AFFINE STANDARD LYNDON WORDS: A-TYPE

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An interesting basis of a finitely generated Lie algebra $\mathfrak{a}$ was constructed in [2] in the 1990s using the combinatorial notion of *standard Lyndon words*. The goal of the present work is to extend this framework to affine root systems. To this end, let us recall an intriguing remark from the very end of [2]:
"*Preliminary computations seem to indicate that it will be very instructive to study root multiplicities for Kac-Moody Lie algebras by way of standard Lyndon words*".

In the paper we generalize an algorithm of Leclerc [1] describing explicitly the bijection of Lalonde-Ram [2] from finite to affine Lie algebras. We focus on the simplest class of affine Lie algebras: the affine Lie algebras of type $A_n^{(1)}$.
The **main result** of the work is the computation of all affine standard Lyndon words in A-type for any order of the simple roots. The key technical tool is the following generalization of Leclerc's algorithm, which demonstrates that standard Lyndon words can be constructed inductively by their length:

Proposition 1. *Using the notations from our section 3, the affine standard Lyndon words are determined inductively by the following rules:*

(a) For simple roots, we have $\mathrm{SL}(\alpha_i) = [i]$. *For other real* $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, *we have:*

$$\mathrm{SL}(\alpha) = \max\left\{ \mathrm{SL}_*(\gamma_1)\mathrm{SL}_*(\gamma_2) \Big|_{\substack{\alpha=\gamma_1+\gamma_2,\, \gamma_k\in\widehat{\Delta}^+ \\ \mathrm{SL}_*(\gamma_1)<\mathrm{SL}_*(\gamma_2) \\ [\mathsf{b}[\mathrm{SL}_*(\gamma_1)],\mathsf{b}[\mathrm{SL}_*(\gamma_2)]]\neq 0}} \right\}, \tag{1}$$

where $\mathrm{SL}_*(\gamma)$ *denotes* $\mathrm{SL}(\gamma)$ *for* $\gamma \in \widehat{\Delta}^{+,\mathrm{re}}$ *and any of* $\{\mathrm{SL}_k(\gamma)\}_{k=1}^{|I|}$ *for* $\gamma \in \widehat{\Delta}^{+,\mathrm{im}}$.

(b) For imaginary $\alpha \in \widehat{\Delta}^{+,\mathrm{im}}$, *the corresponding* $|I|$ *affine standard Lyndon words* $\{\mathrm{SL}_k(\alpha)\}_{k=1}^{|I|}$ *are the* $|I|$ *lexicographically largest words from the list as in the right-hand side of* (1) *whose standard bracketings are linearly independent.*

The explicit computation of affine standard Lyndon words in A-type allowed us to establish some interesting properties of roots, which can be useful for future constructions. In particular, we proved (see Proposition 5.8) that the order on $\widehat{\Delta}^{+,\mathrm{ext}}$, induced from the lexicographical order on the affine standard Lyndon words, is *preconvex*, i.e. it satisfies:

$$\alpha < \alpha+\beta < \beta \quad \text{or} \quad \beta < \alpha+\beta < \alpha \qquad \forall\ \alpha,\beta,\alpha+\beta \in \widehat{\Delta}^{+,\mathrm{re}}. \tag{2}$$

References

[1] B. Leclerc, *Dual canonical bases, quantum shuffles and q-characters*, Math. Z. 246 (2004), no. 4, 691–732.

[2] P. Lalonde P., A. Ram, *Standard Lyndon bases of Lie algebras and enveloping algebras*, Trans. Amer. Math. Soc. 347 (1995), no. 5, 1821–1830.