B. Verkin Institute for Low Temperature Physics
and Engineering
of the National Academy of Sciences of Ukraine

AMPH 2018

# VI International Conference

# ANALYSIS AND MATHEMATICAL PHYSICS

dedicated to the centennial anniversary of
the National Academy of Sciences of Ukraine and
the 50th anniversary of
the Department of Function Theory

June 18–22, 2018

Book of Abstracts

Kharkiv 2018

**VI International Conference "ANALYSIS AND MATHEMATICAL PHYSICS" dedicated to the centennial anniversary of the National Academy of Sciences of Ukraine and the 50th anniversary of the Department of Function Theory**: Book of abstracts, Kharkiv: B.Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 2018, 36 p.

## Program Committee

V. Marchenko, L. Pastur, E. Khruslov.

## Organizing Committee:

L. Fardigola, G. Feldman, A. Melamedova, M. Myronyuk,

# CONTENTS

# To Hopf's conjecture about metric on the topological product $S^2 \times S^2$ of two 2-spheres

Yuriy Aminov, *Kharkiv, Ukraine*

Hopf's well-known conjecture states that there exists no metric of strictly positive curvature on the topological product $S^2 \times S^2$ of two 2-spheres. Note that by Preissmann's theorem [1], on the topological product $M \times N$ of two compact differentiable manifolds there exists no metric of strictly negative curvature. In [2], M. Berger showed that if the sectional curvature $K$ of a metric on the $S^2 \times S^2$ satisfies the inequalities $\delta \le K \le 1$, then $\delta < \frac{4}{17}$. Two articles by J.-P. Bourguignon, A. Deshamps, and P. Sentenac were devoted to this problem [3], [4]. Here we expose some theorems from our article [5]. We apply the stability theory of minimal surfaces to Hopf's problem. Saks and Uhlenbeck [6] proved that if the second homotopy group $\pi_2(M) \neq 0$, then there exists a set generating $\pi_2(M)$ and consisting of conformally minimal branched immersions of spheres that minimize energy and area in their homotopy classes. The group $\pi_2(S^2 \times S^2$ has two generators. We denote the 2-dimensional minimal cycles realizing them by $F_1$ and $F_2$.

**Theorem 1.** *If globally minimal cycles in the product $M = S^2 \times S^2$ are uniformly stable and the metric satisfies the condition of orthogonality, then, at the orthogonal intersection point of the cycles, the curvature of $M$ is non positive, at least for some area element.*

**Theorem 2.** *If, on the topological product $M = S^2 \times S^2$ with a Riemannian metric, there exist orthogonal coordinates $u^1, \dots, u^4$ and all 2-dimensional coordinate surfaces $u^1 = const, u^2 = const$ and $u^3 = const, u^4 = const$ are minimal, then the integral inequality*

$$\int_M (K_{13} + K_{23} + K_{14} + K_{24}) dV \le 0$$

*holds, where $dV$ is the volume element of $M$ and $K_{ij}$ is the curvature of coordinate surface.*

We mention that by the existence of orthogonal coordinates we mean the existence of two two-dimensional foliations by 2-spheres and orthogonal coordinates on each leaf with singular point at which $g_{ii}$ may vanish. The usual product $S^2 \times S^2$ of standard spheres gives an example of such manifold.

We give examples of metrics on $S^2 \times S^2$ with globally minimal uniformly stable cycles.

[1] A. Preissmann, Quelques proprietes globales des espaces de Riemann, Comment. Math. Helv. **15** (1943), 175–216.

[2] M. Berger, Sur les varietes (4/23)-pincees de dimension 5, Comp. Rend. Acad. Sci. Paris. **257** (1963), 4122–4125.

[3] J.-P. Bourguignon, A. Deschamps et P. Sentenac, Conjecture de H.Hopf sur les Produits de Varietes, Ann. Scient. Ec. Norm. Sup., $4^e$ serie **5** (1972), 277–302.

[4] J.-P. Bourguignon, A. Deschamps et P.Sentenac, Quelques variations particulieres d'un produit de metriques, Ann. scient. Ec. Norm. Sup., $4^e$ Serie **6** (1973) 1–16.

[5] Yu.A. Aminov, Stable and Unstable Minimal Surfaces and Hopf's Problem, Dokl.Math. **93** (2016), 211–215.

[6] J. Saks and K. Uhlenbeck, Ann. Math.**113** (1981), 1-4.

[7] K. Kenmotsu and C. Xia, Kodai Math. J. **18** (1995), 242-249 .

[8] A.A. Borisenko, Intrinsic and Extrinsic Geometry of Multidimensional Submanifolds, Ekzamen, Moscow, 2003 (Russian).


# Exact null controllability of retarded and mixed time-delay systems


Pavel Barkhayev, *Kharkiv, Ukraine*
Rabah Rabah, *Nantes, France*
Grigory Sklyar, *Szczecin, Poland*


For a broad class of retarded and mixed time-delay control systems

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^{0} [A_2(\theta)\dot{z}(t+\theta) + A_3(\theta)z(t+\theta)]\,\mathrm{d}\theta + Bu(t), \quad (1)$$

with $\det A_{-1} = 0$, we study the problem of exact null controllability which may be stated as an infinite vector moment problem.

For neutral type systems ($\det A_{-1} \neq 0$), the moment problem approach proved to be a powerful tool for investigation controllability problem due to the fact that spectrum of the system's operator belongs to a vertical strip and the corresponding family of exponentials forms a Riesz basis of its closure on an appropriate interval. In [3] we prove a criterion of exact controllability in terms of some rank conditions and give the exact controllability time.

In [2], for the subclass of systems with point-wise commensurable delays it was shown that, in the case of retarded (and mixed) systems, one of the rank conditions may be weaken. The technique of the proof used there was essentially based on the particular form of systems.

In this work we prove that the same conditions of controllability hold for systems with distributed delays (1). Since spectrum of retarded systems lies on half-plane, the corresponding family of exponentials does not form a Riesz basis, and so we had to find another technique of investigation. The approach used is essentially based on the property of minimality of the operator's family of exponentials and as a consequence, existing of biorthogonal family. This allows to construct steering controls and solve moment problem for each state of the model space. The obtained result also proves the conjecture from [1].

[1] R. Rabah, G.M. Sklyar, and P.Y. Barkhayev, *Exact null controllability, complete stabilizability and continuous final observability of neutral type systems*, Int. J. Appl. Math. Comp. Sci. 27 (2017), 489–499.

[2] R. Rabah, G.M. Sklyar, and P.Y. Barkhayev, *On exact controllability of neutral time-delay systems*, Ukrainian Math. Journal 68 (2016), 800–815.

[3] V.E. Khartovskiĭ, and A.T. Pavlovskaya, *Complete controllability and controlability for linear autonomous systems of neutral type*, Automation and Remote Control 79(2013), 769–784.

## Synthesis problem for nonlinear systems with power principal part

Maxim Bebiya, *Kharkiv, Ukraine*

We address the problem of bounded control synthesis for nonlinear systems with uncontrollable unstable first approximation. Namely, we consider systems of the form

$$\begin{cases} \dot{x}_1 = u, \quad |u(x)| \leq d, \\ \dot{x}_2 = c_1 x_1^{2k_1+1} + f_1(t,x,u), \\ \dot{x}_3 = c_2 x_2^{2k_2+1} + f_2(t,x,u), \end{cases} \tag{1}$$

where $u \in \mathbb{R}$ is a control, $d > 0$ is a given number, $k_i = \frac{p_i}{q_i}$ ($p_i > 0$ is an integer, $q_i > 0$ is an odd integer, $i = 1,2$), $c_i \neq 0$ $(i = 1,2)$ are real numbers, $f_i(t,x,u)$ $(i = 1,2)$ are continuous real-valued functions with $f_i(t,0,0) = 0$ for all $t \geq 0$.

Our objective is to construct a class of bounded controls such that for any initial point $x_0 \in U(0)$ the solution $x(t,x_0)$ of the corresponding closed-loop system is well-defined and ends at 0 in a finite-time $T(x_0) < +\infty$, i.e. $\lim_{t\to T(x_0)} x(t,x_0) = 0$.

The stabilization problem for more general system in $n$–dimensional case was investigated in [1]. The synthesis problem for the case when $f_i(t,x,u) = 0$ $(i = 1,2)$ and $k_1 = 0$, $k_2 > 0$ was solved in [2]. The approach which was proposed in [2] for $n$–dimensional systems with one power nonlinearity is based on the controllability function method [3]. Under some additional growth conditions imposed on function $f_i(t,x,u)$ we develop this approach to construct a class of bounded finite-time stabilizing controls $u = u(x)$ for system (1). We also construct a class of controllability functions $\Theta(x)$ such that the inequality $\dot{\Theta}(x) \leq -\beta \Theta^{1-\frac{1}{\alpha}}(x)$ holds for some $\alpha \geq 1$, $\beta > 0$. The last inequality guarantees that the trajectory of the closed-loop system steers any initial point $x_0 \in U(0)$ to the origin in some finite time $T(x_0)$.

[1] M.O. Bebiya and V.I. Korobov, *On Stabilization Problem for Nonlinear Systems with Power Principal Part*, Journal of Mathematical Physics, Analysis, Geometry **12** (2016), 113–133.

[2] M.O. Bebiya, *Global synthesis of bounded controls for systems with power nonlinearity*, Visnyk of V.N. Karazin Kharkiv National University, Ser. Mathematics, Applied Mathematics and Mechanics **81** (2015), 36–51.

[3] V.I. Korobov, *The method of controllability function*, R&C Dynamics, M.-Izhevsk, 2007 (Russian).

# Foliations of bounded absolute total curvature

Dmitry V. Bolotov, *Kharkiv, Ukraine*

Let $(M, g)$ be a complete non-compact surface equipped with a smooth riemannian metric. The total curvature of $M$ is the improper integral $\int_M K d\mu$ of the Gaussian curvature $K$ with respect to the volume element $d\mu$ of $(M, g)$. If $\int_M K d\mu$ exists, then $\int_M K d\mu \leq 2\pi\chi(M)$ [1]. From [2] follows that if the absolute total curvature $\int_M |K| < \infty$, then $\int_M K d\mu$ exists and $M$ is homeomorphic to a compact Riemann surface with finitely many punctures. Moreover, in this case the area of a geodesic ball of radius $r$ at a fixed point must grow at most quadratically in $r$ [3]. The following theorem describes a topological structure of $C^\infty$-foliations on closed 3-Manifolds with leaves of bounded absolute total curvature in the induced riemannian metric (BATC-foliations).

**Theorem 1.** *Let $\mathcal{F}$ be a transversally orientable BATC-foliation of class $C^\infty$ on a closed orientable riemannian 3-Manifold $M$. Then the following holds:*

1. *$\mathcal{F}$ is a foliation almost without holonomy;*
2. *At least one of the following holds: a) $\mathcal{F}$ is a surface bundle over the circle; b) $\mathcal{F}$ consists of dense leaves which are homeomorphic to a typical leaf $L$. In this case the manifold $M$ is homeomorphic to a torus bundle over the circle. c) $M$ is divided by a finite set of compact leaves $\{K_i\}$, which are homeomorphic to torus $T^2$, into pieces $\{A_j\}$, which are fibered over the circle.*
3. *If the curvature of leaves is nonpositive then $M$ is aspherical and the division above defines a graph $G$ of fundamental groups $\pi_1(A_j)$ (vertexes) and $\pi_1(K_i)$ ( edges); the fundamental group $\pi_1(M)$ is isomorphic to a fundamental group of the graph $G$;*
4. *Every closed 3-Manifold admits BATC-foliation for some riemannian metric $g$ on $M$ with leaves of nonpositive or nonnegative curvature in the induced metric.*

[1] S. Cohn-Vossen *Küzeste Wege und Totalkrümmung auf Flächen*, Compositio Math. **2** (1935), 69–133.
[2] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.
[3] Ph. Hartman, *Geodesic parallel coordinates in the large*, Amer. J. Math. **86** (1964), 705–727.

# Close geodesics on regular tetrahedra in hyperbolic space

Alexander Borisenko, *Kharkiv, Ukraine*
Darya Sukhorebska, *Kharkiv, Ukraine*

The talk deals with the question of the existence of closed geodesies on the regular tetrahedra in the hyperbolic 3-space. Properties of closed geodesics on the regular tetrahedron in the hyperbolic space differ from one in Euclidean space.

We present necessary conditions that simple close geodesics on regular tetrahedra in the 3-dimensional hyperbolic space satisfy.

**Theorem 1.** *Simple close geodesics on regular tetrahedra in the hyperbolic 3-space pass through the middle of two pairs of opposing edges.*

It is known that in a Euclidean space closed geodesics on a regular tetrahedron don't have self-intersection, don't pass through the vertices of a tetrahedron, but can pass at an arbitrarily close distance from them [1]. However, in the hyperbolic space a simple closed geodesic can't pass close to the vertices of the tetrahedron.

**Theorem 2.** *Let $\gamma$ be a simple closed geodesic on the regular tetrahedron in the hyperbolic space. Let $d$ is the smallest of the distances from the vertices of the tetrahedron to $\gamma$. Then*

$$th(d) > \cos(\frac{3\alpha}{2})\frac{\sqrt{2\cos(\alpha)-1}}{\cos(\alpha)},$$

*where $\alpha$ is the flat angle of the tetrahedron faces.*

Furthermore, we explicitly describe three classes of simple closed geodesics on regular tetrahedra in the hyperbolic 3-space. These are so-called 2-homogeneous, 3-homogeneous and (3,2)-homogeneous geodesics. Up to a rigid motion of a tetrahedron there exists a unique close geodesic in each class.

[1] D. Fuchs and E. Fuchs, *Closed geodesics on regular polyhedra,* Moskow mathematical journal, 2007, 265–279.
[2] J. Cannon, W. Floyd, R. Kenyon, and W. Parry, *Hyperbolic Geometry*, Flavors of Geometry, MSRI Publications, **31**, 1997.
[3] V. Protasov, *Closed geodesics on the surface of a simplex,* Mat. Zbornik **198** (2007), 103–120 (Russian).

# Generalized $Fup$-functions and their applications

Iryna Brysina, *Kharkiv, Ukraine*
Victor Makarichev, *Kharkiv, Ukraine*

Consider the function $f(x) \in L_2(\mathbb{R})$ such that $\operatorname{supp} f(x) = [-1, 1]$, $f(x)$ is an even function, $f(x) \geq 0$ for any $x \in [-1, 1]$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. Denote by $F(t)$ the Fourier transform of this function.

The function $f_{N,m}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{itx} \left( \frac{\sin\left(\frac{t}{N}\right)}{\frac{t}{N}} \right)^{m+1} F\left(\frac{t}{N}\right) dt$, where $N > 0$ and $m = 2, 3, 4, \ldots$ is called a generalized $Fup$-function [1].

Generalized $Fup$-functions have many convenient properties. In particular it was shown in [2] that $\operatorname{supp} f_{N,m}(x) = \left[-\frac{m+2}{N}, \frac{m+2}{N}\right]$. In other words, $f_{N,m}(x)$ has a compact support. In addition, it was shown in terms of the Kolmogorov width that generalized $Fup$-functions are asymptotically extremal for approximation of periodic differentiable functions in the norm of $L_2[-\pi, \pi]$.

Let $V_{N,m}$ be the space of $2\pi$-periodic functions

$$f(x) = \sum_k c_k \cdot f_{N,m}\left(\frac{x}{\pi} - \frac{2k}{N} + 1 + \frac{m+2}{N}\right), x \in [\pi, \pi].$$

By $\widetilde{W}^r_\infty$ we denote the class of functions $f \in C^r_{[-\pi,\pi]}$ such that $f^{(k)}(-\pi) = f^{(k)}(\pi)$ for any $k = 0, 1, \ldots, r-1$ and $\left\|f^{(r)}\right\|_{C[-\pi,\pi]} \leq 1$. Let $E_x(A, L) = \sup\limits_{\phi \in A} \inf\limits_{\psi \in L} \|\phi - \psi\|_X$ and also let $d_N(K, X) = \inf\limits_{\dim\ L = N} E_X(K, L)$ be the Kolmogorov width.

**Theorem 1.** *If $m \geq r - 1$ and $m \leq N - 2$ then*

$$E_{C[-\pi,\pi]}\left(\widetilde{W}^r_\infty, V_{N,m}\right) \leq d_N\left(\widetilde{W}^r_\infty, C[-\pi, \pi]\right) \cdot \left(1 + \bar{o}(2^{-m-r})\right).$$

We note that this statement is a generalization of Theorem 5 [3].


[1] I.V. Brysina and V.A. Makarichev, *On the asymptotics of the generalized Fup-functions*, Adv. Pure Appl. Math. **5** (2014), 131–138.

[2] I.V. Brysina and V.A. Makarichev, *Approximation properties of generalized Fup-functions*, Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh. **84** (2016), 61–92.

[3] V.A. Rvachev, *Compactly supported solutions of functional-differential equations and their applications*, Russian Math. Surveys **45** (1990), 87–120.


# PT-symmetric eigenvalues of homogeneous potentials

Alexandre Eremenko, *West Lafayette, USA*

We consider one-dimensional Schrödinger equations with potential $x^{2M}(ix)^\varepsilon$, where $M \geq 1$ is an integer, and $\varepsilon$ is real, under appropriate PT-symmetric boundary conditions. We prove the phenomenon which was discovered by Bender and Boettcher by numerical computation: as $\varepsilon$ changes, the real spectrum suddenly becomes non-real in the sense that all but finitely many eigenvalues become non-real. We find the limit arguments of these non-real eigenvalues $E$ as $E \to \infty$.

The proofs are based ons an asymptotic expansion of the spectral determinant of an auxiliary self-adjoint problem, which is of independent interest.

## Transformation operators and wave equations in bounded and unbounded domains

Larissa Fardigola, *Kharkiv, Ukraine*

In the talk, a transformation between the wave equations:

$$w_{tt} = \frac{1}{\rho}\left(k w_x\right)_x + \gamma w, \qquad x \in (0, d),\ t \in [0, T], \tag{1}$$

and

$$z_{tt} = z_{\xi\xi} - q^2 z, \qquad \xi \in (0, l),\ t \in [0, T], \tag{2}$$

is studied under some assumptions on the variable coefficients $\rho$, $k$, $\gamma$ and the constant $q \geq 0$ where $d \in (0, +\infty]$ and $l = \lim_{\xi \to d^-} \int_0^\xi \sqrt{\rho(\mu)/k(\mu)}\,d\mu \in (0, +\infty]$, $T > 0$.

The behavior of solutions to equation (1) essentially depends on the type of $l$. If $l \in (0, +\infty)$, equation (1) replicates properties of equation (2) with $q = 0$ on the bounded domain $(0, l)$. If $l = +\infty$, equation (1) replicates properties of equation (2) on the half-axis $(0, +\infty)$. In the both cases the domain $(0, d)$ can be either bounded or unbounded.

Equation (2) is considered in spaces $\widetilde{H}^{-m}(0, l)$ of the Sobolev type, and equation (1) is considered in some modified Sobolev spaces $\widetilde{\mathbb{H}}^{-m}(0, d)$ determined by the coefficients $\rho$ and $k$, $m = 0, 1, 2$.

An isometric isomorphizm of the spaces $\widetilde{H}^{-m}(0, l)$ and $\widetilde{\mathbb{H}}^{-m}(0, d)$ transforming each solution to (2) into a solution to (1) (and vice versa) is constructed and studied, $m = \overline{-2, 2}$. This isomorphism is called a transformation operator.

Applying this operator, it is proved that equation (1) replicates controllability properties of equation (2) and vice versa [1–3]. Thus controllability properties of (1) are obtained from the ones of (2) [1–4].

[1] L.V. Fardigola, *Transformation operators in controllability problems for the degenerate wave equation with variable coefficients*, Ukr. Mat. Zh., to appear.

[2] L.V. Fardigola, *Transformation operators of the Sturm–Liouville problem in controllability problems for the wave equation with variable coefficients on a half-axis controlled by the Dirichlet boundary control,* Mathematical Control and Related Fields **5** (2015), 31–53.

[3] L.V. Fardigola, *Transformation operators of the Sturm–Liouville problem in controllability problems for the wave equation on a half-axis,* SIAM J. Control Optim. **51** (2013), 1781–1801.

[4] L.V. Fardigola, *Controllability problems for the 1-D wave equation on a half-axis with the Dirichlet boundary control*, ESAIM: Control, Optim. Calc. Var. **51** (2012), 748–773.


# On the controllability problems for the heat equation on a half-axis


Larissa Fardigola, *Kharkiv, Ukraine*
Kateryna Khalina, *Kharkiv, Ukraine*


Consider the control system

$$w_t(x,t) = w_{xx}(x,t), \quad w(0,t) = u(t), \qquad x \in (0,+\infty),\ t \in (0,T), \tag{1}$$

$$w(x,0) = w^0(x), \qquad x \in (0,+\infty), \tag{2}$$

where $T > 0$ is given, $u \in L^\infty(0,T)$ is the control, the state $w(\cdot,t)$, $t \in (0,T)$, and the initial state $w^0$ belong to the space $\widetilde{H}_0^0(0,+\infty)$ of the Sobolev type.

A state $w^0 \in \widetilde{H}_0^0(0,+\infty)$ is called *approximately controllable at the time* $T$ if for any $w^T \in \widetilde{H}_0^0(0,+\infty)$ and for any $N \in \mathbb{N}$ there exists a control $u_N \in L^\infty(0,T)$ such that for the solution $w_N(x,t)$ to system (1), (2) with $u = u_N$ we have $\left\|w^T - w_N(\cdot,T)\right\| < 1/N$.

Note, that controllability problems for the parabolic equation in bounded domains are well investigated (see, e. g., [1] and the references therein). In unbounded domains, these problems have not been sufficiently studied. In [2], it is indicated that one can prove the approximate controllability directly both in the case of bounded and unbounded domains by using adjoint system. But there is no proof of this fact in an unbounded domain in [2]. Instead, the authors study the class of initial data that may be transferred to zero in finite time via the boundary $L^2$-control. In [2], using similarity variables and weighted Sobolev spaces, it was proven that no initial datum belonging to any Sobolev space of negative order may be driven to zero in finite time. Nevertheless, it was shown that there exist initial data with exponentially growing Fourier coefficients such that null-controllability holds in finite time with $L^2$-controls.

In the present talk we prove that a state $w^0 \in \widetilde{H}_0^0(0,+\infty)$ of system (1), (2) on a half-axis can be steered to each small neighborhood of a state $w^T \in \widetilde{H}_0^0(0,+\infty)$ at the time $T$. Moreover, the controls solving the approximate controllability problems are constructed.

[1] S. Micu, E. Zuazua, *Regularity issues for the null-controllability of the linear 1-d heat equation,* Systems Control Lett. **60** (2011), 406–413.

[2] S. Micu, E. Zuazua, *On the lack of null-controllability of the heat equation on the half-line,* Trans. Amer. Math. Soc. **353** (2001), 1635–1659.

## Pullback attractors for processes generated by fluid-plate interaction systems.

Tamara Fastovska, *Kharkiv, Ukraine*

We consider a non-autonomous fluid-structure interaction system for the fluid velocity field $v = v(x,t) = (v^1(x,t); v^2(x,t); v^3(x,t))$, the pressure $p(x,t)$, and the transversal displacement of the plate

$$\begin{aligned} &\varepsilon(t)v_t - \nu\Delta v + \nabla p = f(x,t) && \text{in } \mathcal{O}\times(\tau,+\infty),\\ &\operatorname{div} v = 0 && \text{in } \mathcal{O}\times(\tau,+\infty),\\ &\delta(t)u_{tt} + \Delta^2 u + F(u) = p|_\Omega + g(x,t) && \text{in } \Omega\times(\tau,\infty) \end{aligned} \tag{1}$$

for any $\tau \in R$. Here $\nu > 0$, $O \subset R^3$ is a bounded domain representing a vessel filed with viscous incompressible fluid with a smooth boundary $\partial\mathcal{O} = \overline{\Omega} \cup \overline{S}$, where $\Omega \cap S = \emptyset$. The flat domain $\Omega$ represents an elastic plate while $S$ is a rigid wall. Equations (1) are supplemented with the boundary

$$\begin{aligned} &v = 0 \quad \text{on } S; \qquad v \equiv (v^1; v^2; v^3) = (0; 0; u_t) \quad \text{on } \Omega,\\ &u|_{\partial\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \end{aligned} \tag{2}$$

and initial conditions

$$v(x,\tau) = v_\tau(x), \quad u(x,\tau) = u^0_\tau(x), \quad u_t(x,\tau) = u^1_\tau(x). \tag{3}$$

**Theorem 1.**

- *Let* $f \in L^2_{loc}(\mathbb{R}; Y')$, $g \in L^2_{loc}(\mathbb{R}; H^{-1/2}(\Omega))$ *and there exist* $\sigma_0, C_{f,g} > 0$, *such that for any* $t \in \mathbb{R}$, $\sigma \in [0, \sigma_0]$

$$\int_{-\infty}^{t} e^{-\sigma(t-s)} \left( \|f(s)\|^2_{Y'} + \|g(s)\|^2_{-1/2,\Omega} \right) ds \le C_{f,g}.$$

- *Let* $\varepsilon(t), \delta(t) > 0$ , $\varepsilon(t), \delta(t) \in C^1(\mathbb{R})$ *be decreasing. There exists* $L > 0$ $\sup\limits_{t\in\mathbb{R}}(|\varepsilon(t)| + |\varepsilon'(t)| + |\delta(t)| + |\delta'(t)|) \le L$, $\lim\limits_{t\to+\infty} \varepsilon(t) = 0$, $\lim\limits_{t\to+\infty} \delta(t) = 0$.

- *There exists* $\epsilon > 0$, $\|\mathcal{F}(\eta_1) - \mathcal{F}(\eta_2)\|_{-1/2,\Omega} \leq C_R\|\eta_1 - \eta_2\|_{2-\epsilon,\Omega}$, $\eta_1, \eta_2 \in H_0^2(\Omega)$, $\|\eta_i\|_{2,\Omega} \leq R$. *There exists* $C^1$*-functional* $\Pi(\eta)$ *on* $H_0^2(\Omega)$, $\mathcal{F}(\eta) = \Pi(\eta)$, $\Pi(\eta) \leq Q(\|\eta\|_{2,\Omega})$, $Q$*-increasing. There exists* $0 < \nu < 1$, $C \geq 0$, $(1-\nu)\|\Delta\eta\|^2 + \Pi(\eta) + C \geq 0$, $\forall \eta \in H_0^2(\Omega)$ $(\mathcal{F}(\eta), \eta) \geq a_1\Pi(\eta) - a_2 - (1-\nu)\|\Delta\eta\|_\Omega^2$, $a_1, a_2 \geq 0$.

*Then problem (1), (2) generates a process possessing a pullback attractor.*

# Pólya theorem about entire integer-valued functions and uniqueness of an integer solution of an implicit linear difference equation

Sergey Gefter, *Kharkiv, Ukraine*
Anna Goncharuk, *Kharkiv, Ukraine*

Let us consider the following linear difference equation

$$c_k y_{n+k} = c_{k-1} y_{n+k-1} + \ldots + c_1 y_{n+1} + c_0 y_n + f_n, \tag{1}$$

where $c_i, f_n \in \mathbb{Z}$, $c_k, c_0 \neq 0$. We are interested in integer solutions of this equation.

First we note the obvious condition for uniqueness of this kind of solution.

**Theorem 1.** *The equation (1) has no more than one integer solution if absolute values of roots of its characteristic polynomial are less than 1.*

Using the Pólya theorem about integer-valued entire functions (see [1], section 6.3 and [2], section 1.1), we obtain the following result:

**Theorem 2.** *The equation (1) has no more than one integer solution if roots of its characteristic polynomial belong to* $[0, 2] \setminus \{1\}$.

For example, it is obvious, that the equation $5y_{n+2} = 7y_{n+1} - y_n + f$, where f is integer, has a solution $y_n = f$. Its characteristic polynomial is $5\lambda^2 - 7\lambda + 1$. Roots of this polynomial are $0 < \frac{7-\sqrt{29}}{10} < 1$ and $1 < \frac{7+\sqrt{29}}{10} < 2$. They do not satisfy condition of Theorem 1, but belong to $[0, 2] \setminus \{1\}$. So, there is not more integer solutions of this equation.

[1] Ludwig Bieberbach *Analytische Fortsetzung*, Springer-Verlag, Berlin–Heidelberg, 1955.

[2] Paul-Jean Cahen, Jean-Luc Chabert, *Integer-valued polynomials*, Mathematical Surveys and Monographs, AMS, **48** (1997).

# On a local Darlington synthesis problem

Leonid Golinskii, *Kharkiv, Ukraine*

A function $s$ ($2\times 2$ matrix function $S$) on the unit disk $\mathbb{D}$ is called *contractive* if

$$|s(z)| \le 1, \qquad (I - S^*(z)S(z) \ge 0), \qquad z \in \mathbb{D}.$$

We write $s \in \mathcal{S}$ ($S \in \mathcal{S}^{(2)}$). A function $s \in \mathcal{S}$ (a matrix function $S \in \mathcal{S}^{(2)}$) is said to be *inner (matrix) function* if its boundary values which exist almost everywhere on the unit circle $\mathbb{T}$ are unimodular (unitary).

Given $s \in \mathcal{S}$, the Darlington synthesis problem asks whether there exists an inner matrix function $S = \|s_{ij}\|_{i,j=1}^2$, so that $s = s_{22}$. A seminal result of Arov [1] and Douglas–Helton [2] states that a contractive function $s$ admits the Darlington synthesis if and only if it possesses a meromorphic pseudocontinuation of bounded type across $\mathbb{T}$.

We give a local version of this result. Let $\gamma \subset \mathbb{T}$ be an arc of the unit circle (the case $\gamma = \mathbb{T}$ is not excluded). Denote by $\mathcal{S}_\gamma$ ($\mathcal{S}_\gamma^{(2)}$) the class of contractive functions (matrix function), unimodular (unitary) a.e. on $\gamma$.

**Theorem 1.** *Let $s \in \mathcal{S}\backslash\mathcal{S}_\gamma$. The following conditions are equivalent.*

1. *There is a matrix function $S = \|s_{ij}\|_{i,j=1}^2 \in \mathcal{S}_\gamma^{(2)}$ so that $s_{22} = s$;*
2. *$s$ possesses a pseudocontinuation of bounded type across $\gamma$, and the Boyd condition holds $\displaystyle\int_{\mathbb{T}} \log(1-|s(t)|^2)m(dt) > -\infty$.*

[1] D.Z. Arov, *Realization of matrix-valued functions according to Darlington*, Izv. Akad. Nauk SSSR, Ser. Mat. **7** (1973), 1295–1326.

[2] R.G. Douglas and J.W. Helton, *Inner dilations of analytic matrix functions and Darlington synthesis*, Acta Sci. Math. (Szeged) **34** (1973), 61–67.

# A Riemann–Hilbert approach to the modified Camassa–Holm equation with nonzero boundary conditions

Iryna Karpenko, *Kharkiv, Ukraine*
Dmitry Shepelsky, *Kharkiv, Ukraine*

We study the initial value problem for the modified Camassa–Holm (mCH) equation

$$\begin{aligned} &m_t + \big((u^2 - u_x^2)m\big)_x = 0, \quad m = u - u_{xx}, \quad -\infty < x < \infty,\ t \ge 0, \\ &u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\quad -\infty < x < \infty, \end{aligned} \tag{1}$$

in the class of functions satisfying, for all $t \geq 0$, the nonzero boundary conditions, which, without loss of generality, are as follows: $u(x,t) \to 1$ as $|x| \to \infty$. This equation was introduced as a formal new integrable system by Fuchssteiner and Olver and Rosenau. It also arises in the theory of nonlinear water waves as a model equation and from an (intrinsic) arc-length preserving invariant planar curve flow in Euclidean geometry.

We present the Inverse Scattering Transform approach for (1) based on the matrix Riemann–Hilbert problem formalism and applied to equation

$$\tilde{m}_t + \left((\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})\tilde{m}\right)_x = 0 \tag{2}$$

on the zero background ($\tilde{u} \to 0$ as $|x| \to \infty$), where $u$ and $\tilde{u}$ are related by $u(x,t) = \tilde{u}(x-t,t) + 1$ and $\tilde{m} = \tilde{u} - \tilde{u}_{xx} + 1$. We introduce the Lax pair equations $\Phi_x = U\Phi$, $\Phi_t = V\Phi$ for (2), with $U = \frac{1}{2}\begin{pmatrix} -1 & \lambda\tilde{m} \\ -\lambda\tilde{m} & 1 \end{pmatrix}$ and

$$V = \begin{pmatrix} \lambda^{-2} + \frac{(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})}{2} & -\lambda^{-1}(\tilde{u} - \tilde{u}_x + 1) - \frac{\lambda(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})\tilde{m}}{2} \\ \lambda^{-1}(\tilde{u} + \tilde{u}_x + 1) + \frac{\lambda(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})\tilde{m}}{2} & -\lambda^{-2} - \frac{(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})}{2} \end{pmatrix}.$$

Using this formalism, we construct a parametric representation of the solution of the Cauchy problem for (2) (and thus for (1)) in terms of the solution of an associated Riemann–Hilbert problem, which can be efficiently used for further studying of properties of the solution (particularly, for studying the long-time behavior of the solution of the Cauchy problem).

Also, using the Riemann–Hilbert formalism, we present smooth and certain non-smooth soliton solutions of (2).

## The time-optimal problem and vector min-moment problem.

Valery Korobov, *Kharkiv, Ukraine*

The time-optimal problem for a linear completely controllable system is considered.

Its solution is based on the introduction of a vector min-moment problem, which is a further development of the min-moment problem [1, 2].

A power vector min-moment problem is considered in detail. The corresponding examples are given.

[1] V.I. Korobov and G.M. Sklyar, *The problem of moments on the minimum possible interval.*, Report of the USSR Academy of Sciences, **308** (1989), 525–528.

[2] V.I. Korobov and G.M. Sklyar, *Optimal time-optimal and power moment problem.*, Mathematical collection. **134(176)** (1987), 186–206.

# Time-optimal problem solution for the linear system in the case of getting on the subspace.

Valery Korobov, *Kharkiv, Ukraine*
Anna Pivnenko, *Kharkiv, Ukraine*
Anastasia Tsarevskaia, *Kharkiv, Ukraine*

We consider the time-optimal problem for a linear control system

$$\dot{x} = Ax + bu, \tag{1}$$

with the target subspace $G = \{x \in \mathbb{R}^n : Hx = 0\}$, where $H$ is a constant real matrix. Here $A, b$ are constant real matrices of the sizes $n \times n$, $n \times 1$, resp., $x \in \mathbb{R}^n$, $u \subset \Omega \subset \mathbb{R}^1$, $\Omega = \{u : |u| \leq 1\}$,

The solution of this problem comes to the Markov min-moment problem [1]. Problem (1) is new and quite difficult to solve. We consider the particular case of the system (1) where $n = 4$:

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3 \end{cases}, \qquad |u| \leq 1. \tag{2}$$

In this case, we find the numerical values of the switching moments $T_1$ and $T_2$ and the time $\theta$ of getting into the subspace $G$ from a given initial point. For the system (2), we consider several types of subspace $G$, in other words, the matrices $H = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \end{pmatrix}$. In our case, the problem is solved by reducing it to a power moment problem. The talk provides a general method for finding the switching moments $T_1$ and $T_2$ and the time-optimal value $\theta$. For clarity, the starting point is chosen and $T_1, T_2, \theta$ are found numerically. We choose the starting point $x_0 = (1, 2, 0, 1)^T$. Solving the system (2), we get numerical values of time $\theta = 11.5286$ and switching moments $T_1 = 3.454691267$, $T_2 = 8.718991267$. Thus, for the system (2), the time-optimal problem of getting into the subspace is solved.

[1] V.I. Korobov and G.M. Sklyar,, *Time-optimality and the power moment problem*, Mat. Sb. (N.S.) **134(176)** (1987), 186–206; Engl. transl.: Math. USSR-Sb. **62** (1989), 185–206.

# Several approaches for solving feedback synthesis for the perturbed double integrator

Valerii Korobov, *Kharkiv, Ukraine*
Tetiana Revina, *Kharkiv, Ukraine*

Let us consider the feedback synthesis for the perturbed double integrator:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (1+r(t,x_1,x_2))x_2 \\ u \end{pmatrix}. \tag{1}$$

Here $t \geq 0$, $(x_1,x_2) \in \mathbb{R}^2$ is a state, $u$ is a scalar control satisfying the constraint $|u| \leq 1$, $r(t,x_1,x_2)$ is *unknown* continuous perturbation which satisfy the imposed constraint $r_1 \leq r(t,x_1,x_2) \leq r_2$. Our approach is based on the controllability function method created by V.I. Korobov in 1979. In [1] the control $u(x)$ which solves the feedback synthesis problem for system (1) without perturbation is given. This control is such that:

1) $|u(x)| \leq 1$;

2) the trajectory $x(t)$ of the closed system $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u(x) \end{cases}$ starting from an arbitrary initial point $x(0) \in \mathbb{R}^2$ ends at the origin at a certain finite period of time.

*The goal of our research* is to find such $r_1 < 0$ and $r_2 > 0$ that for any perturbations $r_1 \leq r(t,x_1,x_2) \leq r_2$ the trajectory $x(t)$ of the closed system (1) with this $u(x)$ starting from an arbitrary initial point $x(0) \in \mathbb{R}^2$ ends at the origin at a certain finite period of time.

**Theorem 1.** *Let* $a_1 < -4.5,\ 0 < \gamma_1 < 1,\ \gamma_2 > 1.$ *The controllability function* $\Theta = \Theta(x_1,x_2)$ *is defined for* $x \neq 0$ *as a unique positive solution of the equation*

$$\frac{(4+a_1)\Theta^4}{a_1(3+a_1)} - a_1x_1^2 + 4\Theta x_1x_2 + \Theta^2x_2^2 = 0. \tag{2}$$

*Let* $r_1^0 = \max\{(1-\gamma_1)\tilde{r}_1^0;\ (1-\gamma_2)\tilde{r}_2^0\}$, $r_2^0 = \min\{(1-\gamma_1)\tilde{r}_2^0;\ (1-\gamma_2)\tilde{r}_1^0\}$,

$$\tilde{r}_1^0 = \frac{2}{a_1} - 2\sqrt{2}\sqrt{-\frac{4+a_1}{a_1^2}},\ \tilde{r}_2^0 = \frac{2}{a_1} + 2\sqrt{2}\sqrt{-\frac{4+a_1}{a_1^2}}.$$

*Denote the control as* $u(x) = \dfrac{a_1x_1}{\Theta^2(x_1,x_2)} - \dfrac{3x_1}{\Theta(x_1,x_2)}.$

*Then for all* $r_1 \le r(t, x_1, x_2) \le r_2$ *such that* $[r_1; r_2] \subset (r_1^0; r_2^0)$ *the trajectory of the closed system starting from an arbitrary initial point* $x(0) = x_0 \in \mathbb{R}^n$ *ends at the point* $x(T) = 0$ *at a certain finite time* $T = T(x_0, r_1, r_2)$ *satisfying the estimate* $\Theta(x_0)/\gamma_2 \le T(x_0, r_1, r_2) \le \Theta(x_0)/\gamma_1$.

[1] A. E. Choque Rivero, V. I. Korobov, and V. A. Skoryk, *The controllability function as the time of motion 1,* J. of Math. Phys., Anal., Geom. **11** (2004), 208 - 225 (Russian).

# The approximate solutions of a complete second-kind hypersingular integral equation

Oleksii Kostenko, *Kharkiv, Ukraine*

The two-dimension problems of electromagnetic wave diffraction on the periodic impedance objects can be reduced to the systems of boundary integral equation [1]. One of these equations is the complete hypersingular integral equation of the second kind. This hypersingular equation is complete in the sense that it includes the complete set of special components. There are hypersingular integral, singular and logarithmic ones.

An algorithm for obtaining approximate solutions of this complete second-kind hypersingular integral equation is proposed. The algorithm is projective in nature and is based on analytical regularization [2] and discretization with using quadrature formulas of the interpolation type [3].

The proposed algorithm is justified. The existence and uniqueness theorem for solution of the complete second-kind hypersingular integral equation is proved. The convergence rate of the sequence of approximate solutions to the exact ones is obtained.

For illustration of the proposed algorithm the model problem on the basis of results [4] was constructed. This is the special hypersingular integral equation of second kind and its exact solution. The comparison of the exact and approximate solutions obtained according to the proposed algorithm shows the features of its applying.

[1] O.V. Kostenko, *Mathematical models of diffraction on prefractal electrodynamics structures,* PhD thesis, A.N. Podgorny institute for mechanical engineering problems of the National academy of sciences of Ukraine, Kharkiv, 2017 (Russian).

[2] Yu.V. Gandel, S.V. Eremenko, and T.S. Polyanskaya, *Mathematical problems of the method of discrete currents. Justification of the numerical method of discrete singularities of the solution of two-dimensional problems of diffraction of electromagnetic waves. Part 2,* M. Gorky Kharkiv State University, Kharkiv, 1992 (Russian).

[3] Yu.V. Gandel, *Introduction to the methods of calculation of singular and hypersingular integrals,* V.N. Karazin Kharkiv National University, Kharkiv, 2001 (Russian).

[4] G.N. Pykhteev, Accurate methods for calculating the Cauchy-type integrals, Nauka, Novosibirsk, 1980 (Russian).

# A Baker–Akhiezer function and finite-gap solutions of the Maxwell–Bloch equations

Vladimir Kotlyarov, *Kharkiv, Ukraine*

The Maxwell–Bloch (MB) equations can be written in the form:

$$\begin{aligned} &\mathcal{E}_t + \mathcal{E}_x = \langle\rho\rangle, \\ &\qquad \langle\rho\rangle = \int\limits_{-\infty}^{\infty} n(\lambda)\rho(t,x,\lambda)d\lambda \\ &\rho_t + 2i\lambda\rho = \mathcal{N}\mathcal{E}, \\ &\qquad \mathcal{N}_t = -\frac{1}{2}(\mathcal{E}^*\rho + \mathcal{E}\rho^*), \end{aligned}$$

where $\mathcal{E} = \mathcal{E}(t,x)$, $\mathcal{N} = \mathcal{N}(t,x,\lambda)$ and $\rho = \rho(t,x,\lambda)$ are unknown functions while the weight function $n(\lambda)$ is given.

The Baker–Akhiezer function theory was successfully developed many years ago, in the middle of 70-th. This theory concerns of completely integrable nonlinear equations. Later this theory was reproduced for the Ablowitz—Kaup-Newell–Segur hierarchies. However, nothing is known about the Baker–Akhiezer function for the Maxwell-Bloch system or for the Karpman–Kaup equations which contain prescribed weight functions. The main goal of the talk is to present a construction of matrix Baker–Akhiezer function associated with the Maxwell–Bloch equations. Using different Riemann–Hilbert problems on the complex plane with cuts we propose such a matrix function that has a unit determinant and takes an explicit form through the Cauchy integrals, hyperelliptic integrals and theta functions. This function solves the AKNS equations associated with the Maxwell-Bloch system and generates a quasi-periodic finite-gap solution to the Maxwell–Bloch equations. We discuss two ways of determining such Baker–Akhiezer functions. The suggested function will be useful for studying of the long time asymptotic behavior of solutions of different initial-boundary value problems for the MB equations using the Deift–Zhou method of steepest descent and for an investigation of rogue waves of the Maxwell–Bloch equations.

[1] V. Kotlyrov, *A matrix Baker–Akhiezer function associated with the Maxwell–Bloch equations and their finite-gap solutions*, SIGMA (submitted), arXiv:1802.01622.

# On some properties of the characteristic vector field of $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds

Tetiana Kovalova, *Kharkiv, Ukraine*

We study geometrical properties of the characteristic vector field $\xi$ of $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds ($\alpha$ and $\beta$ are smooth functions, see [1]) by considering geometry of section $\xi : M \to (T_1M, g_S)$, where $g_S$ is the Sasaki metric. We say that the unit vector field $\xi$ is *minimal* or *totally geodesic*, if the section $\xi(M) \subset (T_1M, g_S)$ is minimal or totally geodesic, respectively.

For the Sasakian manifold ($\alpha = 1$) it is known, that the characteristic vector field is totally geodesic and hence is minimal [2]. In case of $\alpha$-Sasakian manifold we prove the following theorem.

**Theorem 1.** *The characteristic vector field of $\alpha$-Sasakian manifold is minimal if and only if $\alpha = const$ and totally geodesic if and only if $\alpha = 1$.*

In case of $\beta$-Kenmotsu manifold (of the dimension $2n+1$), we introduce the orthonormal frame $\{\xi, e_1, ..., e_{2n}\}$, where $\xi$ is the characteristic unit vector field.

**Theorem 2.** *The characteristic vector field of $\beta$-Kenmotsu manifold is minimal if and only if $e_k(\beta) = 0$ and totally geodesic if and only if*

$$\xi(\beta) = \frac{\beta^2(1+\beta^2)}{1-\beta^2}, \quad e_k(\beta) = 0$$

*for all $k = 1, \ldots, 2n$.*

[1] D.E. Blair and J.A. Oubiña, *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publicacions Matemátiques **34** (1990), 199–207.

[2] A. Yampolsky, *A totally geodesic property of Hopf vector field*, Acta Math. Hungar. **101** (1–2) (2003), 93–112.

# On eigenvalues and singular values of adjacency matrices of regular directed random graphs

Anna Lytova, *Opole, Poland*

We consider adjacency matrices of random $d$-regular directed graphs, that is, matrices uniformly distributed on the set of all $0/1$-valued $n \times n$ matrices such that each row and each column of a matrix has exactly $d$ ones. We are interested in invertibility of such matrices, in quantitative estimates of their singular values, in the structure of their kernels, and in convergence of their empirical spectral distributions to the circular law as $n$ and $d$ tends to infinity. This is a joint work with A. Litvak, K. Tikhomirov, N. Tomczak-Jaegermann, and P. Youssef.

# The XYZ theorem is sharp

Vitalii Marchenko, *Kharkiv, Ukraine*

Vector and subspace Riesz bases are very convenient tools of modern functional analysis, infinite-dimensional linear systems theory and signal processing. If the operator $A$ on a Hilbert space $H$ generates a Riesz basis of $A$-invariant subspaces, then it is possible to split the system $\dot{x}(t) = Ax(t)$ into subsystems and make conclusions on the behavior of the trajectory of the whole system on the basis of the study of projections of the trajectory in these $A$-invariant subspaces, since the series of norms of projections are square summable, see, e.g., [1,2]. So the question to find general conditions, under which $A$ has a Riesz basis of $A$-invariant subspaces, becomes the key question. A breakthrough result, which sums up approximately 30 years of research in linear systems theory, was obtained by G. Q. Xu, S.P. Yung and H. Zwart

**XYZ Theorem** ( [1,2]). *Suppose that the following three conditions hold.*

1. *The operator $A$ generates the $C_0$-group on a Hilbert space $H$.*
2. *The eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of $A$ is a union of $K < \infty$ interpolation sequences $\Lambda_k$, $1 \leq k \leq K$. In other words, $\{\lambda_n\}_{n=1}^{\infty} = \bigcup\limits_{k=1}^{K} \Lambda_k$, where*
$$\min_k \inf_{\lambda_n, \lambda_m \in \Lambda_k :\, n \neq m} |\lambda_n - \lambda_m| > 0. \tag{1}$$
3. *The span of the generalized eigenvectors of $A$ is dense.*

*Then there exists a sequence of spectral projections $\{P_n\}_{n=1}^{\infty}$ of $A$ such that $\{P_n H\}_{n=1}^{\infty}$ forms a Riesz basis of subspaces of $H$ with $\sup\limits_{n \in \mathbb{N}} \dim P_n H \leq K$.*

In [3] the authors presented the class of generators of $C_0$-groups with eigenvalues not satisfying (1) and dense minimal eigenvectors not forming a Schauder basis. The following can be proved on the basis of the results of [3].

**Theorem 1.** *XYZ theorem is sharp. None of its conditions can be weakened.*

[1] G. Q. Xu and S. P. Yung, *The expansion of a semigroup and a Riesz basis criterion*, J. Differential Equations **210** (2005), 1–24.

[2] H. Zwart, *Riesz basis for strongly continuous groups*, J. Differential Equations **249** (2010), 2397–2408.

[3] G. M. Sklyar and V. Marchenko, *Hardy inequality and the construction of infinitesimal operators with non-basis family of eigenvectors*, J. Funct. Anal. **272** (2017), 1017–1043.

# Transformation operators and wave equations in bounded and unbounded domains

Alexander Minakov, *Trieste, Italy*

The initial boundary value problem for a model of stimulated Raman Scattering

$$2\mathrm{i}q_t(x,t) = \mu(x,t),$$

$$\mu_x(x,t) = 2\mathrm{i}\nu(x,t)q(x,t),$$
$$\nu_x(x,t) = \mathrm{i}\left(\overline{q(x,t)}\mu(x,t) - \overline{\mu(x,t)}q(x,t)\right),$$

with the initial data and boundary data

$$q(x,0) = 0, \quad x \in (0,+\infty),$$
$$\mu(0,t) = p\mathrm{e}^{\mathrm{i}\omega t}, \quad \nu(0,t) = l,$$
$$p^2 + l^2 = 1, \quad -1 < l < 0, \quad p > 0.$$

was considered in [1]. The authors showed that in the long-time range $t \to +\infty$ the quarter $x > 0$, $t > 0$ plane is divided into 3 regions with qualitatively different asymptotic behavior of the solution: a region of a finite amplitude plane wave, a modulated elliptic wave region and a vanishing dispersive wave region. The asymptotics in the modulated elliptic region was studied under an implicit assumption of the solvability of the corresponding Whitham type equations. Here we establish the existence of these parameters, and thus justify the results in [1]. This is a joint work with R. Aydagulov.

[1] E. Moskovchenko and V. Kotlyarov. *Periodic boundary data for an integrable model of stimulated Raman scattering: long-time asymptotic behavior,* J. Phys. A **43** (2010), 055205, 31 pp.

# Symmetric extensions of symmetric linear relations (operators) preserving the multivalued part

Vadim Mogilevskii, *Poltava, Ukraine*

Let $\mathfrak{H}$ be a Hilbert space and let $A$ be a symmetric linear relation (in particular nondensely defined operator) in $\mathfrak{H}$. By using the concept of a boundary triplet for $A^*$ we characterize symmetric extensions $\widetilde{A} \supset A$ preserving the multivalued part of $A$. Such a characterization is given in terms of an abstract boundary parameter and the Weyl function of the boundary triplet. Application of these results to the Hamiltonian system $Jy' - B(t)y = \lambda H(t)y$ enabled us to describe its matrix solutions generating the generalized Fourier transform with the nonempty set of respective spectral functions.

The talk is based on the paper [1].

[1] V.I. Mogilevskii, *Symmetric extensions of symmetric linear relations (operators) preserving the multivalued part*, Methods of Functional Analysis and Topology **24** (2018), no. 2.

# On the conditions for special entire functions to belong to the Laguerre–Pólya class

Thu Hien Nguyen, *Kharkiv, Ukraine*
Anna Vishnyakova, *Kharkiv, Ukraine*

The paper [1] answered the question: for which $a > 1$ the partial theta-function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$ belongs to the Laguerre–Pólya class. It is proved, in particular, that there is a constant $q_\infty, (q_\infty \approx 3.2336)$, such that the function $g_a$ (and all its odd Taylor sections $S_{2n+1}(z,a) = \sum_{j=0}^{2n+1} \frac{z^j}{a^{j^2}}$ ) belongs to the Laguerre–Pólya class if and only if $a^2 \geq q_\infty$. A wonderful paper [3] among the other results explains the role of the constant $q_\infty$ in the study of the set of entire functions with positive coefficients having Taylor sections with only real zeros.

We present the following results.

**Theorem 1.** *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, be an entire function. Suppose that the quotions $\frac{a_{n-1}^2}{a_{n-2}a_n}$ are decreasing in $n$, and $\lim_{n\to\infty} \frac{a_{n-1}^2}{a_{n-2}a_n} = b \geq q_\infty$. Then all the zeros of $f$ are real and negative, in other words $f \in \mathcal{L} - \mathcal{P}$.*

**Theorem 2.** *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, be an entire function. Suppose that the quotions $\frac{a_{n-1}^2}{a_{n-2}a_n}$ are increasing in $n$, and $\lim_{n\to\infty} \frac{a_{n-1}^2}{a_{n-2}a_n} = c < q_\infty$. Then the function $f$ does not belong to the Laguerre–Pólya class.*


[1] O. Katkova, T. Lobova and A. Vishnyakova, *On power series having sections with only real zeros,* Comput. Methods Funct. Theory, **3** (2003), 425–441.

[2] O. Katkova, T. Lobova, and A. Vishnyakova, *On entire functions having Taylor sections with only real zeros,* J. Math. Phys., Anal., Geom., **11** (2004), 449–469.

[3] V.P. Kostov and B.Shapiro, *Hardy–Petrovitch–Hutchinson's problem and partial theta function,* Duke Math. J. **162** (2013), 825–861.

[4] Thu Hien Nguyen and Anna Vishnyakova, On the entire functions from the Laguerre–Pólya class having the decreasing second quotients of Taylor coefficients, J. Math. Anal. Appl., to appear.


# K-group identification of supergravity solutions

Tetiana Obikhod, *Kiev, Ukraine*

$D$-branes of superstring theory can be considered as supergravity solutions of type IIB. In the well-known article of Maldacena [1] was argued that the 't Hooft large $N$ limit of $N = 4SU(N)$ gauge theory is dual to Type IIB strings on $AdS^5 \times S^5$. In the article [2] was presented the duality between Type IIB string theory on $AdS^5 \times S^5$ and $D3$-branes at a conical singularity, $(SU(2) \times SU(2))/U(1)$. $Dp$-brane solutions of type IIB are a direct analog of the Schwarzschild black charged hole-black $p$-branes. The black $p$-brane has a mass equal to the charge and therefore, it is BPS state. $p$-brane due to the mass and charge distorts the geometry and its metric is the following

$$ds^2 = H^{-1/2}(r)[-dt^2 + d\vec{x}^2] + H^{1/2}(r)[dr^2 + r^2 g_{ij} dx^i dx^j],$$

where $H(r) = 1 + \frac{L^4}{r^4}$, $\quad L^4 = 4\pi g_s N(\alpha')^2$. and represents a generalization of the solution of the Kerr charged black hole. A black $p$-brane with such a metric has an $RR$ charge $N$, creating a flow through its surrounding $(8-p)$-sphere: $\int\limits_{S^{8-p}} F_{8-p} = N$. From the article [2] we see the coincidence of the black $p$-brane metric and ten-dimensional metric of $N$-parallel $D3$-branes at conical singularity. Using $D$-brane probe on a conifold from the viewpoint of the Azumaya structure on $D$-branes [3], connected with deformation of the classical moduli space from a conifold to a deformed conifold [4], we can classify Hilbert space of $N$ coinciding $Dp$-branes as vector bundles through $K$-functor. For example in [5], was calculated twisted $K$-group for $D6$-brane, $K_1(S^3, n[H]) = Z_n$ with $[H] \in H^3(X, Z)$ - the Dixmier Douady invariant. Thus, $D$-branes on a conifold are classified by twisted $K$-functors and their Hilbert space coincides with the microstates of the black hole.

[1] J. Maldacena, *The Large N limit of superconformal field theories and supergravity*, hep-th/9711200.

[2] Igor R. Klebanov, Edward Witten, *Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity*, hep-th/9807080.

[3] Chien-Hao Liu and Shing-Tung Yau, *Azumaya structure on D-branes and deformations and resolutions of a conifold revisited: Klebanov-Strassler-Witten vs. Polchinski-Grothendieck*, arXiv:0907.0268 [math.AG].

[4] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascade and $\chi$SBresolution of naked singularities*, J. High Energy Phys. **052** (2000), 35 pp., (arXiv:hep-th/0007191).

[5] Yu.M. Malyuta, *Nonlinear analysis and homological algebra*, Nonlinear boundary value problems (2003), issue 13, 114–117.

# On the persistence probability for Kac polynomials and truncations of random orthogonal matrices

Mihail Poplavskyi, *London, United Kingdom*

In this talk we discuss current progress in persistence probability study for Kac polynomials via using applications to Random Matrix Theory. Problems considering random polynomials first appeared in $18^{\text{th}}$ century and have got rapid development in $1940$'s starting from seminal works of Kac, Littlewood & Offord, Hammersley, Ibragimov & Maslova, and others. We consider classical Kac model consisting of random polynomials

$$f_N(z) = \sum_{k=0}^{N} c_k z^k, \text{ with } c_k \text{ being i.i.d. standard Gaussian random variables.}$$

We study the probability for a random real polynomial being positive on the real line, which is obviously connected to the probability of having no real roots

$$p_N := \mathbb{P}\left[f_N(x) > 0, \forall x \in \mathbb{R}\right] = \frac{1}{2}\mathbb{P}\left[f_N(x) \neq 0, \forall x \in \mathbb{R}\right].$$

It was shown recently by Dembo et al. that for even $N = 2n$ persistence probability $p_N$ decays as a power law $n^{-4\theta}$ with the exponent $\theta$ given in terms of the persistence probability for Gaussian stationary stochastic process with covariance kernel $R(t) = \operatorname{sech}(t/2)$. Despite of the fact being explicitly defined, exponent $\theta$ was yet unknown. We use connection between Kac polynomials and truncations of random orthogonal matrices, suggested by Forrester and based on ideas of Krishnapur, and prove

**Theorem 1.** *Let $M_{2n}$ being a random matrix obtained by removing last column and row from uniformly distributed orthogonal matrix $O \in O(2n+1)$. Then*

$$\mathbb{P}\left[M_{2n} \text{ has no real eigenvalues}\right] = \det\left\{\delta_{j,k} - \frac{1}{\pi\left(j+k+3/2\right)}\right\}_{j,k=1}^{n} \sim n^{-\frac{3}{8}}.$$

This confirms prediction $\theta = 3/16$ suggested earlier in physics literature. This is joint project with G. Schehr (LPTMS CNRS) and M. Gebert (QMUL).

[1] G. Schehr and M. Poplavskyi, *Exact persistence exponent for the $2d$-diffusion equation and related Kac polynomials*, to appear, 2018.

[2] M. Gebert and M. Poplavskyi *On pure complex spectrum for truncations of random orthogonal matrices and Kac polynomials*, to appear, 2018.

# The Toda rarefaction problem

Anton Pryimak, *Kharkiv, Ukraine*

The Toda rarefaction wave is the Cauchy problem solution for the doubly infinite Toda lattice

$$\begin{aligned} \dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \end{aligned} \qquad (n,t) \in \mathbb{Z} \times \mathbb{R}_+. \tag{1}$$

The following steplike initial data is considered

$$\begin{aligned} &a(n,0) \to a, \quad b(n,0) \to b, \quad \text{as } n \to -\infty, \\ &a(n,0) \to \frac{1}{2}, \quad b(n,0) \to 0, \quad \text{as } n \to +\infty, \end{aligned} \tag{2}$$

$$1 < b - 2a, \quad a > 0, \quad b \in \mathbb{R}. \tag{3}$$

This problem was studied by Deift et all in [1] in a small transition region $\xi \approx 0$ by applying the nonlinear steepest descent method for the vector oscillatory Riemann-Hilbert problem. In [2] the first and the second terms of the asymptotical expansion for the solution with respect to large $t$ were obtained in all principal regions of $(n,t)$ half plane by use the same approach as in [1] under an additional condition

$$\sum_{n=1}^{\infty} \mathrm{e}^{\nu n}\big(|a(-n,0) - a| + |b(-n,0) - b| + |a(n,0) - \tfrac{1}{2}| + |b(n,0)|\big) < \infty \tag{4}$$

for some $\nu > 0$. The second terms of order $O(t^{-1})$ were obtained in [2] under a conjecture, that these terms got contributions from main transformations of the RH-problem approach only. In [2] any proof that the obtained asymptotics were indeed the asymptotics for the solution of (1)–(4) was absent.

The goal of this work is to justify rigorously the asymptotics obtained in [2], and to prove the conjecture that solution of the parametrix problem indeed does not contribute in the second terms of asymptotic expansion.To this end we solve the parametrix problem and perform in all details a conclusive asymptotic analysis. Moreover, in the present talk the Toda rarefaction problem (1)–(3) is solved completely in the middle principal regions for the general case which admits resonances at the edges of the background spectra.

[1] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49** (1996), 35–83.

[2] I. Egorova, J. Michor and G. Teschl *Rarefaction waves for the Toda equation via nonlinear steepest descent*, Discrete Contin. Dyn. Syst. **38** (2018), 2007-2028.

# Plurisubharmonic geodesics, holomorphic hulls and interpolation

Alexander Rashkovskii, *Stavanger, Norway*

Plurisubharmonic geodesics on a bounded hyperconvex domain $D \subset \mathbb{C}^n$ [1], [2] are local counterparts of weak geodesics in the space of metrics on compact Kähler manifolds due to Mabuchi and Donaldson. In the talk, we relate them to some known notions from complex and convex geometry.

Let $u_t$, $0 \le t \le 1$, be the geodesic whose endpoints $u_0$ and $u_1$ are relative extremal functions of compact, polynomially convex subsets $K_0$ and $K_1$ of $D$, respectively. We characterize the sets $K_t = \{z \in D : u_t(z) = -1\}$ as sections by $\{|\zeta| = e^t\}$ of the holomorphic hull of the set $(K_0 \times A_0) \cup (K_1 \times A_1) \subset \mathbb{C}^{n+1}$ with respect to the collection of all functions holomorphic on $\mathbb{C}^n \times (\mathbb{C} \setminus \{0\})$; here $A_j = \{\zeta \in \mathbb{C} : |\zeta| = e^j\}$, $j = 0, 1$.

In the particular case when $K_0$, $K_1$ are multicircular subsets of the unit polydisk, we have $K_t = K_0^{1-t} K_1^t$ and the geodesic $u_t$ can be represented in terms of the Legendre transform of the convex functions generated by $u_0$ and $u_1$. Furthermore, in this case the relative capacity of $K_t$ is proved to be a logarithmically convex function of $t$.

Part of the results are obtained in collaboration with Dario Cordero-Erausquin.

[1] R.J. Berman and B. Berndtsson, *Moser-Trudinger type inequalities for complex Monge-AmpΓËre operators and Aubin's "hypothèse fondamentale"*, arXiv:1109.1263.

[2] A. Rashkovskii, *Local geodesics for plurisubharmonic functions*, Math. Z. **287** (2017), 73–83.

[3] A. Rashkovskii, *Copolar convexity*, Ann. Polon. Math. **120** (2017), 83–95.

# Partial differential equations with state-dependent delays: different types of solutions

Alexander Rezounenko, *Kharkiv, Ukraine*

A class of non-linear partial differential equations with general type of bounded time delays is investigated. The general form of delay system is

$$\frac{d}{dt}u(t) + Au(t) + du(t) = B(u_t), \tag{1}$$

where, as usual in a delay system with (maximal) delay $h > 0$, for a function $v(t), t \in [a - h, b] \subset \mathbb{R}$, $b > a$, we denote the history segment $v_t = v_t(\theta) \equiv v(t+\theta)$, $\theta \in [-h, 0]$, $t \in [a, b]$. In (1), $A$ is an unbounded linear operator in a Banach space $X$, $B : C \equiv C([-h, 0]; X) \to X$ is a nonlinear (delay) map.

Initial conditions, in general, are

$$u|_{[-h,0]} = \varphi \in C \equiv C([-h, 0]; X). \tag{2}$$

For particular cases, the set of initial functions could be a subset (not necessarily linear) of the space $C$.

A motivating example is reaction-diffusion equations and systems (in bounded domains) with delays in reaction terms. Particular interest is in the case of presence of discrete state-dependent delays. This type of delay is the most relevant to real-world applications and most difficult from mathematical point of view. For a survey on the ODE theory see [1]. The well-posedness in the sense of Hadamard and long time asymptotic behaviour of different types of solutions to (1)-(2) are studied (see, e.g., [2–4]). A recent study is connected to biological problems such as population and viral in-host infection ones with state-dependent delays [4].

[1] F. Hartung, T. Krisztin, H.-O. Walther and J. Wu, *Functional differential equations with state-dependent delays: Theory and applications*, In: *Canada, A., Drabek., P. and A. Fonda (Eds.) Handbook of Differential Equations, Ordinary Differential Equations,* Elsevier Science B.V., North Holland **3** (2006), 435–545.

[2] A.V. Rezounenko, *A condition on delay for differential equations with discrete state-dependent delay*, Journal of Mathematical Analysis and Applications 385 (2012), 506–516.

[3] I.D. Chueshov, A.V. Rezounenko, *Finite-dimensional global attractors for parabolic nonlinear equations with state-dependent delay*, Communications on Pure and Applied Analysis **14/5** (2015), 1685–1704.

[4] A. Rezounenko, *Viral infection model with diffusion and state-dependent delay: Stability of classical solutions*, Discrete and Continuous Dynamical Systems - Series B **23** (2018), 1091–1105.

# Long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation with step-like initial data

Yan Rybalko, *Kharkiv, Ukraine*
Dmitry Shepelsky, *Kharkiv, Ukraine*

We study the initial value problem for the integrable nonlocal nonlinear Schrödinger (NNLS) equation

$$\begin{array}{ll} iq_t(x,t)+q_{xx}(x,t)+2q^2(x,t)\bar{q}(-x,t)=0, & -\infty<x<\infty,\ t\geq 0,\\ q(x,0)=q_0(x), & -\infty<x<\infty, \end{array} \tag{1}$$

in the class of functions satisfying, for all $t\geq 0$, the following nonzero boundary conditions: $q(x,t)\to 0$ as $x\to-\infty$ and $q(x,t)\to A$ as $x\to\infty$, where $A>0$ is an arbitrary constant. The NNLS equation was recently introduced by Ablowitz and Musslimani, see [1].

The main aim of our work is to study the long-time behavior of the solution of this problem. Our approach is based on the asymptotic analysis of the associated matrix Riemann–Hilbert (RH) problem using the ideas of the nonlinear steepest decent method [2]. Under certain assumptions on the initial data expressed in the spectral terms (zeros of the spectral functions and the index of a certain scalar Riemann–Hilbert problem), our main result is that there are two regions in the half-plane $-\infty<x<\infty$, $t>0$, where the asymptotics has qualitatively different form:

1. $q(x,t)=o(1)$, $\quad t\to\infty$, $x<0$,
2. $q(x,t)=A\delta^2(\xi,0)+o(1)$, $\quad t\to\infty$, $x>0$,

where $\xi=\frac{x}{4t}$, $\delta(\xi,k)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{-\xi}\frac{\ln(1+r_1(\zeta)r_2(\zeta))}{\zeta-k}\,d\zeta\right\}$, and $r_j(k)$, $j=1,2$, are the reflection coefficients determined by the initial data. Moreover, using our approach it is possible to obtain explicitly the slow decaying corrections to the main asymptotic terms above, of order $t^{-\frac{1}{2}+\nu(\xi)}$, where $\nu(\xi)$ is determined in terms of the spectral functions associated with the initial data.

A distinctive feature of the matrix RH problem associated with problem (1) is that the jump matrix has a singularity on the jump contour.

[1] M.J. Ablowitz and Z.H. Musslimani, *Integrable nonlocal nonlinear Schrödinger equation,* Phys. Rev. Lett. **110** (2013) 064105.

[2] P.A. Deift and X. Zhou, *A steepest descend method for oscillatory Riemann–Hilbert prob- lems. Asymptotics for the MKdV equation,* Ann. Math. **137** (1993), 295–368.

# Modification of quasistability method for application in study of non-autonomous processes

Iryna Ryzhkvoa-Gerasymova, *Kharkiv, Ukraine*

Quasistability method developed by Igor Chueshov and Irena Lasiecka (see, e.g., [1]) is a powerful tool for investigation of asymptotical behaviour of dynamical systems, generated by autonomous PDEs.

It appeared, that quasistability method can also be used (after appropriate modification) in study of long-time behaviour of non-autonomous equations with translation compact symbols. We consider the following nonlinear non-autonomous wave equation as a representative example

$$u_{tt} - \Delta u + \alpha(t)u_t + \beta(t)g(u) = f(t, x).$$

The time-depending coefficient (time symbol) is called translation compact if the completion of $\{\alpha(t+s),\ s \in \mathbb{R}\}$ (time symbol space) is compact in an appropriate functional space (e.g. $C_b(\mathbb{R})$).

We modify quasistability inequality to account for time symbol, so that it's essential consequences which hold for autonomous equations take place for non-autonomous equations also. We establish asymptotical compactness and improved smoothness of individual trajectories. In the case when time symbol space is of finite fractal or Hausdorff dimension, we find the condition under which a uniform attractor has finite fractal dimension.

[1] I. Chueshov and I. Lasiecka *Well-posedness and long time behavior in nonlinear dissipative hyperbolic-like evolutions with critical exponents* Nonlinear Hyperbolic PDEs, Dispersive and Transport Equations, HCDTE Lecture Notes, Part I. / Ed. by G. Alberti et al. AIMS on Applied Mathematics, vol.6, AIMS, Springfield, 2013, 1–96.

# On $U_q\mathfrak{sl}_2$-symmetries for certain quantum domains

Sergey Sinel'shchikov, *Kharkiv, Ukraine*

The symmetries ($U_q\mathfrak{sl}_2$-module algebra structures) of quantum bounded symmetric domains are discussed, with $U_q\mathfrak{sl}_2$ being the universal enveloping algebra for the Lie algebra $\mathfrak{sl}_2$. Formerly, it was a single such structure being implicit to a specific quantum domain, with no hints on uniqueness of the symmetry in question.

By now several classification works [1–5] disprove the above uniqueness conjecture. Instead, complete lists of quantum symmetries have been made available for the simplest quantum bounded symmetric domains as follows.

1. The quantum plane $\mathbb{C}_q[x, y]$, i.e., the unital algebra given by the two generators $x$, $y$, and a single relation $yx = qxy$.

2. The Laurent extension $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ of (the standard) quantum plane, which differs from the above by allowing the generators $x$, $y$ to be invertible.

3. Certain multidimensional versions $\mathbb{C}_q[x_1, \ldots, x_n]$ of the standard quantum plane, see [2].

4. The quantum disk $Pol(\mathbb{D})_q$, i.e., the unital algebra generated by $z$, $z^*$ subject to the relation $z^*z = q^{-2}zz^* + 1 - q^{-2}$.

It is worthwhile to note that the complete list of $U_q(\mathfrak{sl}_2)$-symmetries on the standard quantum plane (case 1) is rather poor, although it contains an uncountable collection of non-isomorphic symmetries. This list is formed by finitely many (6) series, each of those determined by a single pair of weight constants [1]. On the other hand, it turns out that a passage to the Laurent extension $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ (case 2) essentially increases the collection of symmetries. In particular, the set of pairs of weight constants involved in this case becomes infinite (in fact, uncountable).


[1] S. Duplij and S. Sinel'shchikov, *Classification of $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane*, J. Math. Phys., Anal., and Geom. **6** (2010), 406–430.

[2] S. Duplij, Y. Hong, and F. Li, *$U_q(\mathfrak{sl}_{m+1})$-module algebra structures on the coordinate algebra of a quantum vector space*, J. Lie Theory **25** (2015), 327–361.

[3] S. Sinel'shchikov, *Classification of symmetries on the Laurent extension of quantum plane*, J. Math. Phys., Anal., and Geom. **11** (2015), 333–358.

[4] S. Sinel'shchikov, *The Laurent extension of quantum plane: a complete list of symmetries*, in preparation.

[5] S. Sinel'shchikov, *Classification of $U_q(\mathfrak{sl}_2)$-symmetries on the quantum disk*, in preparation.


## Linearizability of control systems of the class $C^1$


Grigory Sklyar, *Szczecin, Poland*

Kateryna Sklyar, *Szczecin, Poland*

Svetlana Ignatovich, *Kharkiv, Ukraine*


A control system $\dot{x} = f(x, u)$ is called feedback linearizable if there exists a change of variables $z = F(x)$ and a change of control $v = g(x, u)$ which reduce it to a linear form $\dot{z} = Az + Bv$. If $v = u$, the system is called linearizable.

First results in the field were obtained in 1973. Namely, V.I. Korobov [1] introduced a special class of nonlinear systems ("triangular systems") which are feedback linearizable. This class have important applications; it was originated by satellite control problems. Within this approach, triangular systems of the class $C^1$ were treated. On the other hand, A. Krener [2] studied linearizability for systems of the general form of the class $C^\infty$; he proposed to apply the Lie algebraic technique. Later, the linearizability problem in the class $C^\infty$ was completely studied by B. Jakubczyk and W. Respondek [3] and other authors.

In [4] we considered systems of general form *and* of the class $C^1$. It turned out that in this case the conditions of feedback linearizability for systems of the class $C^\infty$ [3] are neither necessary nor sufficient. The new ideas were proposed induced by the original technique of triangular systems. In particular, it was proposed to use some other vector fields instead of Lie brackets which may not exist in the class $C^1$. In the talk we give an overview of the results of [4] and their further development [5], [6].


[1] V.I. Korobov, *Controllability, stability of some nonlinear systems*, Differ. Uravn. 9 (1973), 614–619 (in Russian); English transl.: Differential Equations **9** (1973), 466–469 (1975).

[2] A. Krener, *On the equivalence of control systems and the linearization of non-linear systems*, SIAM J. Control **11** (1973), 670–676.

[3] B. Jakubczyk and W. Respondek, *On linearization of control systems*, Bull. Acad. Sci. Polonaise Ser. Sci. Math. **28** (1980), 517–522.

[4] G.M. Sklyar, K.V. Sklyar, and S.Yu. Ignatovich, *On the extension of the Korobov's class of linearizable triangular systems by nonlinear control systems of the class $C^1$*, Systems and Control Lett. **54** (2005), 1097–1108.

[5] K.V. Sklyar and S.Yu. Ignatovich, *Linearizability of systems of the class $C^1$ with multi-dimensional control*, Systems and Control Lett. **94** (2016), 92–96.

[6] K.V. Sklyar, S.Yu. Ignatovich, and G.M. Sklyar, *Verification of feedback linearizability conditions for control systems of the class $C^1$*, Proceedings of the 25th Mediterranean Conference on Control and Automation (MED), 163–168, Valetta, Malta, 2017.

# Spectral measures of finitely valued stationary sequences and an approximation problem on the circle

Mikhail Sodin, *Tel Aviv, Israel*

We will discuss somewhat striking spectral properties of finitely valued stationary sequences and a related approximation problem on the unit circle. The talk is based on joint works with A. Borichev, A. Nishry, and B. Weiss (arXiv:1409.2736, arXiv:1701.03407) and on a work in progress with A. Borichev and A. Kononova.

# Ambitwistor space realization of $SU(2,2)$ positive energy unitary irreducible representations corresponding to massless fields on anti-de Sitter space

Dmitriy Uvarov, *Kharkiv, Ukraine*

Using the isomorphism [1] of the algebras of quantized twistors [2] and $SU(2)\oplus SU(2)$ oscillators [3] we examine the realization of a class of positive energy unitary irreducible representations of $SU(2,2)$ [4] as homogeneous functions on ambitwistor space [5] and compare it with the oscillator realization [3]. Examined $SU(2,2)$ representations are those with $E=j_1+j_2+2$ and (i) $j_1=j_2\geq 0$ or (ii) $|j_1-j_2|=1/2$ or (iii) $j_1j_2\neq 0$, $|j_1-j_2|>1/2$ [6], where $E$ is the anti-de Sitter energy (conformal dimension) and non-negative (half-)integers $j_{1,2}$ label associated finite-dimensional representation of the Lorentz subalgebra $SO(1,3)$. Penrose transform of respective ambitwistor functions $f(Z^{\boldsymbol{\alpha}},\bar W_{\boldsymbol{\beta}})$ homogeneous of degree $2j_1-2$ in $Z^{\boldsymbol{\alpha}}$ and of degree $2j_2-2$ in $\bar W_{\boldsymbol{\beta}}$ ($\boldsymbol{\alpha},\boldsymbol{\beta}=1,\ldots,4$) yields massless (i) totally symmetric bosonic, (ii) totally symmetric fermionic, (iii) mixed-symmetry bosonic and fermionic fields on 4-dimensional (complexified conformally-compactified) Minkowski space-time that correspond [7] to boundary values of the non-normalizable solutions of the (Fang-)Fronsdal equations [8,9] for massless fields on 5-dimensional anti-de Sitter space.

[1] P. Claus et.al., *Supertwistors as quarks of $SU(2,2|4)$*, J. High Energy Phys. **9905** (1999), 019.

[2] R. Penrose and W. Rindler, *Spinors and Space-time*. V2.: *Spinor and Twistor Methods in Space-time Geometry*, Cambridge Univ. Press, Cambridge, 1986.

[3] M. Gunaydin and N. Marcus, *The spectrum of the $S^5$ compactification of the chiral $N=2$, $D=10$ supergravity and the unitary supermultiplets of $U(2,2|4)$*, Class. Quantum Grav. **2** (1985), L11.

[4] G. Mack, *All unitary ray representations of the conformal group $SU(2,2)$ with positive energy*, Comm. Math. Phys. **55** (1977), 1.

[5] M.G. Eastwood, *Supersymmetry, twistors, and the Yang-Mills equations*, Trans. AMS **301** (1987), 615.

[6] D.V. Uvarov, *Ambitwistors, oscillators and massless fields on $AdS_5$*, Phys. Lett. B **762** (2016), 415.

[7] R.R. Metsaev, *Shadows, currents and $AdS$*, Phys. Rev. D **78** (2008) 106010.

[8] C. Fronsdal, *Singletons and massless, integral-spin fields on de Sitter space*, Phys. Rev. D **20** (1979), 848.

[9] J. Fang and C.Fronsdal, *Massless, half-integer-spin fields in de Sitter space*, Phys. Rev. D **22** (1980), 1361.

# Pointwise estimates for solutions of some high-order nonlinear elliptic equations in the terms of Wolff's potentials

Mykhailo Voitovych, *Sloviansk, Ukraine*

Let $m, n \in \mathbb{N}$, $m \geqslant 3$ and $n > 2(m-1)$. Let $\Omega$ be a bounded open set of $\mathbb{R}^n$. We consider a $2m$-th order nonlinear partial differential equation of the form:

$$\sum_{1\leqslant|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_m u) = f(x), \quad x \in \Omega, \tag{1}$$

where $f \in L^1(\Omega)$, $\nabla_m u = \{D^\alpha u : 1 \leqslant |\alpha| \leqslant m\}$. We assume that the coefficients $\{A_\alpha : 1 \leqslant |\alpha| \leqslant m\}$ are Carathéodory functions satisfying some growth and coercivity conditions (see [2], [3]) suitable for the energy space $W^{1,q}_{m,p}(\Omega) = W^{1,q}(\Omega) \cap W^{m,p}(\Omega)$, where $2n(m-2)/[n(m-1)-2] < p < n/m$, $\max(\overline{p}, mp) < q \leqslant n$, $\overline{p} = 2p/[p(m-1) - 2(m-2)]$.

Let $u \in W^{1,q}_{m,p}(\Omega)$ be an arbitrary generalized (in the sense of distributions) solution of the given equation, $x_0 \in \Omega$ be a Lebesgue point of the function $u$, and let $B_{2R}(x_0) \subset \Omega$ be an open ball with center $x_0$ and radius $2R < 2$. Let $q - 1 < \gamma < \frac{n(q-1)}{n-q+1}$. The main result of our report is the following estimate:

$$|u(x_0)| \leqslant C\left(R^{-n}\int_{B_R(x_0)} |u(x)|^\gamma dx\right)^{1/\gamma} + C\,\mathbf{W}^f_{1,q}(x_0; 2R) + CR^\theta, \tag{2}$$

where $\mathbf{W}^f_{1,q}(x_0; 2R) = \int_0^{2R} \left(r^{q-n}\int_{B_r(x_0)} |f(x)|dx\right)^{1/(q-1)} r^{-1} dr$ is the Wolff potential of $f$, $\theta = \theta(n, m, p, q) \in (0, 1)$, the constant $C > 0$ depends only on $n$, $m$, $p$, $q$, $\gamma$ and on some constants from the structure conditions on the coefficients of Eq. (1). The proof of inequality (2) is based on the development of Kilpeläinen–Malý method proposed in [1] for second-order equations.

**Acknowledgements.** This research was supported by the State Fund for Fundamental Research of Ukraine (Project No. 0116U007160, 0117U006053).

[1] T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), no. 1, 137–161.

[2] A. A. Kovalevsky, *Entropy solutions of Dirichlet problem for a class of nonlinear elliptic high-order equations with $L^1$-data*, Nonlinear Bound. Value Probl. **12** (2002), 119–127.

[3] I. V. Skrypnik, *Higher order quasilinear elliptic equations with continuous generalized solutions*, Differential Equations **14** (1978), no. 6, 786–795.

# Caustics by reflection in the Euclidean n-space

Alexander Yampolsky, *Kharkiv, Ukraine*

*Caustic by reflection* or catacaustic is the envelope of the family of rays emitted from a given point $O$ (source) and reflected by smooth hypersurface $F^n$ (mirror) [1]. We prove the following theorem.

**Theorem 1.** *Let us be given a regular hypersurface $F^n$ parameterized by position-vector*

$$\vec{r} : D^n(u^1, \dots, u^n) \to F^n \subset E^{n+1}.$$

*Suppose $\vec{a} = \frac{1}{r}\vec{r}$ $(r \neq 0)$ is a direction of incidental rays and $\vec{n}$ is a field of unit normals of the surface. Decompose $\vec{a} = \vec{a}_t + \cos\theta\,\vec{n}$ into tangent and normal components. Denote by $k_i^*$ the (always real) real roots of*

$$\left(k^* + \frac{1}{r}\right)^n + \sum_{m=1}^{n} 2^m \cos^{m-2}\theta \left(\binom{n}{m} H_m - \sin^2\theta\, k_{n|m}(a_t)\right) \left(k^* + \frac{1}{r}\right)^{n-m} = 0. \quad (1)$$

*where $k_{n|m}(a_t)$ is the $m$-th normal curvature of $F^n$ in the direction of $\vec{a}_t$ and $H_m$ is the $m$-th mean curvature of $F^n$, respectively. Then over each local domain where $k_i^* \neq 0$ the caustics of reflected front can be parameterized by*

$$\vec{\xi}_i^* = \vec{r} + \frac{1}{k_i^*}\vec{b},$$

*where $\vec{b} = \vec{a} - 2(\vec{a}, \vec{n})\vec{n}$ is the direction of reflected rays.*

The equation (1) is applicable to the case $O = \infty$, i.e. $r \to \infty$. In this case $\vec{a}$ is a unit parallel vector field in $E^{n+1}$. In case $F^2 \subset E^3$ the equation (1) takes the form

$$\left(k^* + \frac{1}{r}\right)^2 + 2\cos\theta\Big(2H + k_n(a_t)\tan^2\theta\Big)\left(k^* + \frac{1}{r}\right) + 4K = 0,$$

where $H$, $K$ and $k_n(a_t)$ are mean, Gaussian and normal curvatures of the surface, respectively. In case of $O = \infty$ the equation simplifies to

$$(k^*)^2 + 2\cos\theta\Big(2H + k_n(a_t)\tan^2\theta\Big)(k^*) + 4K = 0.$$

Some supporting examples and computer simulations will be presented.

[1] J. W. Bruce, P. J. Giblin and C. G. Gibson, *On caustics by reflexion*, Topology 21(1982), no 2, 179 — 199.

# Analytical forms of deuteron wave function for potentials Nijmegen group and density distribution

Viktor Zhaba, *Uzhgorod, Ukraine*

In the short range the structure of a deuteron is visually described by means of density distribution $\rho_d^{M_d}(r', \theta)$ [1] for a projections $M_d = 0;\pm1$ of full angular momentum

$$\begin{cases} \rho_d^0 = \dfrac{4}{\pi}\left[C_0(2r') - 2C_2(2r')P_2(\cos\theta)\right]; \\ \rho_d^{\pm1} = \dfrac{4}{\pi}\left[C_0(2r') + C_2(2r')P_2(\cos\theta)\right]; \end{cases}$$

where $C_0 = R_0^2 + R_2^2$; $C_2 = \sqrt{2} R_0 R_2 - \frac{1}{2} R_2^2$ are components of density distribution; $R_0 = u/r$; $R_2 = w/r$ - the radial functions for S- and D- states; $P_2$ is a Legendre's polynom; $r'$ is the distance from the center of masses; $\theta$ is the polar angle to $r'$; $r = 2r'$ is the between partial distance.

Values of density distribution $\rho_d^{M_d}$ and transition density $\rho_{tr}^{\pm 1}$ [1] are calculated using the earlier obtained coefficients of DWFs [2, 3] analytical forms in coordinate representation for a nucleon-nucleon potentials Nijmegen group (Nijm1, Nijm2, Nijm93).

Results of similar calculations of density distribution and transition density for Argonne v18 potentials are quoted in paper [4]. Depending on a choice of approximation for DWF the calculated values $\rho_d^{M_d}$ and $\rho_{tr}^{\pm 1}$ differ only in the area at 0–0.3 fm. In fact, this indicates, which of the approximations applied is the "best" for beginning of coordinates, despite the absence of redundant knots of the radial DWF.


[1] J.L. Forest et al., *Femtometer toroidal structures in nuclei*, Phys. Rev. C **54** (1996), 646–667.

[2] V.I. Zhaba, *New analytical forms of a deuteron wave function for potentials Nijmegen groups*, Nucl. Phys. Atom. Energy **17** (2016), 22–26.

[3] V.I. Zhaba, *Analytical forms of the wave function in the coordinate space and the tensor polarization of the deuteron for potentials Nijmegen group*, J. Phys. Stud. **20** (2016), 3101.

[4] V.I. Zhaba, *Analytical forms of deuteron wave function and density distribution*, World Scientific News **98** (2018), 206–213.


# Model representations of non-selfadjoint operator bundles

Vladimir Zolotarev, *Kharkiv, Ukraine*

Model representations of quadratic non-selfadjoint operator bundles are obtained. They are expressed in terms of Hilbert and Stieltjes transforms. The explicit form of characteristic function is obtained and its connection with the Riemann matrix boundary problem is determined. Spectrum of the quadratic bundle is described in terms of this model.

# AUTHOR INDEX


| | |
|---|---|
| *Aminov, Yu.* | aminov@ilt.kharkov.ua |
| *Barkhaev, P.* | barkhayev@ilt.kharkov.ua |
| *Bebiya, M.* | m.bebiya@karazin.ua |
| *Bolotov, D.* | bolotov@ilt.kharkov.ua |
| *Borisenko, A.* | borisenko@ilt.kharkov.ua |
| *Brysina, I.* | iryna.brysina@gmail.com |
| *Eremenko, A.* | eremenko@math.purdue.edu |
| *Fardigola, L.* | fardigola@ilt.kharkov.ua |
| *Fastovska, T.* | fastovskaya@karazin.ua |
| *Gefter, S.* | gefter@karazin.ua |
| *Golinskii, L.* | leonid.golinskii@gmail.com |
| *Goncharuk, A.* | angoncharuk@ukr.net |
| *Ignatovich, S.* | ignatovich@ukr.net |
| *Karpenko, I.* | inic.karpenko@gmail.com |
| *Khalina, K.* | khalina@meta.ua |
| *Korobov, V.* | korobov@karazin.ua |
| *Kostenko, O.* | kostenko@ilt.kharkov.ua |
| *Kotlyarov, V.* | vkot44@gmail.com |
| *Kovalova, T.* | t.m.kovalyova@gmail.com |
| *Lytova, A.* | anna.lytova@gmail.com |
| *Makarichev, V.* | victor.makarichev@gmail.com |
| *Marchenko, V.* | v.marchenko@ilt.kharkov.ua |
| *Minakov, A.* | minakov.ilt@gmail.com |
| *Mogilevskii, V.* | vadim.mogilevskii@gmail.com |
| *Nguyen, T.H.* | nguyen.hisha@gmail.com |
| *Obikhod, T.* | tanyamaliuta@gmail.com |
| *Pivnenko, A.* | antroshina14@gmail.com |
| *Poplavskyi, M.* | mpoplavskyi@gmail.com |
| *Pryimak, A.* | pryimakaa@gmail.com |
| *Rabah, R.* | abah.rabah@mines-nantes.fr |
| *Rashkovskii, A.* | alexander.rashkovskii@uis.no |
| *Revina, T.* | t.revina@karazin.ua |
| *Rezounenko, A.* | rezounenko@karazin.ua |
| *Rybalko, Ya.* | rybalkoyan@gmail.com |
| *Ryzhkova-Gerasymova, I.* | iryonok@gmail.com |
| *Shepelsky, D.* | shepelsky@yahoo.com |
| *Sinel'shchikov, S.* | sinelshchikov@ilt.kharkov.ua |
| *Sklyar, G.* | sklar@univ.szczecin.pl |
| *Sklyar, K.* | sklar@univ.szczecin.pl |
| *Sodin, M.* | sodin@post.tau.ac.il |
| *Sukhorebska, D.* | suhdaria0109@gmail.com |
| *Tsarevskaia, A.* | tsarevskaia.nastya@gmail.com |
| *Uvarov, D.* | d_uvarov@hotmail.com |
| *Voitovych, M.* | voitovichmv76@gmail.com |
| *Yampolsky, A.* | a.yampolsky@karazin.ua |
| *Zhaba, V.* | viktorzh@meta.ua |
| *Zolotarev, V.* | zolotarev@ilt.kharkov.ua |