On the Free Energy of the Two Dimensional $U(n)$ Gauge Field Theory on the Sphere.

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Abstract

The partition function of the two dimensional $U(n)$- gauge field theory in the limit $n \to \infty$ is found by a rigorous mathematical way. the results obtained and the methods used are motivated by the recent studies in the theory of random matrices combined with traditional tools of statistical mechanics.

1 Introduction.

Two dimensional non-abelian gauge field theory (QCD_2) is a quite useful laboratory to study the mathematical structure of corresponding theories in physical relevant dimensions (string aspects, confining phase, behaviour of various invariant quantities, etc.). One of the important objects of non-abelian theories is the partition function (more generally, the Wilson loop expectations) on two dimensional surfaces. These functions are kernels (or their traces) of the heat equation on the gauge group manifolds as expected from the path integral ideas. In many interesting cases they can be explicitly expressed via characteristics of the gauge group irreducible representations and give rise to numerous studies of the QCD_2 and related topics of quantum field theory and mathematical physics.

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In the paper [1] the partition function on the sphere corresponding to the $U(n)$-gauge theory was analyzed in the large n (planar) limit. Authors found that in this limit there is a phase transition of the third kind as one varies the area A of the sphere. Since the coupling constant g enters in the partition function only as g^2A, the transition can be interpreted as a certain change of the large n asymptotic behaviour corresponding to the weak coupling (trivial) and the strong coupling regimes. We refer the reader to the papers [1],[2] for the discussion of the QCD aspects of the transition and references.

In this paper we present the rigorous derivation of the free energy found in [1]. Our derivation is based on the ideas that are traditional in statistical mechanics. Namely, we combine the scheme of the proof of the equivalence of canonical and grand canonical ensembles and the scheme of justification of the mean field limit.

The partition function studied in [1] has the form of the sum over all irreducible representation R of the group $U(n)$:

$$Z(n,A)=\sum_R(\dim R)^2\exp\{-\frac{A}{2n}C_2(R)\}. \tag{1.1}$$

Here $\dim R$ is the dimension of the representation R, $C_2(R)$ is the eigenvalue of the quadratic Casimir operator (the Laplacian), and A is the area of the sphere. By using the standard parametrization of the irreducible representation of $U(n)$ by their weights, on can write the partition function (1.1) in the form [1]:

$$Z(n,A)=\frac{1}{n!}e^{-\frac{A}{24}(n^2-1)}\sum_{h_1,\dots,h_n=-\infty}^{\infty}\exp\{-\frac{An}{2}\sum_{i=1}^{n}(\frac{h_i}{n})^2\}\prod_{1\le i<j\le n}(h_i-h_j)^2, \tag{1.2}$$

where the summation is over all integer $h_1,\dots,h_n$. It is easy to see that up to the trivial multiplier $\frac{1}{n!}e^{-\frac{A}{24}(n^2-1)}$ this expression is very similar to that for the partition function of the unitary invariant ensemble of Hermitian random matrices [4]

$$\hat{Z}_n=\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\exp\{-n\sum_{i=1}^{n}V(x_i)\}\prod_{1\le i<j\le n}(x_i-x_j)^2\prod_{i=1}^{n}dx_i \tag{1.3}$$

with $V(x) = Ax^2/2$. Indeed, if we replace the summation in (1.2) by the integration we obtain the expression (1.3). The asymptotic behaviour of $\hat{Z}_n$ was found by physicists many years ago (see [5], [6]) and recently in a rigorous mathematical way in [7]. However, the direct application of methods, proposed in these papers leads to the expression for (1.2) which is correct only for small A (weak coupling phase). Thus we are faced with some new type phase transition which does not occur in the continuous version of (1.2), i.e. in the matrix model. To understand the mathematical nature of this phase transition consider the calculation of "toy" partition functions

$$\hat{Z}_n^a = \int_{x_i \neq x_j} \exp\{-n \sum_{i=1}^{n} V(x_i)\} \prod_{i=1}^{n} dx_i; \qquad Z_n^a = \sum_{h_i \neq h_j} \exp\{-n \sum_{i=1}^{n} V(h_i)\}, \quad (1.4)$$

where we replace the "interaction" terms $\prod_{i<j}(h_i - h_j)^2$ and $\prod_{i<j}(x_i - x_j)^2$ by much more simple conditions

$$x_i \neq x_j, \quad or \quad h_i \neq h_j, \quad for \quad i \neq j. \qquad (1.5)$$

It is easy to see that, while the continuous partition function $\hat{Z}_n^a$ from (1.4), is factorized

$$\hat{Z}_n^a = \prod_{i=1}^{n} \int e^{-nV(x_i)} dx_i,$$

the discrete one Z_n^a generally can not be written in this way, i.e. even asymptotically for large n the equality

$$Z_n^a = \prod_{i=1}^{n} \sum e^{-nV(\frac{h_i}{n})}$$

is valid only for some special choices of $V(x)$, which corresponds to the "weak coupling phase". This phenomenon is explained by the fact that in the continuous case condition (1.5) has no influence on the integral, while in the discrete case this condition is important and precisely in the situation of the phase transition. One can also compare the present problem with the problem of computing the free energy of the ideal Bose-gas where the difference between the discrete and continuous case (summation and integration) results in the Bose-Einstein condensation.

Thus, to find the asymptotic behaviour of Z_n^a we need to use some technique, which allows us to take into account automatically conditions (1.5). The technique of such type is well known in statistical mechanics. This is the method which allows to prove the equivivalence of canonical and grand canonical ensembles by introducing the chemical potential. By using analogous ideas we prove the following theorem, which is a rigorous version of the result of [1].

Theorem 1 *Consider the discrete partition function of the form*

$$Z_n = \sum_{h_i \neq h_j} \exp\left\{ \sum_{i \neq j} \log |h_i - h_j| - \sum_{i=1}^{n} nV(\frac{h_i}{n}) \right\}, \tag{1.6}$$

where $V(x)$ satisfies the condition

$$|V(x)| > (2+\epsilon) \log |x|. \tag{1.7}$$

Then, there exists the limiting free energy

$$f \equiv \lim_{n \to \infty} n^{-2} \log Z_n =$$

$$\int \log |x - x'| \rho(x) \rho(x') dx dx' - \int \rho(x) V(x) dx,$$

where the density $\rho(x)$ is uniquely defined by the conditions

(i)

$$\rho(x) \geq 0, \qquad \int \rho(x) dx = 1, \tag{1.8}$$

(ii)

$$\rho(x) \leq 1, \tag{1.9}$$

(iii) there exists constant z such that

$$supp \rho(x) \in \{x : u(x) \geq -z\} \tag{1.10}$$

and

$$\rho(x) = 1, \quad if \qquad u(x) > -z, \tag{1.11}$$

where by definition

$$u(x) \equiv 2 \int \log |x - x'| \rho(x') dx' - V(x). \tag{1.12}$$

Remark

It is easy to show that the free energy f and density $\rho(x)$ can be obtained as a solution of the following variational problem

$$f = \sup_{\rho(x)\leq 1} \left\{ \int \log|x-x'|\rho(x)dx\rho(x')dx' - \int V(x)\rho(x)dx \right\} \tag{1.13}$$

where we look for the supremum on the set of all densities $\rho(x)$ under condition (ii) On the other hand, the solution of continuous problem (1.3) is (see [7] for the proof and discussion)

$$\hat{f} \equiv \lim_{n\to\infty} n^{-2} \log \hat{Z}_n =$$

$$\sup_{\hat{\rho}(x)} \left\{ \int \log|x-x'|\hat{\rho}(x)dx\hat{\rho}(x')dx' - \int V(x)\hat{\rho}(x)dx \right\}, \tag{1.14}$$

where we look for the supremum over all densities without condition (ii). Thus, one can see that as far as the solution of problem (1.14) $\hat{\rho}(x)$ satisfies the condition (ii) we have the same solution for both continuous and discrete problems (weak coupling phase). But as soon as for some $A = A_c$ we obtain $\hat{\rho}(x_0) = 1$ at some point x_0 , then for $A > A_c$ we get the different solutions of the problems (1.2) and (1.3) (strong coupling phase of (1.2)).

In the simplest case when $V(x)$ is an even function with only one minimum (e.g. $V(x) = x^2$) to find $\rho(x)$ in the weak coupling phase (or $\hat{\rho}(x)$) we have to solve the singular integral equation

$$V'(x) = 2\int_{-a}^{a} \frac{\rho(y)dy}{x-y}. \tag{1.15}$$

This equation has a bounded solution for any a (see book [8]) and then a can be found from the normalizing condition (i). But if we are in the strong coupling phase, then, according to (i)-(iii), the function $\rho(x)$, satisfies the conditions

$$\rho(x) = 0, \ if \ |x| \geq a,$$

$$0 \leq \rho(x) \leq 1, \ if \ a > |x| > b,$$

$$\rho(x) = 1, \ if \ |x| \leq b. \tag{1.16}$$

Therefore we have to solve another singular integral equation (cf. (1.15))

$$V'(x) = 2\int_{-a}^{-b} \rho(y)dy \left(\frac{1}{x-y} + \frac{1}{x+y}\right) + 2\int_{-b}^{b} \frac{dy}{x-y} \tag{1.17}$$

and then find a and b from the normalizing condition (i) and equation

$$u(-b) = u(b),$$

which follows from the condition (iii) for the function $u(x)$ defined by formula (1.12).

In the case when $V(x)$ has m minima, the support σ of the density $\rho(x)$ may consist of $k \leq m$ intervals. For any of this interval (a_i, a_{i+1}), $(i = 1, 3, \ldots, 2k-1)$ we consider a small interval $(b_i, b_{i+1}) \subset (a_i, a_{i+1})$ in which $\rho(x) = 1$ and find $\rho(x)$ on the set $\sigma' = \cup_{i=1}^{2k-1}((a_i, b_i) \cup (b_{i+1}, a_{i+1}))$ as a solution of singular integral equation

$$V'(x) = 2\int_{\sigma} \frac{\rho(y)dy}{x-y}, \; if \; x \in \sigma'.$$

According to the theory of singular integral equation [8], this equation has a bounded solution if the function $V(x)$ satisfies $2k$ conditions on the set σ'. This gives us $2k$ equations on the endpoints $a_1, \ldots, a_{2k}$, $b_1, \ldots, b_{2k}$. The other equations can be found from the normalizing condition (i) and from the equations

$$u(b_1) = u(b_2), \; \ldots u(b_{2k-1}) = u(b_{2k})$$

$$u(a_2) = u(a_3), \; \ldots u(a_{2k-2}) = u(b_{2k-1})$$

that follows from conditions (1.10)-(1.11).

2 Proofs

Let us note that, by virtue of Lemma (see below), there exists positive numbers $L > 2$ and $\delta > 0$ such that

$$\left|\frac{1}{n^2}\log Z_n - \frac{1}{n^2}\log Z_n^L\right| \leq e^{-n\delta}, \tag{2.1}$$

where

$$Z_n^L = \sum_{h_i \neq h_j, |h_j| \leq nL} \exp\left\{\sum_{i \neq j} \log|h_i - h_j| - \sum_{i=1}^{n} nV(\frac{h_i}{n})\right\}.$$

One can see that Z_n^L can be considered as the partition functional of the one dimensional Ising-type model. The Hamiltonian of this model can be written in the form

$$H(\nu) = -\frac{1}{n} \sum_{r \neq r', |r| < nL} \log|\frac{r - r'}{nL}|\nu_r \nu_{r'} + \sum_{|r| < nL} V(\frac{r}{n})\nu_r - \frac{1}{n} \sum_{|r| < nL} (\log\frac{1}{nL} + 1)\nu_r \tag{2.2}$$

$\nu_r = 0, 1$ are "occupation numbers". Thus

$$Z_n^L = \sum_{\{\sum \nu_r = n\}} \exp\{-nH(\nu) + n(n-1)\log Ln + n(\log nL - 1)\}. \tag{2.3}$$

To eliminate condition $\sum \nu_r = n$ we introduce a parameter z (the "chemical potential") in the Hamiltonian (this procedure is standard in statistical mechanics):

$$H(\nu, z) = \sum_{|r|, |r'| < nL} w(\frac{r - r'}{nL})\nu_r \nu_{r'} + \sum_{|r| < nL} (V(\frac{r}{n}) - z)\nu_r + nz, \tag{2.4}$$

where

$$w(r) = \begin{cases} -n^{-1}\log|r|, & r \neq 0 \\ -n^{-1}(\log nL - 1) & r = 0 \end{cases} \tag{2.5}$$

and consider

$$Z_n(z) = \sum_{\{\nu_r\}} \exp\{-nH(\nu, z)\} \tag{2.6}$$

Now, if we determine z from the condition

$$\left\langle \frac{1}{n} \sum_{|r| < nL} \nu_r \right\rangle_{H(z)} = 1 \tag{2.7}$$

and prove that

$$\left\langle \left(\frac{1}{n} \sum_{|r| < nL} \nu_r - 1\right)^2 \right\rangle_{H(z)} \to 0, \quad as \quad n \to \infty, \tag{2.8}$$

then, as usually in statistical mechanics we obtain that

$$\lim_{n\to\infty}\left|\frac{1}{n^2}\log Z_n^L-\frac{1}{n^2}\log Z_n(z)-\log nL\right|=0. \qquad (2.9)$$

Thus, to calculate Z_n we have to calculate $Z_n(z)$, to find z from equation (2.7) and to check (2.8). To solve these problems we use the mean field theory approach in the form which was proposed in [9] to solve similar problems. Consider so-called approximating Hamiltonian of the form

$$H_a(\nu,c,z)=2\sum_{|r|,|r'|<nL}w(\frac{r-r'}{nL})\nu_r c_{r'}-\sum_{|r|,|r'|<nL}w(\frac{r-r'}{nL})c_r c_{r'}+$$
$$\sum_{|r|<nL}(V(\frac{r}{n})-z)\nu_r+nz. \qquad (2.10)$$

Then

$$H(\nu,z)=H_a(\nu,c,z)+\sum_{|r|,|r'|<nL}w(\frac{r-r'}{nL})(\nu_r-c_r)(\nu_{r'}-c_{r'})=H_a+R. \qquad (2.11)$$

According to the Bogolyubov [10] inequality, we have

$$\frac{1}{n^2}\langle R\rangle_{H(z)}\le\frac{1}{n^2}\log Z_n(z)-\frac{1}{n^2}\log Z_n(c,z)\le\frac{1}{n^2}\langle R\rangle_{H_a(c,z)}, \qquad (2.12)$$

where $Z_n(c,z)$ is the partition function corresponding to the Hamiltonian $H_a(c,z)$. For any z let us choose $c^{(n)}=\{c_r^{(n)}\}$ as a minimum point of the function $\Phi(c,z)=\frac{1}{n^2}\log Z_n(c,z)$. Since in [7] it was proved that $w(\frac{r-r'}{nL})$ is positive defined matrix, function Φ is "strictly convex" with respect to c, z (i.e. its second derivative in any direction is strictly positive). Besides, it grows as $c\to\infty$. Therefore $\Phi(c,z)$ for any z takes its minimum with respect to c in the unique point $c^{(n)}$ which is the solution of the equations:

$$c_r^{(n)}=\langle\nu_r\rangle_{H_a(c,z)}=\frac{\exp\{n(u_n(r/n)+z)\}}{1+\exp\{n(u_n(r/n)+z)\}}, \qquad (2.13)$$

where

$$u_n(\frac{r}{n})=-\sum_{|r'|<L}w(\frac{r-r'}{nL})c_{r'}^{(n)}+V(\frac{r}{n}). \qquad (2.14)$$

Thus, if we take in (2.10) $c=c^{(n)}$ and use the fact that

$$\langle\nu_r\nu_{r'}\rangle_{H_a(c,z)}=\langle\nu_r\rangle_{H_a(c,z)}\langle\nu_{r'}\rangle_{H_a(c,z)}\ (r\neq r'),$$

we get

$$\frac{1}{n^2}\langle R\rangle_{H(z)} \le \frac{1}{n^2}\log Z_n(z) - \frac{1}{n^2}\log Z_n(c,z) \le \frac{1}{n}w(0) = \frac{\log nL + 1}{n}. \tag{2.15}$$

Since due to the positivity of the matrix $w(\frac{r-r'}{nL})$ the left hand side of this inequality is positive, we obtain

$$\lim_{n\to\infty}\left|\frac{1}{n^2}\log Z_n(z) - \min_c \frac{1}{n^2}\log Z_n(c,z)\right| = 0 \tag{2.16}$$

and

$$\left\langle \frac{1}{n}\sum_{|r|,|r'|<nL} w(\frac{r-r'}{nL})(\nu_r - c_r)(\nu_{r'} - c_{r'})\right\rangle_{H(z)} \le \frac{\log nL + 1}{n}. \tag{2.17}$$

In [7] it was proved that the last inequality implies

$$\left\langle\left(\frac{1}{n}\sum_{|r|<nL}\nu_r - \frac{1}{n}\sum_{|r|<nL} c_r^{(n)}\right)^2\right\rangle_{H(z)} \le \text{const}\cdot n^{-1/2}\log^{1/2} n.$$

Thus, if we choose z provided the equality

$$\frac{1}{n}\sum_{|r|<nL} c_r^{(n)} = 1, \tag{2.18}$$

we solve equation (2.7) and prove (2.8). But, as it is easy to see, equation (2.18) is just the critical point equation for the functional $\Phi(c,z)$ which, as it was mentioned above, is "strictly convex" with respect to c and z and grows as $z\to\infty$. Thus there exists a unique point $z^{(n)}$ which satisfies (2.18) and according to (2.9) and (2.12)

$$\lim_{n\to\infty}\left|\frac{1}{n^2}\log Z_n^L - \min_{c,z}\Phi_n(c,z) - \log nL\right| = 0. \tag{2.19}$$

Now we shall study $\min_{c,z}\Phi_n(c,z) = \Phi_n(c^{(n)}, z^{(n)})$. Let us note that it follows from (2.13) that

$$0 \le c_r^{(n)} \le 1. \tag{2.20}$$

Consider the measures

$$\mu^{(n)}(\Delta) = \frac{1}{n}\sum_{|r|<nL} c_r^{(n)}\chi_\Delta(\frac{r}{n}), \tag{2.21}$$

where $\chi_\Delta(x)$ is the characteristic function of the interval Δ. Then, by the Helly theorem there exists a subsequence $\mu^{(n_k)}$, a measure μ and a constant z such that $\mu^{(n_k)}(\Delta) \to \mu(\Delta)$ for any Δ and $z^{(n_k)} \to z$ as $k \to \infty$. Since $\mu^{(n_k)}$ have the form (2.21) and $c_r^{(n_k)}$ satisfy (2.18) and (2.20), the limiting measure $\mu(dx)$ is absolutely continuous $\mu(dx) = \rho(x)dx$ and its density $\rho(x)$ satisfies the conditions

$$0 \le \rho(x) \le 1, \quad \int \rho(x)dx = 1. \tag{2.22}$$

Define

$$u(x) = 2\int \log|\frac{x-x'}{L}|\rho(x')dx' - V(x). \tag{2.23}$$

Since $c^{(n)}$ is the solution of (2.13), then (cf.(1.8)-(1.12))

$$\mathrm{supp}\rho(x) \in \{x : u(x) \ge -z\} \tag{2.24}$$

and

$$\rho(x) = 1, \quad if \quad u(x) > -z. \tag{2.25}$$

We shall prove now that the conditions (2.22)-(2.25) determine the function $\rho(x)$ uniquely. To this end assume that there exists another function $\rho_1(x)$ satisfying the same conditions (may be with different $z = z_1$). Consider

$$u^1(x) = 2\int \log|\frac{x-x'}{L}|\rho_1(x')dx' - V(x) \tag{2.26}$$

$$d_n(\frac{r}{n}) = \rho_1(\frac{r}{n}) + n^{-1/2}(c_r^{(n)} - \rho_1(\frac{r}{n}))$$

$$u_n^1(\frac{r}{n}) = -\sum_{|r'|<L} w(\frac{r-r'}{nL})d_n(\frac{r'}{n}) + V(\frac{r}{n}) =$$

$$u^1(\frac{r}{n}) + n^{-1/2}\left(u(\frac{r}{n}) - u^1(\frac{r}{n})\right) + O(n^{-1}\log n), \tag{2.27}$$

where $u_n(\frac{r}{n})$ is defined by (2.14). Since $\Phi_n(c,z)$ is a convex function and $(c^{(n)}, z^{(n)})$ is its minimum point we have for $n = n_k$

$$0 \le \Phi_n(d_n, z_1) - \Phi(c^{(n)}, z^{(n)}) \le$$

$$-\frac{1}{n}\sum_{|r|<nL} d_n(\frac{r}{n})\left(\overline{u}^1(\frac{r}{n}) - \overline{u}(\frac{r}{n})\right) +$$

$$+\frac{1}{n}\sum_{|r|<nL}\Big(\overline{u}^1(\frac{r}{n})-\overline{u}(\frac{r}{n})\Big)\frac{\exp\{n\overline{u}_n^1(r/n)\}}{1+\exp\{n\overline{u}_n^1(r/n)\}}+o(1)=$$

$$\int\big(\overline{u}^1(x)-\overline{u}(x)\big)\,dx\sum_{|r|<nL}\Big(\overline{u}^1(\frac{r}{n})-\overline{u}(\frac{r}{n})\Big)$$

$$\frac{\exp\{n\overline{u}^1(\frac{r}{n})+n^{-1/2}(\overline{u}(\frac{r}{n})-\overline{u}^1(\frac{r}{n}))+o(n^{-1/2}\}}{1+\exp\{n\overline{u}^1(\frac{r}{n})n^{-1/2}(\overline{u}(\frac{r}{n})-\overline{u}^1(\frac{r}{n})\}}+o(1)=$$

$$\int_{\overline{u}^1(x)>0}\big(\overline{u}^1(x)-\overline{u}(x)\big)\,dx-\int_{\overline{u}^1(x)=0}\big(\overline{u}^1(x)-\overline{u}(x)\big)\,dx$$

$$\sum_{\overline{u}^1(\frac{r}{n})>0}\Big(\overline{u}^1(\frac{r}{n})-\overline{u}(\frac{r}{n})\Big)+o(1), \qquad (2.28)$$

where

$$\overline{u}(x)=u(x)+z,\quad \overline{u}^1(x)=u^1(x)+z_1.$$

It follows from the last inequality that for $n=n_k$

$$0\le\Phi_n(d_n,z_1)-\Phi(c^{(n)},z^{(n)})\le-\int_{\overline{u}^1(x)>0,\overline{u}(x)<0}(-\overline{u}(x))\rho^1(x)dx-$$

$$-\int_{\overline{u}^1(x)>0,\overline{u}(x)>0}\overline{u}(x)(1-\rho^1(x))dx+o(1)\le o(1). \qquad (2.29)$$

The second inequality in (2.29) is true because $\rho^1(x)$ satisfies (2.22). Besides, since $\Phi_n(c,z)$ can be represented as

$$\Phi_n(c,z)=w(\frac{r-r'}{nL})c_rc_{r'}-z+F_n(c,z),$$

where $F_n(c,z)$ is also convex function, we get

$$\Phi_n(c,z)-\Phi(c^{(n)},z^{(n)})\ge\sum_{|r|,|r'|<nL}\frac{\partial^2\Phi_n}{\partial c_r\partial c_{r'}}(c_r-c_r^{(n)})(c_{r'}-c_{r'}^{(n)})+$$

$$\sum_{|r|<nL}\frac{\partial^2\Phi_n}{\partial c_r\partial z}(c_r-c_r^{(n)})(z-z^{(n)})+\frac{\partial^2\Phi_n}{\partial^2 z}(z-z^{(n)})^2\ge$$

$$\frac{2}{n}\sum_{|r|,|r'|<nL}w(\frac{r-r'}{nL})(c_r-c_r^{(n)})(c_{r'}-c_{r'}^{(n)}).$$

Thus it follows from (2.29) that

$$\frac{1}{n}\sum_{|r|,|r'|<nL}w(\frac{r-r'}{nL})(c_r-c_r^{(n)})(c_{r'}-c_{r'}^{(n)})=\int\log|\frac{x-x'}{L}|(\rho(x)-\rho_1(x))(\rho(x')-\rho_1(x'))dxdx'+$$

$$+o(1)\le\Phi_n(c,z)-\Phi(c^{(n)},z^{(n)})\le o(1)$$

As a result $\rho(x)=\rho_1(x)$ and $z=z_1$

Lemma

If $V(x)$ satisfies the conditions (1.7), then there exist positive numbers $L > 2$ and $\delta > 0$ such that

(i)

$$\left|\frac{1}{n^2}\log Z_n - \frac{1}{n^2}\log Z_n^L\right| \le e^{-n\delta}, \tag{2.30}$$

(ii) *for* $|r_1|, ..., |r_k| \le nL$

$$|\rho(r_1,\dots,r_k) - \rho_L(r_1,\dots,r_k)| \le \rho_L(r_1,\dots,r_k)e^{-n\delta},$$

(iii) *and for* $|r_1| > nL$ *and any* $r_2, ...r_k$

$$\rho(r_1,\dots,r_k)| \le \exp\{-n\delta[V(x) - \max_{|y|\le 1/2} V(y)]\}.$$

Proof

Let us chose L large enough $(L > 2)$ to provide the condition:

$$V(x) - \max_{|y|\le 1/2} V(y) \ge (2 + \frac{\epsilon}{3})\log|x| + \frac{\epsilon}{3}|V(x)|. \tag{2.31}$$

Consider

$$Z_n = I_0 + \sum_{p=1}^{n} C_n^p \sum_{j_1,\dots,j_p=\pm1,\pm2,\dots} I(j_1, ..., j_p), \tag{2.32}$$

where

$$Z_n^L = \sum_{|r_1|,\dots,|r_n|\le Ln} \exp\{-nH(r_1, ...r_n)\} \tag{2.33}$$

and

$$I(j_1, ..., j_p) = \sum_{|r_1|,\dots,|r_n|\le Ln} \exp\{-nH(r_1{+}j_1nL, ..., r_p{+}j_pnL, r_{p+1}, ..., r_n)\}. \tag{2.34}$$

Now we estimate

$$\Delta(j_1, ..., j_p) = \log I(j_1, ..., j_p) - \log Z_n^L$$

To this end we compare each term in (2.32) with some term in (2.33). Namely, for any configuration $|r_{p+1}|, ..., |r_n| \le nL$ consider $|r_1^*|, ..., |r_p^*| \le nL/2$ which do not coincide with any of $r_{p+1}, ..., r_n$ (it is possible because $L > 2$). Then

$$-nH(r_1 + j_1nL, ..., r_p + j_pnL, r_{p+1}, ..., r_n) + nH(r_1^*, ..., r_p^*, r_{p+1}, ..., r_n) \le$$

$$-n\sum_{1\le i\le p}\left[V(\frac{r_i}{n}+j_iL)-V(\frac{r_i^*}{n})\right]+2\sum_{1\le i<k\le p}\left[\log|\frac{r_i}{n}+j_iL-\frac{r_k}{n}+j_kL|-\log|\frac{r_i^*-r_k^*}{n}|\right]$$

$$\sum_{1\le i\le p, p<k\le n}\left[\log|\frac{r_i}{n}+j_iL-\frac{r_k}{n}|-\log|\frac{r_i^*-r_k}{n}|\right]\le$$

$$-n\left(2+\frac{\epsilon}{3}\right)\sum_{1\le i\le p}\log L(|j_i|-\frac{1}{2})+$$

$$2n\sum_{1\le i\le p}\log L(|j_i|+\frac{1}{2})+npC+O(\log n)\le$$

$$-n\frac{\epsilon}{6}\sum_{1\le i\le p}\log L(|j_i|-\frac{1}{2})+npC+O(\log n). \tag{2.35}$$

Here we have used (2.31) to estimate $V(\frac{r_i}{n}+j_iL)-V(\frac{r_i^*}{n})$, inequality

$$\log|a-b|\le\log|a|+\log|b| \quad (|a|,|b|>1)$$

to estimate the second sum in the r.h.s. of (2.35), and inequality

$$-\frac{1}{n}\sum_{1\le i\le p, i\ne k}\log|\frac{r_i^*-r_k^*}{n}|-\frac{1}{n}\sum_{p+1\le i\le n}\log|\frac{r_i-r_k^*}{n}|\ge-\frac{2}{n}\sum_{1\le i\le n}\log\frac{i}{n}=$$

$$-\int_0^{1/2}\ln x dx+O(n^{-1}\log n)$$

to estimate the last sum in the r.h.s. of (2.35). Substituting this estimate in (2.32), we get

$$|Z_n-Z_n^L|\le Z_n^L ne^{-n\delta}\left(1+e^{-n\delta}\right)^n$$

where δ depends on L and ϵ. This inequality proves the inequality (i) of Lemma. Inequalities (ii) and (iii) can be proved similarly.

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