,\varepsilon_N^*}*\prod\chi_{N,h}$ of this function with the product of $\chi_{N,h}(x_k)$ (recall that $(f*g)(\overline{x})\equiv\int f(\overline{x}-\overline{x}')g(\overline{x}')d\overline{x}'$). Let us check, that for $0\le x_k\le N^{1/2+d}$

$$\begin{aligned}\prod_{k=1}^{N}\theta(x_k)\theta(N^{1/2+d}-x_k)&\le(1-e^{-h^2/2\lambda})^{-N}l_N^{p/2}\\\cdot(D_{\lambda,\varepsilon_N^*}*\prod\chi_{N,h})(x_1,...,x_N)&\det{}^{1/2}\{\mathbf{I}+\frac{\varepsilon_N^*}{\lambda l_N}\mathbf{J}\}\exp\{\frac{\varepsilon_N^*}{2l_N}\sum_{j,k=1}^{N}J_{jk}\}.\end{aligned}\tag{3.2}$$

Indeed, by definition of composition

$$(D_{\lambda,\varepsilon_N^*}*\prod\chi_{N,h})(x_1,...,x_N)\det{}^{1/2}\{\lambda\mathbf{I}+\frac{\varepsilon_N^*}{l_N}\mathbf{J}\}\exp\{\frac{\varepsilon_N^*}{2l_N}\sum_{j,k=1}^{N}J_{jk}\}l_N^{p/2}$$

$$=\frac{1}{(2\pi)^{N/2}}\int\exp\{-\frac{1}{2}\sum_{j,k=1}^{N}(\lambda\mathbf{I}+\frac{\varepsilon_N^*}{l_N}\mathbf{J})_{jk}^{-1}(x_j-x_j')(x_k-x_k')\}\prod_{k=1}^{N}\chi_{N,h}(x_k')dx_k'$$

$$\begin{aligned} &\geq \frac{1}{(2\pi)^{N/2}} \int \exp\{-\frac{1}{2\lambda}\sum_{k=1}^{N}(x_k - x_k')^2\} \prod_{k=1}^{N} \chi_{N,h}(x_k')dx_k' \\ &\geq (\frac{1}{\sqrt{2\pi}} \int dx' \exp\{-\frac{(x-x')^2}{2\lambda}\}\chi_{N,h}(x'))^N. \end{aligned} \tag{3.3}$$

But for $x \in (0, N^{1/2+d})$

$$\begin{aligned} I_1 &= \int dx' \exp\{-\frac{(x-x')^2}{2\lambda}\}(1-\chi_{N,h}(x')) \\ &= \int_{-\infty}^{-h} \exp\{-\frac{(x-x')^2}{2\lambda}\}dx' + \int_{N^{1/2+d}+h}^{\infty} \exp\{-\frac{(x-x')^2}{2\lambda}\}dx' \\ &\leq \int_{-\infty}^{-h} \exp\{-\frac{(x')^2}{2\lambda}\}dx' + \int_{h}^{\infty} \exp\{-\frac{(x')^2}{2\lambda}\}dx' \leq \frac{2\lambda}{h}e^{-\frac{h^2}{2\lambda}}. \end{aligned}$$

So for $h > (\frac{2\lambda}{\pi})^{1/2}$

$$\frac{1}{\sqrt{2\pi}} \int dx' \exp\{-\frac{(x-x')^2}{2\lambda}\}\chi_{N,h}(x') = (\sqrt{\lambda} - \frac{I_1}{\sqrt{2\pi}}) \geq \sqrt{\lambda}(1 - e^{-h^2/2\lambda}).$$

Thus, we have proved (3.2) for $x_k \in (0, N^{1/2+d})$. Besides, using the inequality $\log(1+x) \leq x$, we get

$$\begin{aligned} \det{}^{1/2}\{\mathbf{I} + \frac{\varepsilon_N^*}{\lambda l_N}\mathbf{J}\} &= \exp\{\frac{1}{2}\sum_{\lambda_i \in \sigma(\mathbf{J})} \log(1 + \frac{\varepsilon_N^*}{\lambda l_N}\lambda_i\} \\ \leq \exp\{\frac{1}{2}\sum_{\lambda_i \in \sigma(\mathbf{J})} \frac{\varepsilon_N^*}{\lambda l_N}\lambda_i\} &= \exp\{\frac{\varepsilon_N^*}{2\lambda l_N}\mathrm{Tr}\mathbf{J}\} = \exp\{\frac{\varepsilon_N^*\alpha}{2\lambda l_N}N\}. \end{aligned} \tag{3.4}$$

Here $\sigma(\mathbf{J})$ is a spectrum of the matrix $\mathbf{J}$.

Therefore, it follows from (3.2) and (3.4) that for $x_k \in (0, N^{1/2+d})$

$$\begin{aligned} &\prod_{k=1}^{N} \theta(x_k)\theta(N^{1/2+d} - x_k) \leq (1 - e^{-h^2/2\lambda})^{-N} l_N^{p/2} \\ &\cdot \exp\{\frac{\varepsilon_N^*\alpha N}{2\lambda l_N}\}(D_{\lambda,\varepsilon_N^*} * \prod \chi_{N,h})(x_1, ..., x_N) \exp\{\frac{\varepsilon_N^*}{2l_N}\sum_{j,k=1}^{N} J_{jk}\}. \end{aligned} \tag{3.5}$$

But for all the other values of $\{x_k\}$ the l.h.s. of this inequality is zero, while the r.h.s. is positive, so we can extend (3.5) to all $\{x_k\} \in \mathbf{R}^N$.

Besides, according to the Chebyshev inequality,

$$\begin{aligned} \mathrm{Prob}\{\sum J_{jk} \leq N(\varepsilon_N^*)^{-1/2}\} &\leq \min_{\tau > O} e^{-\tau(\varepsilon_N^*)^{-1/2}N} E\{e^{\tau\sum J_{jk}}\} \\ &= \min_{\tau > O} e^{-\tau(\varepsilon_N^*)^{-1/2}N} E^p\{\exp\{\tau\sum\frac{1}{N}\xi_j^1\xi_k^1\}\} \\ &\leq \min_{1 > \tau > O} \exp\{-\tau(\varepsilon_N^*)^{-1/2}N - \frac{p}{2}\log(1-\tau)\} \\ &\leq \exp\{-\,\mathrm{const}\,(\varepsilon_N^*)^{-1/2}N\}. \end{aligned} \tag{3.6}$$

Here we have used the standard trick, valid for $\tau < 1$,

$$E\{\exp\{\tau\sum\frac{1}{N}\xi_j^1\xi_k^1\}\} = (2\pi)^{-1/2}E\{\int dx\exp\{-\sqrt{\tau}x\frac{1}{\sqrt{N}}\sum\xi_i^1 - \frac{x^2}{2}\}$$
$$= (2\pi)^{-1/2}\int dx(\cosh\frac{x\sqrt{\tau}}{\sqrt{N}})^N e^{-x^2/2} = (1-\tau)^{-1/2}(1+O(N^{-1})).$$

Therefore finally, on the basis (3.1), (3.5) and (3.6), we get

$$\begin{gathered}\langle\prod_{k=1}^N\theta(\tilde{x}_k - a_k)\rangle \leq e^{-N(\varepsilon_N^*)^{-1/2}}\text{const}\\ +\frac{e^{\text{const}\,N(\varepsilon_N^*)^{1/2}}l_N^{p/2}}{(1-e^{-h^2/2\lambda})^N}\langle(D_{\lambda,\varepsilon_N^*} * \prod_{k=1}^N\chi_{N,h})(\tilde{x}_1 - a_1,...,\tilde{x}_N - a_N)\rangle.\end{gathered} \tag{3.7}$$

Now to finish the proof of Lemma 1 we are left to find the Fourier transform $\hat{D}_{\lambda,\varepsilon_N^*}$ of the function $D_{\lambda,\varepsilon_N^*}$.

$$\hat{D}_{\lambda,\varepsilon_N^*}(\overline{\zeta}) = (2\pi)^{-N/2}\int d\overline{x}e^{i(\overline{x},\overline{\zeta})}D_{\lambda,\varepsilon_N^*}(\overline{x}) = l_N^{-p/2}\exp\{-\frac{\lambda}{2}(\overline{\zeta},\overline{\zeta})$$
$$-\frac{\varepsilon_N^*}{2l_N N}\sum_\mu(\sum_k\xi_k^\mu\zeta_k)^2 - \frac{\varepsilon_N^*}{2l_N N}\sum_\mu(\sum_k\xi_k^\mu)^2\}$$
$$= l_N^{-p/2}\exp\{-\frac{\lambda}{2}(\overline{\zeta},\overline{\zeta}) - \frac{\varepsilon_N^*}{2l_N}\sum_\mu((\tilde{u}^\mu)^2 + (\tilde{v}^\mu)^2)\},$$

where $\tilde{u}^\mu$ and $\tilde{v}^\mu$ are defined by (2.2). Then

$$\begin{gathered}\langle(D_{\lambda,\varepsilon_N^*} * \prod_{k=1}^N\chi_{N,h})(\tilde{x}_1 - a_1,...,\tilde{x}_N - a_N)\rangle\\ = (2\pi)^{-N}\int\prod_{k=1}^N d\zeta_k\hat{\chi}_{N,h}(\zeta_k)\exp\{-ia_k\zeta_k\}\cdot\langle\hat{D}_{\lambda,\varepsilon_N^*}(\overline{\zeta})\exp\{i\sum_{k=1}^N\zeta_k\tilde{x}_k\}\rangle\\ = l_N^{-p/2}(2\pi)^{-N}\int\prod_{k=1}^N d\zeta_k\hat{\chi}_{N,h}(\zeta_k)\exp\{-ia_k\zeta_k - \frac{\lambda}{2}\zeta_k^2\}\\ \cdot\prod_\mu\langle\exp\{-\frac{\varepsilon_N^*}{2l_N}(\tilde{u}^\mu)^2 + (\tilde{v}^\mu)^2 + i\tilde{u}\tilde{v}\}\rangle.\end{gathered} \tag{3.8}$$

Let us use the representation (cf. (2.3))

$$\langle\exp\{-\frac{\varepsilon_N^*}{2l_N}((\tilde{u}^\mu)^2 + (\tilde{v}^\mu)^2) + i\tilde{u}^\mu\tilde{v}^\mu\}\rangle$$
$$= \frac{l_N^{1/2}}{2\pi}\int du^\mu dv^\mu\langle\exp\{-\frac{\varepsilon_N^*}{2}((u^\mu)^2 + (v^\mu)^2) - il_N u^\mu v^\mu + iu^\mu\tilde{u}^\mu + iv^\mu\tilde{v}^\mu\}\rangle,$$

where we have taken into account, that by definition (see Lemma 1) $l_N = l_N^2 + (\varepsilon_N^*)^2$. Substituting this representation into (3.8), we get

$$\begin{aligned} &\langle (D_{\lambda,\varepsilon_N^*} * \prod_{k=1}^N \chi_{N,h})(\tilde{x}_1 - a_1, \dots, \tilde{x}_N - a_N)\rangle \\ &= (2\pi)^{-N-p} \int \prod_{k=1}^N d\zeta_k \hat{\chi}_{N,h}(\zeta_k) \exp\{-\frac{\lambda}{2}\zeta_k^2 - ia_k\zeta_k\} \prod_\mu \int du^\mu \\ &\cdot dv^\mu \exp\{-iu^\mu v^\mu - \frac{\varepsilon_N^*}{2}(u^\mu)^2 - \frac{\varepsilon_N^*}{2}(v^\mu)^2\} \prod_{k=1}^N \cos\frac{u^\mu\zeta_k + v^\mu}{\sqrt{N}} = P_N^1. \end{aligned} \tag{3.9}$$

Inequality (3.7) and this representation prove Lemma 1.

Proof of Lemma 2

Take $L = \frac{\pi}{6\varepsilon_N^2}$ and consider an intermediate functions:

$$\begin{aligned} &F_{cL}^{(m)}(\overline{u}_1, \overline{v}_1, \overline{u}_2, \overline{v}_2) \equiv \int_{-L}^{L} d\zeta_k \hat{\chi}_{N,h}(\zeta_k) e^{-\lambda\zeta_k^2/2 - ia\zeta_k} \\ &\cdot \prod_{\mu\le m} \cos\frac{u^\mu\zeta_k + w^\mu}{\sqrt{N}} \exp\{-\frac{1}{N}(\overline{u}_2, \overline{w}_2)\zeta_k - \frac{1}{2N}(\overline{u}_2, \overline{u}_2)\zeta_k^2\} \prod_{\nu>m}^p \cos\frac{w^\nu}{\sqrt{N}}; \\ &F_{NL}(\overline{u}_1, \overline{v}_1, \overline{u}_2, \overline{v}_2) \equiv \int_{-L}^{L} d\zeta_k \hat{\chi}_{N,h}(\zeta_k) e^{-\lambda\zeta_k^2/2 - ia\zeta_k} \prod_{\mu=1}^p \cos\frac{u^\mu\zeta_k + w^\mu}{\sqrt{N}}. \end{aligned} \tag{3.10}$$

Denote also $F_c^{(m)}$ by the same formula as $F_{cL}^{(m)}$ with $L = \infty$.

Then

$$\begin{aligned} R^{(m)} \equiv F_N - F^{(m)} = (F_N - F_{NL}) + (F_{NL} - F_{cL}^{(m)}) \\ +(F_{cL}^{(m)} - F_c^{(m)}) + (F_c^{(m)} - F^{(m)}). \end{aligned} \tag{3.11}$$

One could easily estimate $(F_N - F_{NL})$ by using the simple inequalities

$$|(F_N - F_{NL})(\overline{u}, \overline{w})| \le \frac{e^{\frac{(\overline{\lambda}_2, \overline{\lambda}_2)}{N}}}{2\pi} \int_{|\zeta_k|>L} e^{-\zeta_k^2/2\lambda} d\zeta_k \le e^{\frac{(\overline{\lambda}, \overline{\lambda})}{N}} e^{-\operatorname{const}\varepsilon_N^{-4}}. \tag{3.12}$$

Let us estimate $R_*^{(m)} \equiv F_{NL} - F_{cL}^{(m)}$. To this end we consider

$$f(\zeta_k) = \sum_{\nu>m} \log\cos\frac{u^\nu\zeta_k + w^\nu}{\sqrt{N}} + \frac{\zeta_k^2}{2}\tilde{U}^2 + \frac{\zeta_k}{N}(\overline{u}_2, \overline{w}_2)$$

and use the inequality

$$|e^{f(\zeta_k)} - e^{f(0)}| \le |f(\zeta_k) - f(0)|(|e^{f(\zeta_k)}| + |e^{f(0)}|).$$

Then, since $|\frac{\xi u^\nu}{\sqrt{N}}|, |\frac{v^\nu}{\sqrt{N}}| \le L\varepsilon_N^2 \le \frac{\pi}{6}$ and $|u^\nu|, |v^\nu|, |\lambda^\nu| \le \varepsilon_N\sqrt{N}$, we get

$$\begin{aligned}
&|f(\zeta_k) - f(0)| \le |\zeta_k||f'(\xi)| \\
&= |\zeta_k|| \sum_{\nu>m} [-\frac{u^\nu}{\sqrt{N}} \mathrm{tg} \frac{\xi u^\nu + w^\nu}{\sqrt{N}} + \xi\frac{(u^\nu)^2}{N} + \frac{u^\nu w^\nu}{N}]| \\
&\le |\zeta_k| \,\mathrm{const} \sum_{\nu>m} |\frac{u^\nu}{\sqrt{N}}||\frac{\xi u^\nu + w^\nu}{\sqrt{N}}|^3 \\
&\le \varepsilon_N^2 |\zeta_k| \,\mathrm{const}\, (\tilde{U}^2|\zeta_k|^3 + \frac{1}{N}\sum_{\nu>m}(|v^\nu|^2 + |\lambda^\nu|^2)).
\end{aligned} \tag{3.13}$$

To estimate $|e^{f(\zeta_k)}|$ we use the inequality, valid for $|\Re z| \le \frac{\pi}{2}$

$$\Re(\log\cos z + \frac{1}{2}z^2) \le \frac{1}{2}(\Im z)^2. \tag{3.14}$$

(The proof of this inequality is given at the end of the proof of Lemma 2.) It follows from (3.14) that

$$\begin{aligned}
&\Re f(\zeta_k) = \Re\{ \sum_{nu=m+1}^{p} [\log\cos\frac{\zeta_k u^\nu + w^\nu}{\sqrt{N}} + \frac{(\zeta_k u^\nu + w^\nu)^2}{2N} - \frac{(w^\nu)^2}{2N}]\} \\
&\le \sum_{\nu>m} \frac{(\Im\{\zeta_k u^\nu + w^\nu\})^2}{2N} - \sum_{\nu>m}\frac{\Re\{(w^\nu)^2\}}{2N} = -\frac{(\overline{v}_2, \overline{v}_2)}{2N} + \frac{(\overline{\lambda}_2, \overline{\lambda}_2)}{2N}.
\end{aligned} \tag{3.15}$$

Therefore we derive from (3.13) and (3.15) that

$$\begin{aligned}
&|\prod_{\nu>m}^{p} \cos\frac{u^\nu\zeta_k + w^\nu}{\sqrt{N}} - \exp\{-\frac{(\overline{u}_2, \overline{v}_2)\zeta_k}{N} - \frac{(\overline{u}_2, \overline{u}_2)\zeta_k^2}{2N}\} \prod_{\nu>m}\cos\frac{w^\nu}{\sqrt{N}}| \\
&= \exp\{-\frac{1}{N}(\overline{u}_2, \overline{v}_2)\zeta_k - \frac{1}{2N}(\overline{u}_2, \overline{u}_2)\zeta_k^2\}|e^{f(\zeta_k)} - e^{f(0)}| \\
&\le \mathrm{const}\,\varepsilon_N^2|\zeta_k|(\tilde{U}^2|\zeta_k|^3 + \frac{(\overline{v}_2, \overline{v}_2) + (\overline{\lambda}_2, \overline{\lambda}_2)}{N}) \\
&\cdot[\exp\{-\frac{\zeta_k^2\tilde{U}^2}{2} - \zeta_k\frac{(\overline{u}_2, \overline{v}_2)}{N} - \frac{(\overline{v}_2, \overline{v}_2)}{N} + \frac{(\overline{\lambda}_2, \overline{\lambda}_2)}{N}\} \\
&+ \exp\{-\frac{\zeta_k^2\tilde{U}^2}{2} - \zeta_k\frac{(\overline{u}_2, \overline{v}_2)}{N}\}\prod_{\nu>m}|\cos\frac{v^\nu + i\lambda^\nu}{\sqrt{N}}|].
\end{aligned} \tag{3.16}$$

Using inequality (3.14) for $|\cos\frac{v^\nu + i\lambda^\nu}{\sqrt{N}}|$ $(\nu > m)$, we get

$$\begin{aligned}
&|R_*^{(m)}(\overline{u}_1, \overline{v}_1, \overline{u}_2, \overline{v}_2 + i\overline{\lambda}_2)| \le \varepsilon_N^2 \int d\zeta_k(\tilde{U}^2|\zeta_k|^3 + \frac{(\overline{v}_2, \overline{v}_2) + (\overline{\lambda}_2, \overline{\lambda}_2)}{N})e^{-\lambda\zeta_k^2/2} \\
&\cdot \prod_{\mu\le m}|\cos\frac{u^\mu\zeta_k + w^\mu}{\sqrt{N}}|\exp\{-\frac{\zeta_k^2\tilde{U}^2}{2} - \zeta_k\frac{(\overline{u}_2, \overline{v}_2)}{N} - \frac{(\overline{v}_2, \overline{v}_2)}{2N} + \frac{(\overline{\lambda}_2, \overline{\lambda}_2)}{N}\} \\
&\le \varepsilon_N^2\,\mathrm{const}\,(1 + \frac{(\overline{v}_2, \overline{v}_2) + (\overline{\lambda}_2, \overline{\lambda}_2)}{N\sqrt{\tilde{U}^2 + \lambda}})\exp\{-\frac{(\overline{v}_2, \overline{v}_2)}{2N} + \frac{(\overline{u}_2, \overline{v}_2)^2}{N^2(\tilde{U}^2 + \lambda)} + \frac{(\overline{\lambda}, \overline{\lambda})}{N}\}.
\end{aligned} \tag{3.17}$$

Now to obtain the estimate of the form (2.24) we use (3.23) and the inequality

$$\frac{(\overline{v}_2,\overline{v}_2)}{2N} \le \frac{2(\tilde{U}^2+\lambda)}{\lambda}\exp\{\frac{\lambda(\overline{v}_2,\overline{v}_2)}{4N(\tilde{U}^2+\lambda)}\}.$$

Combining them with (3.17), we get

$$\begin{aligned} &|R_*^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u}_2,\overline{v}_2+i\overline{\lambda}_2)| \\ &\le \varepsilon_N^2 \,\mathrm{const}\,(\tilde{U}^2+\lambda)^{1/2}(1+\frac{(\overline{\lambda}_2,\overline{\lambda}_2)}{N})\exp\{-\frac{\lambda(\overline{v}_2,\overline{v}_2)}{4N(\tilde{U}^2+\lambda)}+\frac{(\overline{\lambda},\overline{\lambda})}{N}\}. \end{aligned} \tag{3.18}$$

To estimate $(F_{cL}^{(m)}-F_c^{(m)})$ we use again the inequality (3.14) for $|\cos\frac{v^\nu+i\lambda^\nu}{\sqrt{N}}|$ $(\nu>m)$.

$$\begin{aligned} &|F_{cL}^{(m)}(\overline{u},\overline{w})-F_c^{(m)}(\overline{u},\overline{w})| \le e^{\frac{(\overline{\lambda},\overline{\lambda})}{N}}e^{-\frac{(\overline{v}_2,\overline{v}_2)}{2N}} \\ &\cdot\int_{|\zeta_k|\ge L} d\zeta_k|\hat{\chi}_{N,h}(\zeta_k)|e^{-\lambda\zeta_k^2/2}\exp\{-\frac{1}{N}(\overline{u}_2,\overline{v}_2)\zeta_k-\frac{1}{2N}(\overline{u}_2,\overline{u}_2)\zeta_k^2\} \\ &\cdot\le e^{\frac{(\overline{\lambda},\overline{\lambda})}{N}}\int_{|\zeta_k|\ge L} d\zeta_k e^{-\lambda\zeta_k^2/2} \le \mathrm{const}\, e^{\frac{(\overline{\lambda},\overline{\lambda})}{N}}e^{-\mathrm{const}\,\varepsilon_N^{-4}}. \end{aligned} \tag{3.19}$$

Thus, we are left to estimate the difference

$$\begin{aligned} &F_c^{(m)}(\overline{u}_1,\overline{w}_1,\overline{u}_2,\overline{w}_2)-F^{(m)}(\overline{u}_1,\overline{w}_1,\overline{u}_2,\overline{w}_2) \\ &=\langle H_{N,h,\tilde{U}}(\frac{a-i(\overline{u}_2,\overline{w}_2)-\frac{(\overline{u}_1,\overline{\xi}_1)}{\sqrt{N}}}{\sqrt{\tilde{U}^2+\lambda}})e^{\frac{i(\overline{v}_1,\overline{\xi}_1)}{\sqrt{N}}}\rangle(\prod_{\mu>m}\cos\frac{w^\nu}{\sqrt{N}}-e^{-\frac{(\overline{w}_2,\overline{w}_2)}{2N}}). \end{aligned} \tag{3.20}$$

The last multiplier here can be estimated by the same way as in (3.10)-(3.16). Then we get

$$\begin{aligned} &|\prod_{\mu>m}\cos\frac{w^\nu}{\sqrt{N}}-e^{-\frac{(\overline{w}_2,\overline{w}_2)}{2N}}| \\ &\le \mathrm{const}\,\varepsilon_N^2\frac{|(\overline{w}_2,\overline{w}_2)|}{N}\exp\{-\frac{(\overline{v}_2,\overline{v}_2)}{2N}+\frac{(\overline{\lambda}_2,\overline{\lambda}_2)}{N}\}. \end{aligned} \tag{3.21}$$

To estimate the first multiplier we use the bound $|H_{N,h,\tilde{U}}(a+ic)|\le e^{c^2/2}$. Thus,

$$\begin{aligned} &\langle H_{N,h,\tilde{U}}(\frac{a-i(\overline{u}_2,\overline{w}_2)-\frac{(\overline{u},\overline{\xi}_1)}{\sqrt{N}}}{\sqrt{\tilde{U}^2+\lambda}})e^{\frac{i(\overline{v}_1,\overline{\xi}_1)}{\sqrt{N}}}\rangle \\ &\le \exp\{\frac{(\overline{u}_2,\overline{v}_2)^2}{2N^2(\tilde{U}^2+\lambda)}\}\langle e^{\frac{(\overline{\lambda}_1,\overline{\xi}_1)}{\sqrt{N}}}\rangle \le \exp\{\frac{(\overline{\lambda}_1,\overline{\lambda}_1)}{N}+\frac{(\overline{u}_2,\overline{v}_2)^2}{2N^2(\tilde{U}^2+\lambda)}\}. \end{aligned}$$

By the same way as in (3.16)-(3.18) we can obtain now from (3.20) and (3.21) the bound of the form (2.24).

Now to finish the proof of Lemma 2 we are left to prove inequality (3.14). For $z=x+iy$ $(x,y\in\mathbf{R})$ by the simple algebraic transformations we get that (3.14) is equivalent to the inequality

$$\frac{1}{2}(\cosh 2y+\cos 2x)\le e^{2y^2-x^2} \tag{3.22}$$

Since $\cosh 2y \le e^{2y^2}$, to prove (3.22) it is enough to prove that

$$\cos 2x \le e^{2y^2}(2e^{-x^2} - 1),$$

which evidently follows from

$$\cos 2x \le (2e^{-x^2} - 1) \quad \Longleftrightarrow \quad \cos x \le e^{-x^2/2}.$$

Since the last inequality is valid for $|x| \le \frac{\pi}{2}$, we have proved (3.22) and so (3.14).
Lemma 2 is proven.

Proof of Lemma 3
We use (2.24) to estimate the integral

$$I'_{m,k} \equiv \int_{-\varepsilon_N\sqrt{N}}^{\varepsilon_N\sqrt{N}} d\overline{v}_2 e^{-il_N(\overline{u}_2,\overline{v}_2)} e^{-\frac{\varepsilon_N^*}{2}(\overline{v}_2,\overline{v}_2)} \cdot (F^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u}_2,\overline{v}_2))^{N-k}(R^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u}_2,\overline{v}_2))^k.$$

By using (2.10), which is evidently valid also for $H_{N,h,\tilde{U}}$ we get

$$\begin{aligned} |F^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u}_2,\overline{v}_2)| &\le \exp\{\frac{(\overline{u}_2,\overline{v}_2)^2}{2N^2(\tilde{U}^2+\lambda)} - \frac{(\overline{v}_2,\overline{v}_2)}{2N}\} \\ &\le \exp\{-\frac{\lambda(\overline{v}_2,\overline{v}_2)}{2N(\tilde{U}^2+\lambda)}\} \le \exp\{-\frac{\lambda(\overline{v}_2,\overline{v}_2)}{4N(\tilde{U}^2+\lambda)}\}. \end{aligned} \tag{3.23}$$

The second inequality here can be obtained if we observe that $\frac{(\overline{u}_2,\overline{v}_2)^2}{N^2(\tilde{U}^2+\lambda)} = \frac{\tilde{U}^2}{\tilde{U}^2+\lambda}(\mathbf{P}_u\overline{v}_2,\overline{v}_2)$, where $\mathbf{P}_u$ is the orthogonal projection operator on the unit vector $(\tilde{U})^{-1}N^{-1/2}\overline{u}_2$, and use the trivial inequality $\mathbf{I} - \frac{\tilde{U}^2}{\tilde{U}^2+\lambda}\mathbf{P}_u \ge \frac{\lambda}{\tilde{U}^2+\lambda}\mathbf{I}$. Note also, that we replace in (3.23) 2 in the denominator by 4 in order to have the same factor as in (2.24). Hence, on the basis of Lemma 2, we have

$$\begin{aligned} |I'_{m,k}| &\le \int_{-\varepsilon_N\sqrt{N}}^{\varepsilon_N\sqrt{N}} d\overline{v}_2 |(F^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u}_2,\overline{v}_2))^{N-k}(R^{(m)}(\overline{u}_1,\overline{v}_1,\overline{u}_2,\overline{v}_2))^k| \\ &\le e^{k\,\mathrm{const}}\,\varepsilon_N^{2k}(\tilde{U}^2+\lambda)^{k/2}\int_{-\varepsilon_N\sqrt{N}}^{\varepsilon_N\sqrt{N}} d\overline{v}_2 \exp\{-\frac{\lambda(\overline{v}_2,\overline{v}_2)}{4(\tilde{U}^2+\lambda)} - \varepsilon_N^*\frac{(\overline{v}_2,\overline{v}_2)}{2}\} \\ &+ e^{k\,\mathrm{const}}\,e^{-k\,\mathrm{const}\,\varepsilon_N^{-4}}\int d\overline{v}_2 \exp\{-\frac{\lambda(N-k)(\overline{v}_2,\overline{v}_2)}{4N(\tilde{U}^2+\lambda)} - \varepsilon_N^*\frac{(\overline{v}_2,\overline{v}_2)}{2}\} \\ &\le e^{N\,\mathrm{const}}\,(\tilde{U}^2+\lambda)^{p/2}\varepsilon_N^{2k} + e^{k\,\mathrm{const}}\,(\varepsilon_N^*)^{-p/2}e^{-k\,\mathrm{const}\,\varepsilon_N^{-4}}. \end{aligned} \tag{3.24}$$

Substituting estimate (3.24) in the expression for $I_{m,k}$ integrating over $\overline{u}_1,\overline{v}_1$, and $\tilde{U}$ we get finally

$$|I_{m,k}| \le (\int (\tilde{U}^2+\lambda)^{p/2}\tilde{U}^{p-m}e^{-N\varepsilon_N^* U^2/2}d\tilde{U})e^{N\,\mathrm{const}}\,(\varepsilon_N)^{2k} + e^{k\,\mathrm{const}}\,(\varepsilon_N^*)^{-p}e^{-k\,\mathrm{const}\,\varepsilon_N^{-4}}.$$

Using the Laplace method for the integration with respect to $\tilde{U}$ and taking into account that the second term in the r.h.s. here for $k > k_0$ is much smaller than the first one, we obtain the statement of Lemma 3.

Proof of Lemma 4
To prove (2.29) we use the variables $w^\mu = -i\lambda^\mu + t^\mu$, $(t^\mu \in \mathbf{R})$ $(\mu = 1, ..., m)$ and $w^\nu = -i\frac{u^\nu}{\tilde U}V + t^\nu$, $(t^\nu \in \mathbf{R})$ $(\nu = m_0 + 1, ..., p - n)$ defined in (2.26) and estimate

$$\begin{aligned}
&|G_{m,k,n}(V, \overline{u}_1, \overline{\lambda}_1, \overline{u}_3)| \\
&\le \sum_{k_1+k_2=k} C_k^{k_1}(2\pi)^{-p}\int_{-\varepsilon_N^2\sqrt{N}}^{\varepsilon_N^2\sqrt{N}} d\overline{u}_4 \int_{\prod L_2^\nu} |d\overline{w}_4| \int d\overline{t}_1 \int_{-\varepsilon_N\sqrt{N}}^{\varepsilon_N\sqrt{N}} d\overline{t}_3 \\
&|\langle e^{\frac{(\overline{\lambda}_1,\overline{\xi}_1)}{\sqrt{N}}} H_{N,h,\tilde U}(\frac{a_1 - V\frac{U^2}{\tilde U} - i\frac{(\overline{u}_3,\overline{t}_3)}{N} - i\frac{(\overline{u}_4,\overline{w}_4)}{N} - \frac{(\overline{u}_1,\overline{\xi}_1)}{\sqrt{N}}}{\sqrt{U^2 + \lambda + N^{-1}(\overline{u}_4,\overline{u}_4)}}\rangle|^{[N\delta]-k_1} \\
&\cdot|\langle e^{\frac{(\overline{\lambda}_1,\overline{\xi}_1)}{\sqrt{N}}} H_{N,h,\tilde U}(\frac{a_2 - V\frac{U^2}{\tilde U} - i\frac{(\overline{u}_3,\overline{t}_3)}{N} - i\frac{(\overline{u}_4,\overline{w}_4)}{N} - \frac{(\overline{u}_1,\overline{\xi}_1)}{\sqrt{N}}}{\sqrt{U^2 + \lambda + N^{-1}(\overline{u}_4,\overline{u}_4)}})\rangle|^{N-[N\delta]-k_2} \\
&\quad\cdot|R_m(\overline{u},\overline{w})|^k \exp\{-l_N((\overline{u}_1,\overline{\lambda}_1) + NV\tilde U - \Im(\overline{u}_4,\overline{w}_4)) \\
&\quad -(N-k)(\frac{1}{2N}(\overline{t}_3,\overline{t}_3) - V^2\frac{U^2}{2\tilde U^2} + \frac{1}{2N}\Re(\overline{w}_4,\overline{w}_4)) \\
&-\frac{\varepsilon_N^*}{2}((\overline{u},\overline{u}) + (\overline{t}_1,\overline{t}_1) - (\overline{\lambda}_1,\overline{\lambda}_1) + (\overline{t}_3,\overline{t}_3) - NV^2\frac{U^2}{\tilde U^2} + \Re(\overline{w}_4,\overline{w}_4))\}.
\end{aligned} \tag{3.25}$$

Here we consider $I_{m,k}$ as the sum of terms, in which k_1 remainder functions $R^{(m)}$ come from the first $[\delta N]$ factors in (2.25) and k_2 of $R^{(m)}$ come from the last $N - [\delta N]$ ones. Since $k = o(N)$ we have that $k_{1,2} = o(N)$ and $C_k^{k_1} = e^{o(N)}$.

Now we use (2.10) for $H_{N,h,\tilde U}$ and the inequalities

$$\begin{aligned}
|N^{-1}(\overline{u}_4,\overline{w}_4)| &\le N^{-1}\varepsilon_N\sqrt{N}\sum_{\nu=p-n+1}^{p}|u^\nu| + \frac{V}{N\tilde U}\sum_{\nu=p-n+1}^{p}|u^\nu|^2 \\
&\le n\varepsilon_N^3 + n\frac{V}{\tilde U}\varepsilon_N^4 \le \text{const } \varepsilon_N^{1/2}; \\
0 \le N^{-1}(\overline{u}_4,\overline{u}_4) &= \tilde U^2 - U^2 \le n\varepsilon_N^4 \le \varepsilon_N^{3/2},
\end{aligned} \tag{3.26}$$

which are valid since $n \le \varepsilon_N^{-5/2}$, $|u^\nu| \le \varepsilon_N^2\sqrt{N}$ (see formula (2.14)) and $|w^\nu| < \varepsilon_N\sqrt{N} + \frac{V}{\tilde U}|u^\nu|$ $(\nu = p-n+1, \ldots, p)$. Besides, $\exp\{\frac{\varepsilon_N^*}{2}[(\overline{\lambda}_1,\overline{\lambda}_1) + NV^2\frac{U^2}{\tilde U^2}]\} \le e^{N\,\text{const}\,\varepsilon_N^*} = e^{o(N)}$ because of the chosen bounds on $\overline{\lambda}_1$ and V. Then, using the inequality

$$H_{N,h,\tilde U}(x) \le H(x), \tag{3.27}$$

and the fact that $k_{1,2} = o(N)$, we get from (3.25)

$$\begin{aligned}
&|G_{m,k,n}(V,\overline{u}_1,\overline{\lambda}_1,\overline{u}_3)| \le \frac{(\text{const}\,\varepsilon_N^2\sqrt{U^2+\lambda})^k}{(2\pi)^p}e^{-nN\varepsilon_N^2/4} \\
&\cdot(G_m^*(U,V,\overline{u}_1,\overline{\lambda}_1))^N \exp\{-\frac{\varepsilon_N^*}{2}(\overline{u}_1,\overline{u}_1) - N\frac{\varepsilon_N^*}{2}U^2 + No(1)\} \\
&\cdot\int d\overline{t}_1 d\overline{t}_3\int_{-\varepsilon_N^2\sqrt{N}}^{\varepsilon_N^2\sqrt{N}} d\overline{u}_4 \exp\{-\frac{N-k-n}{2N}[(\overline{t}_3,\overline{t}_3) - \frac{(\overline{u}_3,\overline{t}_3)^2}{N(U^2+\lambda)}] \\
&\qquad -\frac{\varepsilon_N^*}{2}((\overline{t}_1,\overline{t}_1) + (\overline{t}_3,\overline{t}_3) + (\overline{u}_4,\overline{u}_4))\}.
\end{aligned} \tag{3.28}$$

Here the term $(\,\mathrm{const}\,\varepsilon_N^2\sqrt{U^2+\lambda})^k$ is due to Lemma 2 and the last line of (3.26), and the term $e^{-nN\varepsilon_N^2/4}$ is due to the integration with respect to $\overline{w}_4$. On the other hand, we should note that in fact integrals with respect $\overline{t}_1$ and $\overline{u}_4$ can give us only $(\,\mathrm{const}\,)^{m+n}(\varepsilon_N^*)^{-(m+n)}$ as a multiplier. Since $m,n = o(N|\log\varepsilon_N^*|^{-1})$, we take it into account as $e^{o(N)}$. Our main problem is to estimate the integral with respect $\overline{t}_3$, because it contains almost p integrations. To perform this integration let us note, that it is of the Gaussian type with the matrix of the form $\mathbf{A} = (\mathbf{I} - \frac{\tilde{U}^2}{\tilde{U}^2+\lambda}\mathbf{P}_u)$, where $\mathbf{I}$ is a unit matrix and $\mathbf{P}_u$ is the orthogonal projector on the normalized vector $\frac{\overline{u}_3}{\sqrt{N}U}$. Since such a matrix $\mathbf{A}$ has $(p-m-n-1)$ eigenvalues equal to 1 and only one eigenvalue equal to $1-\frac{\tilde{U}^2}{\tilde{U}^2+\lambda} = \frac{\lambda}{\tilde{U}^2+\lambda}$, the integration with respect to $\overline{t}_3$ gives us $(2\pi)^{\frac{p-n-m}{2}}$ const. Thus we obtain (2.29).

Proof of Proposition 1

It follows from (2.39) that

$$\begin{aligned}&\log|G_m^*(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))|\\ &\le N[o(1)-UV+\frac{1}{2}V^2]+C(U,V)-D(U,V)(\overline{u}_1,\overline{u}_1),\end{aligned} \tag{3.29}$$

where

$$C(U,V) = N\delta\log H(\frac{a_1-h-UV}{\sqrt{U^2+\lambda}}) + N(1-\delta)\log H(\frac{a_2-h-UV}{\sqrt{U^2+\lambda}}).$$

On the other hand, using that $H(x)<1$, we get

$$\langle H(\frac{a_{1,2}-h-VU-\frac{(\overline{u}_1,\overline{\xi}_1)}{\sqrt{N}}}{\sqrt{U^2+\lambda}})e^{\frac{(\overline{\lambda}_1,\overline{\xi}_1)}{\sqrt{N}}}\rangle \le \langle e^{\frac{(\overline{\lambda}_1,\overline{\xi}_1)}{\sqrt{N}}}\rangle \le e^{\frac{(\overline{\lambda}_1,\overline{\lambda}_1)}{2N}}.$$

Therefore, taking in (2.28) $\lambda^\mu = u^\mu$ we obtain

$$\log|G_m^*(U,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))| \le N[-UV+\frac{1}{2}V^2]-\frac{1}{2}(\overline{u}_1,\overline{u}_1). \tag{3.30}$$

Inequalities (3.29) and (3.30) give us

$$\begin{aligned}&\log|G_m^*(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))| \le N(o(1)-UV+\frac{1}{2}V^2)\\ &+\min[C(U,V)-D(U,V)(\overline{u}_1,\overline{u}_1);-\frac{1}{2}(\overline{u}_1,\overline{u}_1)].\end{aligned} \tag{3.31}$$

Now, applying the Laplace method, we get

$$\begin{aligned}&\int d\overline{u}_1|G_m^*(U,V,\overline{u}_1,\overline{\lambda}_1(U,V,\overline{u}_1))|\exp\{-\frac{\varepsilon_N^*N}{2}\sum_{\mu=1}^m(u^\mu)^2\}\\ &\le \exp\{N(-UV+\frac{1}{2}V^2+o(1))\\ &+\max_{(\overline{u}_1,\overline{u}_1)}\min[C(U,V)-D(U,V)(\overline{u}_1,\overline{u}_1);\quad -\frac{1}{2}(\overline{u}_1,\overline{u}_1)]\}.\end{aligned} \tag{3.32}$$

But since both functions in the r.h.s. of (3.32) are linear ones with respect to $(\overline{u}_1,\overline{u}_1)$, one can find the maximum value explicitly. It is just the intersection point of two functions $y = -\frac{1}{2}x$ and $y = C(U,V) - D(U,V)x$. It is easy to see that

$$x_{int} = -\frac{C(U,V)}{0.5 - D(U,V)}, \quad y_{int} = \frac{C(U,V)}{1 - 2D(U,V)}.$$

Substituting y_{int} in (3.32) we get the statement of Proposition 1.

Proof of Proposition 2

The inequality $V(U) < U$ follows easily from (2.48), if we take into account, that $A(x) > 0$. To prove that $V(U) \geq \sqrt{\alpha}$ we use the inequalities:

$$0 < A'(x) < 1, \quad A(x+y) < A(x) + y < 1 + y \;\; (x < 0, \; y > 0). \tag{3.33}$$

From the relations

$$A'(x) = \frac{e^{-x^2/2}}{2\pi H^2(x)}(e^{-x^2/2} - x\sqrt{2\pi}H(x)),$$
$$\sqrt{2\pi}H(x)x \leq \int_x^\infty te^{-t^2/2}dt = e^{-x^2/2}$$

it is easy to derive that $A'(x) > 0$. To get the upper bound for $A'(x)$ let us introduce the function $\phi(x) \equiv \log H(x) + \frac{x^2}{2}$. Using the identities

$$\phi(x) = \log \int_0^\infty \frac{dt}{\sqrt{2\pi}} e^{-tx-t^2/2}, \quad \phi''(x) = \langle (t - \langle t \rangle_x)^2 \rangle_x \geq 0,$$

where $\langle ... \rangle_x \equiv \frac{\int_0^\infty (...) e^{-tx-t^2/2}dt}{\int_0^\infty e^{-tx-t^2/2}dt}$, we obtain that $A'(x) \equiv 1 - \phi''(x) < 1$.

The last bound in (3.33) can be obtained as

$$A(x+y) \leq A(x) + y \max_{x \leq s \leq x+y} |A'(s)| < A(x) + y.$$

Taking into account, that $A(x) < \sqrt{\frac{2}{\pi}} < 1$ for $x < 0$, we get the last inequality in (3.33).

Now from the bound $A'(x) < 1$ we get that the r.h.s. of (2.48) is an increasing function with respect to V. Thus, to prove Proposition 2 it is enough to prove, that

$$U > \sqrt{\alpha} + \delta A(\frac{\alpha + p}{U} - \sqrt{\alpha}) + (1 - \delta)A(\frac{\alpha - p}{U} - \sqrt{\alpha}), \tag{3.34}$$

for $U \geq 2\sqrt{\alpha}$. Here and below we denote $p = 1 - 2\delta$.

Using the last inequality in (3.33) with $x = -\sqrt{\alpha}$ and $y = \frac{\alpha+p}{U}$ to estimate the first A, we get

$$\begin{aligned} \delta A(\frac{\alpha+p}{U} - \sqrt{\alpha}) + (1-\delta)A(\frac{\alpha-p}{U} - \sqrt{\alpha}) &< \delta(\frac{\alpha+p}{U} + 1) \\ +0.3\frac{U(1-\delta)}{p - \alpha + 2\alpha} < \delta(\frac{\alpha+p}{2\sqrt{\alpha}} + 1) &+ \frac{0.3U}{1 + O(\alpha)} \\ = 0.3U(1 + O(\alpha)) + o(\alpha^2). \end{aligned} \tag{3.35}$$

Here in order to estimate the second A in (3.34) we have used the bound $\max_x xA(-x) < 0.3$, which can be easily checked numerically. It implies

$$A(-\frac{p-\alpha+\sqrt{\alpha}U}{U}) < 0.3\frac{U}{p-\alpha+2\alpha} \leq \frac{0.3U}{1+O(\alpha)}.$$

So, if $U > 2\sqrt{\alpha}$, then

$$U > \sqrt{\alpha} + 0.3U(1+O(\alpha)) + o(\alpha^2), \tag{3.36}$$

and (3.34) is valid. Thus, we have finished the proof of Proposition 2.

Proof of Proposition 3
Since for any $\tilde{q} > q$ $\mathcal{C}(\tilde{q}) \subset \cap_{j=1}^{[\delta N]}\{\tilde{x}_j^0 \geq q\}$, on the basis of Theorem 1, we have got

$$\mathrm{Prob}\{\cup_{\tilde{q}>q}\mathcal{C}(\tilde{q})\} \leq \exp\{N \max_{U>0}\min_{V} \mathcal{F}_0^D(U,V;\alpha,\delta,q,-\infty) - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2}\}. \tag{3.37}$$

Let us denote

$$f_0(U,V;q,\alpha,\delta) \equiv \mathcal{F}_0(U,V;\alpha,\delta,q,-\infty) + \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2} + C^*(\delta)$$
$$f^D(U,V;q,\alpha,\delta) \equiv \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2} + C^*(\delta) + \alpha\log U - UV + \frac{V^2}{2} + \delta\frac{\log H(a_1^*U^{-1}-V)}{1-2D(U,V)},$$

and consider

$$\begin{aligned}&\max_U\min_V f_0(U,V;q,\alpha,\delta) \leq \max_U f_0(U,U;q,\alpha,\delta)\\ &\leq \max_U\{\alpha\log U - U^2/2 - \frac{\delta}{2}(\frac{a_1^*}{U}-U)^2\} + \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2} + C^*(\delta) \to -\infty,\end{aligned} \tag{3.38}$$

as $a_1^* \to \infty$. Here we have used the inequality $\log H(x) \leq -\frac{x^2}{2}$ $(x>0)$. Similarly, for $f^D(U,V;q,\alpha,\delta)$, when $D(U,V)<0$ we have the bound

$$\begin{aligned}&\max_U\min_V f^D(U,V;q,\alpha,\delta) \leq \max_U f^D(U,U;q,\alpha,\delta)\\ &\leq \max_U\{\alpha\log U - U^2/2 - \frac{U^2}{2}\frac{A(a_1^*U^{-1}-U)}{2U+(1-\delta)A(a_1^*U^{-1}-U)}\}\\ &-\frac{\alpha}{2}\log\alpha + \frac{\alpha}{2} + C^*(\delta) \leq \max_U\{\alpha\log U - U^2/2\\ &-\frac{U^2}{2}\frac{p-U^2}{p(1-\delta)+U^2(1+\delta)}\} - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2} + C^*(\delta) \to -\frac{\alpha}{2}\log 2 + C^*(\delta).\end{aligned} \tag{3.39}$$

Here we have used the inequalities $\log H(x) \leq -A(x)^2/2$ $(x>0)$ and $A(x) \geq x$. Thus, inequalities (3.38) and (3.39) under conditions $\delta \leq 0.6\alpha^2$, $\alpha \leq 0.113$ prove the first statement of Proposition 3. Besides, (3.39) shows, that it is enough to study only f_0. Since $\max_U\min_V f_0(U,V;q,\alpha,\delta)$ for fixed p increases with α and δ, to prove the second statement of Proposition 3 it is enough to check that for $\alpha = 0.113$, $\delta = \delta_{max} = 0.00645$ and $q = q_0 + 2\delta_{min} - 2\delta_{max} = 0.126$ $\max_U\min_V f_0(U,V;q,\alpha,\delta) < 0$. We do this numerically. Thus, we obtain the statement of Proposition 3.

Proof of Proposition 4

Let $I = I_U \times I_\alpha \times I_q \subset \mathbf{R}^3$ with $I_U = [U_1, U_2]$, $I_\alpha = [\alpha_1, \alpha_2]$ and $I_q = [0, q_0]$. Denote by $V(U, q, \alpha)$ the point of minimum of $\mathcal{F}_0(U, V; \alpha, \delta, q, -q)$ and by $U(q, \alpha)$ the point of maximum of $\Phi(U, q, \alpha)$. Let us note, that during the proof of Proposition 4 the variable δ is fixed. So here and below we omit δ as an argument of the functions Φ and Φ_0.

The first statement follows from the relations:

$$\begin{gathered} U(q,\alpha) \in I_U \qquad (q \in I_q,\ \alpha \in I_\alpha), \\ \Phi(U,q,\alpha) \leq \Phi(U,0,\alpha) \leq \Phi(U,0,\alpha_2) \leq \Phi(U(0,\alpha_2),0,\alpha_2) \leq 0 \end{gathered} \tag{3.40}$$

To prove the fist line of (3.40) it is enough to check that in I

$$\frac{\partial^2 \Phi}{\partial U \partial \alpha} \geq 0,\ \frac{\partial^2 \Phi}{\partial U \partial q} \geq 0, \quad (0 \leq q \leq q_0,\, 0.071 \leq \alpha \leq 0.113), \tag{3.41}$$

because in this case we have for any $q \in I_q$, $\alpha \in I_\alpha$

$$\begin{aligned} 0 = \frac{\partial \Phi}{\partial U}(U_1, 0, \alpha_1)) < \frac{\partial \Phi}{\partial U}(U_1, q, \alpha_1) < \frac{\partial \Phi}{\partial U}(U_1, q, \alpha); \\ 0 = \frac{\partial \Phi}{\partial U}(U_2, q_0, \alpha_2) > \frac{\partial \Phi}{\partial U}(U_2, q, \alpha_2) > \frac{\partial \Phi}{\partial U}(U_2, q, \alpha) \end{aligned}$$

and thus $U_1 \equiv U(0, \alpha_1) \leq U(q, \alpha) \leq U(q_0, \alpha_2) \equiv U_2$. Note, that for our choice of $0.0035 \leq \delta \leq 0.00778$, $0.71 \leq \alpha \leq 0.1133$ and $0 \leq q \leq q_0 \leq 0.13$ we get, that $0.25 < U_1 < U_2 < 0.41$.

Let us prove (3.41). To this end we write

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial U \partial \alpha} &= \frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial U \partial \alpha} + V'_\alpha \frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial U \partial V}; \\ \frac{\partial^2 \Phi}{\partial U \partial q} &= \frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial U \partial q} + V'_q \frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial U \partial V}, \end{aligned} \tag{3.42}$$

where $\tilde{\mathcal{F}}_0(U, V; \alpha, \delta, q) \equiv \mathcal{F}_0(U, V; \alpha, \delta, q, -q) - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2}$ and $V'_{q,\alpha}$ are the derivatives with respect to q and α of the function $V(U, q, \alpha)$ defined above. By the standard method, from the equation $\frac{\partial \tilde{\mathcal{F}}_0}{\partial V}(U, V(q, \alpha)) = 0$ we get

$$V'_\alpha = -\left(\frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial V^2}\right)^{-1} \frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial V \partial \alpha}, \quad V'_q = -\left(\frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial V^2}\right)^{-1} \frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial V \partial q}. \tag{3.43}$$

Now let us find the expressions for the derivatives of the function $\tilde{\mathcal{F}}_0$.

$$\begin{aligned} &\frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial V^2} = 1 - \delta U^2 A'_1 - (1-\delta) U^2 A'_2 > 0; \quad \frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial q^2} = -\delta A'_1 - (1-\delta) A'_2 < 0; \\ &\frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial \alpha^2} = -\frac{1}{2\alpha} - \delta A'_1 - (1-\delta) A'_2 < 0; \quad \frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial V \partial \alpha} = \delta U A'_1 + (1-\delta) U A'_2 > 0; \\ &\frac{\partial^2 \tilde{\mathcal{F}}_0}{\partial U \partial \alpha} = \frac{1}{U} + \frac{\delta}{U} A_1 + \frac{(1-\delta)}{U} A_2 + \delta a_1^* A'_1 + (1-\delta) a_2^* A'_2; \end{aligned}$$

$$\begin{aligned}\frac{\partial^2\tilde{\mathcal{F}}_0}{\partial U\partial q} &= \frac{\delta}{U}A_1-\frac{(1-\delta)}{U}A_2+\frac{\delta}{U}a_1^*A_1'-\frac{(1-\delta)}{U}a_2^*A_2';\\ \frac{\partial^2\tilde{\mathcal{F}}_0}{\partial U\partial V} &= -1-\delta a_1^*A_1'-(1-\delta)a_2^*A_2';\\ \frac{\partial^2\tilde{\mathcal{F}}_0}{\partial V\partial q} &= \delta UA_1'-(1-\delta)UA_2';\end{aligned} \tag{3.44}$$

where $A_{1,2}$ are defined in (1.20) and

$$A_{1,2}'\equiv\frac{1}{U^2}A'(\frac{a_{1,2}^*}{U}-V)=A_{1,2}(A_{1,2}-\frac{a_{1,2}^*}{U^2}+\frac{V}{U}),$$

with function $A(x)$ defined by (1.19). We remind here, that from definition (1.15) it follows that

$$1<a_1^*<1.25,\qquad -1.1<a_2^*<-0.85. \tag{3.45}$$

Let us note also, that for $U\le U_2<0.41$

$$0<A_2'=\frac{1}{U^2}A'(\frac{a_2^*}{U})\le\frac{1}{U_2^2}A'(\frac{a_2^*}{U_2})<0.7. \tag{3.46}$$

Thus,

$$\frac{\partial^2\tilde{\mathcal{F}}_0}{\partial U\partial\alpha}>0,\quad \frac{\partial^2\tilde{\mathcal{F}}_0}{\partial U\partial V}<0, \tag{3.47}$$

and using (3.42) -(3.47), we can see immediately that $\dfrac{\partial^2\Phi}{\partial U\partial\alpha}>0$. To obtain the second inequality in (3.41) we write, using (3.44) - (3.47),

$$0<\frac{-\frac{\partial^2\tilde{\mathcal{F}}_0}{\partial U\partial V}}{\frac{\partial^2\tilde{\mathcal{F}}_0}{\partial V^2}}<\frac{1+\delta a_1^*A_1'}{(1-\delta)(1-U^2A_2')}<\frac{1+1.25\delta U^{-2}}{(1-\delta)(1-U^2A_2')}\le 1.5,$$

where we have used also that $U^2A_{1,2}'<1$, bounds (3.45) for $a_{1,2}^*$ and $0.25<U<0.41$. Then,

$$\begin{aligned}\frac{\partial^2\Phi}{\partial U\partial q} &= \frac{-\frac{\partial^2\tilde{\mathcal{F}}_0}{\partial U\partial V}}{\frac{\partial^2\tilde{\mathcal{F}}_0}{\partial V^2}}(\delta UA_1'-(1-\delta)UA_2')\\ &+\frac{\delta}{U}A_1-\frac{(1-\delta)}{U}A_2+\frac{\delta}{U}a_1^*A_1'-\frac{(1-\delta)}{U}a_2^*A_2'\\ &>\frac{(1-\delta)}{U}[A_2'(-a_2^*-1.5U^2)-A_2]>\frac{(1-\delta)}{U}[0.5A_2'-A_2]\\ &=0.5\frac{(1-\delta)A_2}{U}[(A_2-\frac{a_2^*}{U^2}+\frac{V}{U}-2]>0.\end{aligned}$$

Thus, we have finished the proof of the first line of (3.40).

To prove the second line we use the simple statement

Remark 6. If $f_0(x)=\min_y g(x,y)$ and $\dfrac{\partial^2 g}{\partial x^2}\le 0$, then also $\dfrac{\partial^2 f_0}{\partial x^2}\le 0$.

This statement can be easily proved on the basis of the characteristic property of the concave functions $\frac{f(x_1)+f(x_2)}{2} \le f(\frac{x_1+x_2}{2})$.

Then on the basis of the second line of (3.44) we get automatically that $\frac{\partial^2 \Phi}{\partial \alpha^2} \le 0$. Therefore, using (2.68) and (3.41), we get

$$0 < \frac{\partial \Phi}{\partial \alpha}(U_1, 0, \alpha_2) < \frac{\partial \Phi}{\partial \alpha}(U, 0, \alpha_2) < \frac{\partial \Phi}{\partial \alpha}(U, 0, \alpha).$$

And so

$$\Phi(U,0,\alpha) < \Phi(U,0,\alpha_2) \le \Phi(U(0,\alpha_2),0,\alpha_2) < 0 \tag{3.48}$$

Now, observing that $\frac{\partial^2 \Phi}{\partial q^2} \le 0$ (see Remark 6), we conclude, that the second line of (3.40) follows from (3.48), if we prove also, that for $U \in I_U$, $\alpha \in I_\alpha$

$$\frac{\partial \Phi}{\partial q}(U, 0, \alpha) < 0. \tag{3.49}$$

But since we have proved above that $\frac{\partial^2 \Phi}{\partial q \partial U} > 0$ it is enough to prove (3.49) only for $U = U_2$.

The second inequality in (2.68) implies that

$$\frac{A_2(U_2, 0, \alpha_2)}{A_1(U_2, 0, \alpha_2)} < \frac{\delta}{1-\delta}. \tag{3.50}$$

But

$$\begin{aligned} \frac{d}{d\alpha}\frac{A_2}{A_1} &= (\frac{1}{U} - V'_\alpha)\frac{A_2}{A_1}(\frac{A'_2}{A_2} - \frac{A'_1}{A_1}) \\ &= (\frac{1}{U} - V'_\alpha)\frac{A_2}{A_1}((A(x_2) - x_2) - (A(x_1) - x_1)), \end{aligned}$$

where $x_{1,2} = \frac{a^*_{1,2}}{U} - V(U, q, \alpha)$. Since $A(x) - x$ is decreasing function (see (3.33)) and $U^{-1} - V'_\alpha > 0$ (see (3.43) and (3.44)), we have got that

$$\frac{A_2(U_2, 0, \alpha)}{A_1(U_2, 0, \alpha)} < \frac{A_2(U_2, 0, \alpha_2)}{A_1(U_2, 0, \alpha_2)} < \frac{\delta}{1-\delta} \;\Leftrightarrow\; \frac{\partial \Phi}{\partial q}(U_2, 0, \alpha) < 0.$$

Thus we have proved the first statement of Proposition 4.

Now we are left to prove, that inequalities (2.68) and (2.69) implies (1.29). To this end it is enough to check that for $\delta \le k_c \alpha^2$ and $U > \sqrt{\alpha}$, $D(U, V(U)) \ge 0$, because in this case we have, that $\mathcal{F}^{(D)}(U, V(U)) = \mathcal{F}_0(U, V(U))$ $(U > \sqrt{\alpha})$ and so

$$\max_{U \ge \sqrt{\alpha}} \mathcal{F}^{(D)}(U, V(U); q, -q, \alpha, \delta) + C^*(\delta) - \frac{\alpha}{2}\log\alpha + \frac{\alpha}{2} = \max_{U \ge \sqrt{\alpha}} \Phi(U, q, \alpha, \delta)$$

For $U > 0.5$ evidently $D(U, V(U); \delta) > 0$. For $0.5 > U > \sqrt{\alpha}$ we have

$$\begin{aligned} &D(U, V(U); \delta) > D(\sqrt{\alpha}, V(\sqrt{\alpha}); \delta) \\ &\ge D(\sqrt{\alpha}, V(\sqrt{\alpha}); k_c\alpha^2) \ge D(\sqrt{\alpha_c}, V(\sqrt{\alpha_c}); \delta_c). \end{aligned}$$

So, checking numerically that $D(\sqrt{\alpha_c}, V(\sqrt{\alpha_c}); \delta_c) > 0$ we finish the proof of Proposition 4.

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